B1.1 Logic
Lecture 10

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10. Free and bound variables

Recall Example 9.3: The formula

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \to x_0 \doteq x_1)$$

- is true in $\langle \mathbb{Z}; \cdot \rangle$ under any assignment v with $v(x_2) = 2$,
- but false when $v(x_2) = 0$.

Whether or not $\mathcal{A} \models \phi[v]$ depends on $v(x_2)$ but not on $v(x_0)$ or $v(x_1)$.

This is because all occurrences of x_0 and x_1 in ϕ are subordinate to the corresponding quantifiers $\forall x_0$ and $\forall x_1$.
We say that these occurrences are **bound**, while the occurrence of x_2 is **free**.

10.1 Definition

Let $\mathcal{L}$ be a first-order language, ϕ an $\mathcal{L}$-formula, and $x \in \{x_0, x_1, \ldots\}$ a variable.

An occurrence of x in ϕ is **free**, if

(i) ϕ is atomic; or

(ii) $\phi = \neg\psi$ resp. $\phi = (\chi \to \rho)$, and the occurrence of x is free in ψ resp. in χ or in ρ; or

(iii) $\phi = \forall x_i \psi$, and $x \neq x_i$, and the occurrence of x is free in ψ.

The variables which occur free in ϕ are called the **free variables of** ϕ, Free(ϕ) := $\{x_i : x_i \text{ occurs free in } \phi\}$.

An occurrence which is not free is **bound**. In particular, if $\phi = \forall x_i \psi$, then any occurrence of x_i in ϕ is bound.

10.2 Example

$$(\exists x_0 P(\underbrace{x_0}_{bnd}, \underbrace{x_1}_{free}) \vee \forall x_1 (P(\underbrace{x_0}_{free}, \underbrace{x_1}_{bnd}) \to \exists x_0 P(\underbrace{x_0}_{bnd}, \underbrace{x_1}_{bnd})))$$

10.3 Lemma

Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let v_1, v_2 be assignments in $\mathcal{A}$, and let ϕ be an $\mathcal{L}$-formula.

Suppose $v_1(x_i) = v_2(x_i)$ for every variable x_i with a free *occurrence in ϕ.*

Then

$$\mathcal{A} \models \phi[v_1] \text{ iff } \mathcal{A} \models \phi[v_2].$$

Proof:
For ϕ atomic: exercise.

Now use induction on the length of ϕ.
If $\phi = \neg\psi$ or $\phi = (\chi \to \rho)$, this is straightforward.

So say $\phi = \forall x_i \psi$.
IH: Assume the Lemma holds for ψ.

Suppose $\mathcal{A} \models \forall x_i \psi[v_1]$. $(\star)$
We want to show $\mathcal{A} \models \forall x_i \psi[v_2]$. So suppose $v_2^\star$ agrees with v_2 except possibly at x_i;
we want to show $\mathcal{A} \models \psi[v_2^\star]$.

Let $v_1^\star(x_j) := \begin{cases} v_1(x_j) & \text{if } j \neq i \\ v_2^\star(x_i) & \text{if } j = i \end{cases}$
Then $v_1^\star$ agrees with v_1 except possibly at x_i.
So by $(\star)$, $\mathcal{A} \models \psi[v_1^\star]$.

Now suppose x_j occurs free in ψ.
We show $v_2^\star(x_j) = v_1^\star(x_j)$.
If $j = i$, this is by definition of $v_1^\star$.
If $j \neq i$, then x_j occurs free in ϕ, so

$$v_2^\star(x_j) = v_2(x_j) = v_1(x_j) = v_1^\star(x_j).$$

So by IH, $\mathcal{A} \models \psi[v_2^\star]$, as required

$\square$

10.4 Corollary

Let $\mathcal{L}$ be a language, and let $\alpha, \beta \in \text{Form}(\mathcal{L})$. Assume the variable x_i has no free occurrence in α (i.e. $x_i \notin \text{Free}(\alpha)$). Then

$$\models (\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta)).$$

Proof:

Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$ such that

$\mathcal{A} \models \forall x_i(\alpha \to \beta)[v]$. $(\star)$

To show: $\mathcal{A} \models (\alpha \to \forall x_i \beta)[v]$.

So suppose $\mathcal{A} \models \alpha[v]$.
To show: $\mathcal{A} \models \forall x_i \beta[v]$.

So let $v^\star$ be an assignment agreeing with v except possibly at x_i.
To show: $\mathcal{A} \models \beta[v^\star]$.

x_i is *not* free in $\alpha \Rightarrow_{10.3} \mathcal{A} \models \alpha[v^\star]$
$(\star) \Rightarrow \mathcal{A} \models (\alpha \to \beta)[v^\star]$
$\Rightarrow \mathcal{A} \models \beta[v^\star]$. □

10.5 Definition
A formula σ with no free (occurrences of) variables is called a **statement** or a **sentence**.

Then (by 10.3) for any $\mathcal{L}$-structure $\mathcal{A}$, whether or not $\mathcal{A} \models \sigma[v]$ does not depend on the choice of assignment v.

So we write

$$\mathcal{A} \models \sigma$$

if $\mathcal{A} \models \sigma[v]$ for some/all v.

Say: σ is **true** in $\mathcal{A}$, or $\mathcal{A}$ is a **model** of σ.

($\leadsto$ 'Model Theory')

10.6 Example

Let $\mathcal{L} = \{f, c\}$ be a language, where f is a binary function symbol, and c is a constant symbol.

Consider the sentences (writing x, y, z for x_0, x_1, x_2)

$$\begin{array}{ll} \sigma_1: & \forall x \forall y \forall z f(x, f(y, z)) \doteq f(f(x, y), z) \\ \sigma_2: & \forall x \exists y (f(x, y) \doteq c \wedge f(y, x) \doteq c) \\ \sigma_3: & \forall x (f(x, c) \doteq x \wedge f(c, x) \doteq x) \end{array}$$

and let $\sigma = (\sigma_1 \wedge \sigma_2 \wedge \sigma_3)$

Let $\mathcal{A} = \langle A; \cdot, e \rangle$ be an $\mathcal{L}$-structure (i.e. $\cdot$ is an interpretation of f, and e is an interpretation of c).

Then $\mathcal{A} \models \sigma$ iff $\mathcal{A}$ is a group.

10.7 Example

Let $\mathcal{L} = \{E\}$ with E a binary relation symbol. Consider

$$\begin{aligned}
\tau_1 &: \ \forall x E(x,x) \\
\tau_2 &: \ \forall x \forall y (E(x,y) \leftrightarrow E(y,x)) \\
\tau_3 &: \ \forall x \forall y \forall z (E(x,y) \rightarrow (E(y,z) \rightarrow E(x,z)))
\end{aligned}$$

Then for any $\mathcal{L}$-structure $\langle A; R \rangle$:
$\langle A; R \rangle \models \bigwedge_i \tau_i$ iff R is an equivalence relation on A.

Note: Many mathematical concepts can be naturally expressed by first-order formulas.

10.8 Example
Let $<$ be a binary predicate symbol, $\mathcal{L} := \{<\}$. Consider the sentence

$$\begin{aligned}\sigma := \forall x \forall y \forall z\ (\neg x < x \\ &\wedge (x < y \vee x \doteq y \vee y < x) \\ &\wedge ((x < y \wedge y < z) \to x < z) \\ &\wedge (x < y \to \exists w\ (x < w \wedge w < y)) \\ &\wedge \exists w\ w < x \\ &\wedge \exists w\ x < w).\end{aligned}$$

This axiomatises a **dense linear order without endpoints**. In particular, $\langle \mathbb{Q}; < \rangle \models \sigma$ and $\langle \mathbb{R}; < \rangle \models \sigma$.

But: *'Completeness'* of $\langle \mathbb{R}; < \rangle$ is not captured by the first-order language $\mathcal{L}$, but rather in *second-order* terms, meaning that we also allow quantification over *subsets* of $\mathbb{R}$:

$$\forall A, B \subseteq \mathbb{R}(A < B \to \exists c \in \mathbb{R}(A \leq \{c\} \leq B)),$$

writing $A < B$ to mean that $a < b$ for every $a \in A$ and every $b \in B$, similarly for $A \leq B$.