

# Infinite Groups

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## About Maths again

**William Thurston:** “Experience has shown repeatedly that a mathematical theory with a rich internal structure generally turns out to have significant implications for the understanding of the real world, often in ways no one could have envisioned before the theory was developed.”

## Nilpotent Groups: a key property

### Theorem

*Every subgroup  $H$  of a finitely generated nilpotent group  $G$  is finitely generated.*

**Proof** by induction on the class of nilpotency  $k$  of  $G$ .

For  $k = 1$ ,  $G$  is **abelian finitely generated**. Assume the assertion true for  $k$ .

Let  $G$  be nilpotent of class  $k + 1$ , let  $H \leq G$ .

By the inductive assumption,  $H_1 = H \cap C^2G$  is finitely generated.

$H_2 = H/(H \cap C^2G)$  is finitely generated because subgroup of  $G/C^2G$ , abelian finitely generated.

Thus,  $H$  fits into the short exact sequence

$$1 \rightarrow H_1 \rightarrow H \xrightarrow{\pi} H_2 \rightarrow 1,$$

where  $H_1, H_2$  are finitely generated.

Therefore  $H$  is also finitely generated. □

# Nilpotent Groups

We generalize  $[C^i G, G] = C^{i+1} G$  to: the lower central series is graded, that is

## Proposition

For every  $i, j \geq 1$

$$[C^i G, C^j G] \leq C^{i+j} G. \quad (1)$$

First, recall that  $[a, b]^{-1} = [b, a]$ , whence  $[A, B] = [B, A]$ .

## Lemma

If  $A, B, C$  normal subgroups in  $G$ , then  $[A, B, C] \triangleleft G$  and it is generated by  $[a, b, c]$  with  $a \in A, b \in B, c \in C$ .

## Generation of $[A, B, C]$

**Proof.**  $[A, B, C] \triangleleft G$  follows from  $[x, y]^g = [x^g, y^g]$ .

$[A, B, C]$  generated by  $[k, c]$ ,  $c \in C$ ,  $k$  product of  $n$  commutators  $[a, b]$  or inverses.

We prove, by induction on  $n$ , that  $[k, c]$  is a product of finitely many  $[a, b, c]$  and inverses.

$n = 1$ : consider the case  $[t^{-1}, c]$ , where  $t = [a, b]$ .

$$[t^{-1}, c] = [c, t]^{t^{-1}} = [c^{t^{-1}}, t] = [c', t] = [t, c']^{-1} = [a, b, c']^{-1}.$$

## Generation of $[A, B, C]$ 2

Assume the statement is true for  $n$ , let  $k = k_1 t$ , where  $t$  is  $[a, b]$  or  $[a, b]^{-1} = [b, a]$ , and  $k_1$  product of  $n$  commutators.

$$[k_1 t, c] = [t, c]^{k_1} [k_1, c].$$

Both  $[t, c]^{k_1}$  and  $[k_1, c]$  are products of commutators  $[a, b, c]$  and inverses, by the induction assumption and the fact that  $A, B, C$  are normal subgps.  $\square$

### Exercise

*Prove the same result for  $[H_1, \dots, H_n]$ , where all  $H_i$  are normal subgroups of  $G$ .*

## Second Key Lemma

### Lemma

Assume that  $A, B, C$  are normal subgroups in  $G$ . Then

$$[A, B, C] \leq [B, C, A][C, A, B]. \quad (2)$$

**Proof** Uses the previous Lemma and the **Hall identity**:

$$[x^{-1}, y, z]^x [z^{-1}, x, y]^z [y^{-1}, z, x]^y = 1. \quad (3)$$

The latter identity implies

$$[a, b, c]^{a^{-1}} \leq [B, C, A][C, A, B].$$

□

## Proof of Proposition

We prove by induction on  $i \geq 1$  that for every  $j \geq 1$ ,

$$[C^i G, C^j G] \leq C^{i+j} G. \quad (4)$$

For  $i = 1$ : definition of  $C^{j+1} G$ .

Assume true for  $i$  and prove for  $i + 1$ .

Consider  $j \geq 1$  arbitrary.

$$[C^{i+1} G, C^j G] = [C^i G, G, C^j G] \leq [G, C^j G, C^i G][C^j G, C^i G, G] \leq$$

$$[C^{j+1} G, C^i G][C^{j+i} G, G] = [C^i G, C^{j+1} G]C^{j+i+1} G \leq C^{j+i+1} G,$$

since  $[C^i G, C^{j+1} G] \leq C^{j+i+1} G$  by the inductive assumption.

□



## Back to a more global picture

We study the following classes of groups:

Nilpotent finitely generated  $\subset$  Polycyclic  $\subset$  Solvable finitely generated

### Definition

Given a class  $\mathcal{X}$  of groups, a group  $G$  is said to be **poly- $\mathcal{X}$**  if it admits a subnormal descending series:

$$G = G_0 \triangleright G_1 \triangleright \dots \triangleright G_k \triangleright G_{k+1} = \{1\},$$

such that each  $G_i/G_{i+1}$  belongs to the class  $\mathcal{X}$ .

**Polycyclic** if  $\mathcal{X}$  = all cyclic groups.

**Poly- $C_\infty$**  if  $\mathcal{X} = \{\mathbb{Z}\}$ .

**Solvable** if  $\mathcal{X}$  = all abelian groups.

## Differences and similarities

**Nilpotent finitely generated groups** = the only groups with polynomial growth.

**Polycyclic (hence also nilpotent f.g.) groups** = finitely presented, linear (therefore residually finite) while **solvable groups** are not necessarily finitely presented, linear or residually finite.

A different behaviour of the **torsion**:

$$\text{Tor}G = \{g \in G \mid \exists n \geq 1 \text{ s.t. } g^n = 1\}.$$

When  $G$  nilpotent,  $\text{Tor}G$  is a **characteristic subgroup** of  $G$ .

When  $G$  nilpotent and moreover f.g.,  $\text{Tor}G$  is a **finite characteristic subgroup** of  $G$ .

When  $G$  polycyclic,  $\text{Tor}G$  not necessarily a **subgroup of  $G$** , nor a **finite subset of  $G$** . Examples in Ex. Sheet 2.

# Torsion for nilpotent groups

## Theorem

*When  $G$  is nilpotent (not necessarily finitely generated),  $\text{Tor}G$  is a characteristic subgroup.*

**Proof** by induction on the nilpotency class.

A key result for this induction:

## Lemma

*Let  $G$  be nilpotent of class  $k$ . For every  $x \in G$  the subgroup  $H$  generated by  $x$  and  $C^2G$  is a normal subgroup, nilpotent of class  $\leq k - 1$ .*

## Proof that $H = \langle x, C^2G \rangle$ of class $\leq k - 1$ .

Since  $C^2G \triangleleft G$ ,

$$H = \{x^m c \mid m \in \mathbb{Z}, c \in C^2G\}.$$

**$H$  normal:**  $\forall g \in G$ , and  $h \in H$ ,  $h = x^m c$ ,  $ghg^{-1} = x^m [x^{-m}, g] g c g^{-1}$ .

The last two factors are in  $C^2G \Rightarrow$  the product is in  $H$ .

We prove  $C^2H \leq C^3G$  (implying  $H$  is of class  $\leq k - 1$ ).

Let  $h = x^m c_1$ ,  $h' = x^n c_2$  with  $c_i \in C^2G$ .

$$[h, h'] = [h, x^n c_2] = [h, x^n] [x^n, [h, c_2]] [h, c_2].$$

The last term is in  $C^3G$ , hence the middle term is in  $C^4G$ .

The first term can be rewritten as

$$[h, x^n] = [x^m c_1, x^n] = [x^m, [c_1, x^n]] [c_1, x^n].$$

The last term is in  $C^3G$  and the first in  $C^4G$ . □

# Torsion for nilpotent groups

## Theorem

*When  $G$  is nilpotent (not necessarily finitely generated),  $\text{Tor}G$  is a characteristic subgroup.*

**Proof** by induction on the nilpotency class, using:

## Lemma

*Let  $G$  be nilpotent of class  $k$ . For every  $x \in G$  the subgroup  $H$  generated by  $x$  and  $C^2G$  is a normal subgroup, nilpotent of class  $\leq k - 1$ .*