Infinite Groups

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Torsion for nilpotent groups

Theorem

When G is nilpotent (not necessarily finitely generated), $\mathrm{Tor}G$ is a characteristic subgroup.

Proof by induction on the nilpotency class, using:

Lemma

Let G be nilpotent of class k. For every $x \in G$, the subgroup H generated by x and C^2G is a normal subgroup, nilpotent of class $\leqslant k-1$.

For $k = 1$, G is abelian, statement immediate.

$\mathrm{Tor}\, G$ is a subgroup

Proof of Theorem continued.

Assume statement true for nilpotent groups of class $\leqslant k$, consider a $(k+1)$–step nilpotent group G.

For two elements a, b of finite order in G, we prove ab is of finite order.

$B = \langle b, C^2 G \rangle$ is nilpotent of class $\leqslant k$.

The inductive assumption $\Rightarrow \mathrm{Tor} B$ is a characteristic subgroup of B.

$B \lhd G \Rightarrow \mathrm{Tor} B \lhd G$.

Assume a is of order m.

Then

$$(ab)^m = aba^{-1}a^2ba^{-2}a^3b \cdots a^{-m+1}a^m ba^{-m}\,.$$

The right-hand side is a product of conjugates of $b \Rightarrow$ it is in $\mathrm{Tor} B$.

We conclude that $(ab)^m$ is of finite order. □

Torsion of nilpotent f.g. groups

A torsion group G= all elements of finite order (i.e. $G = \mathrm{Tor}\, G$).

A torsion-free group G= a group with $\mathrm{Tor}\, G = \{1\}$.

Proposition

A finitely generated nilpotent torsion group is finite.

Proof by induction on the nilpotency class k.

For $k = 1$ it follows from the classification of f. g. abelian groups.

Assume true for nilpotent groups of class $\leqslant n$, consider G f. g. torsion group that is $(n+1)$–step nilpotent.

C^2G and G/C^2G are finite, by the inductive assumption, whence G finite. □

Mal'cev's Theorem

Corollary

If G is nilpotent finitely generated then $\mathrm{Tor}\, G$ is a finite subgroup.

Proposition

If G is nilpotent then $G/\mathrm{Tor}\, G$ is torsion-free.

Theorem (Mal'cev)

Every finitely generated torsion-free nilpotent group G of class k embeds as a discrete subgroup in a simply-connected nilpotent Lie group L of class k, s.t. L/G compact.

Generalizes the case of finitely generated torsion-free abelian groups $\mathbb{Z}^n \leqslant \mathbb{R}^n$.

Another theorem of Mal'cev

Theorem (Mal'cev)

Let G be a nilpotent group. The following are equivalent:

(a) $Z(G)$ is torsion-free;

(b) Each quotient $Z_{i+1}(G)/Z_i(G)$ is torsion-free;

(c) G is torsion-free.

Remark

The above characterization of "torsion-free" is not true if we replace the upper central series by the lower central series (Ex. Sheet 3).

Mal'cev's Theorem on torsion 2

(a)$\Rightarrow$ (b). By induction on the nilpotency class n of G. Clear for $n = 1$.
Assume true for nilpotent groups of class $< n$.
We first prove that the group $Z_2(G)/Z_1(G)$ is torsion-free.
We show that for each non-trivial $\bar{x} \in Z_2(G)/Z_1(G)$, there exists a homomorphism $\varphi\colon Z_2(G)/Z_1(G) \to Z_1(G)$ such that $\varphi(\bar{x}) \neq 1$.
Let $x \in Z_2(G)$ be an element which projects to $\bar{x} \in Z_2(G)/Z_1(G)$.Thus $x \notin Z_1(G)$, therefore there exists $g \in G$ such that $[g, x] \in Z_1(G) \setminus \{1\}$.
Define the map $\tilde{\varphi} : Z_2(G) \to Z_1(G)$ by:

$$\tilde{\varphi}(y) := [y, g].$$

Clearly, $\tilde{\varphi}(x) \neq 1$; $\tilde{\varphi}$ is a homomorphism (exercise).
Since $Z_1(G)$ is the center of G, $\tilde{\varphi}$ descends to a homomorphism $\varphi : Z_2(G)/Z_1(G) \to Z_1(G)$.
Since $Z_1(G)$ is torsion-free, $Z_2(G)/Z_1(G)$ is torsion-free.

Mal'cev's Theorem on torsion 3

We replace G by the group $\bar{G} = G/Z_1(G)$.
Since $Z_2(G)/Z_1(G)$ is torsion-free, the group $\bar{G}$ has torsion-free center.
By the inductive assumption, $Z_i(\bar{G})/Z_{i-1}(\bar{G})$ is torsion-free for every $i \geq 1$.

$$Z_i(\bar{G})/Z_{i-1}(\bar{G}) \cong Z_{i+1}(G)/Z_i(G),$$

for every $i \geqslant 1$.
Thus, every group $Z_{i+1}(G)/Z_i(G)$ is torsion-free, proving (b).
(b)⇒ (c). Let k be the nilpotency class, i.e. $G = Z_k(G)$.
$G = \bigsqcup_{i=1}^{k} [Z_i(G) \setminus Z_{i-1}(G)] \sqcup \{1\}$.
For each i, each $x \in Z_i(G) \setminus Z_{i-1}(G)$ and each $m \neq 0$ we have that $x^m \notin Z_{i-1}(G)$.
Thus $x^m \neq 1$.
Therefore, G is torsion-free. □

Polycyclic groups

Definition

Let $\mathcal{X}$ be a class of groups.
G is poly-$\mathcal{X}$ if it admits a subnormal descending series:

$$G = N_0 \rhd N_1 \rhd \ldots \rhd N_k \rhd N_{k+1} = \{1\}, \quad (1)$$

such that each N_i/N_{i+1} belongs to $\mathcal{X}$, up to isomorphism.

Polycyclic if $\mathcal{X} =$ all cyclic groups.
Poly-C_∞ if $\mathcal{X} = \{\mathbb{Z}\}$.
Cyclic series of G= a series as in (1) with $\mathcal{X}$ set of cyclic groups. Its length is the number of non-trivial groups.
The length $\ell(G)$ of a polycyclic group is the least length of a cyclic series of G.
C_∞ series of G= a series as in (1) with $\mathcal{X} = \{\mathbb{Z}\}$.
By convention, $\{1\}$ is poly-C_∞.

Properties of polycyclic groups

Remark

1. *If G is poly-C_∞ then $N_i \simeq N_{i+1} \rtimes \mathbb{Z}$ for every $i \geqslant 0$; thus, the group G is obtained from $N_n \simeq \mathbb{Z}$ by successive semidirect products with $\mathbb{Z}$.*
2. *The above is no longer true for polycyclic groups (with $\mathbb{Z}$ replaced by "cyclic"). However: every polycyclic group contains a normal subgroup of finite index which is poly-C_∞.*

Proposition

A polycyclic group has the bounded generation property. More precisely, let G be a group with a cyclic series of length n and let t_i be such that $t_i N_{i+1}$ is a generator of N_i/N_{i+1}. Then every $g \in G$ can be written as $g = t_1^{k_1} \cdots t_n^{k_n}$, with $k_1, \dots, k_n$ in $\mathbb{Z}$.

Proof by induction on the length of the series. □

Properties of polycyclic groups 2

Corollary

A polycyclic torsion group is finite.

Remark

1. *It is not true that, for G polycyclic, $\mathrm{Tor}(G)$ is either a subgroup or a finite set. Example: D_∞.*
2. *However, every polycyclic group is virtually torsion-free.*

Definition

A group is said to have property * virtually if some finite-index subgroup of it has the property *.

Further properties of polycyclic groups

Proposition

1. *Any subgroup H of a polycyclic group G is polycyclic (hence, finitely generated).*
2. *If $N \lhd G$, then G/N is polycyclic.*
3. *If $N \lhd G$ and both N and G/N are polycyclic then G is polycyclic.*
4. *Properties (1) and (3) hold with 'polycyclic' replaced by 'poly-C_∞', but not (2): $\mathbb{Z}_k$ is a quotient of $\mathbb{Z}$.*