Infinite Groups

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Carl Friedrich Gauss

Carl Friedrich Gauss: "It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment."

Carl Friedrich Gauss: "Theory attracts practice as the magnet attracts iron."

Properties of polycyclic groups

Proposition

1. *Any subgroup H of a polycyclic group G is polycyclic (hence, finitely generated).*
2. *If $N \lhd G$, then G/N is polycyclic.*
3. *If $N \lhd G$ and both N and G/N are polycyclic then G is polycyclic.*
4. *Properties (1) and (3) hold with 'polycyclic' replaced by 'poly-C_∞', but not (2): $\mathbb{Z}_k$ is a quotient of $\mathbb{Z}$.*

Proof. (1). Given a cyclic series for G as above, the intersections $H \cap N_i$ define a cyclic series for H.

Properties of polycyclic groups 2

(2). Proof by induction on the length $\ell(G) = n$.
For $n = 1$, G cyclic, any quotient of G is cyclic.
Assume true for all $k \leqslant n$, consider G with $\ell(G) = n + 1$.
Let N_1 be the first term $\neq G$ in a cyclic series of length $n + 1$.

By induction, $N_1/(N_1 \cap N) \simeq N_1 N/N$ is polycyclic.

$N_1 N/N \lhd G/N$ and $(G/N)/(N_1 N/N) \simeq G/N_1 N$ is cyclic, as quotient of G/N_1. It follows that G/N is polycyclic.

(3). Consider the cyclic series

$$G/N = Q_0 \geqslant Q_1 \geqslant \cdots \geqslant Q_n = \{\bar{1}\}$$

and

$$N = N_0 \geqslant N_1 \geqslant \cdots \geqslant N_k = \{1\}\,.$$

Given $\pi : G \to G/N$ and $H_i := \pi^{-1}(Q_i)$, a cyclic series for G is:

$$G \geqslant H_1 \geqslant \ldots \geqslant H_n = N = N_0 \geqslant N_1 \geqslant \ldots \geqslant N_k = \{1\}\,.$$

Two key examples of polycyclic groups

Proposition

Every finitely generated nilpotent group is polycyclic.

Proof Consider the finite descending series with terms $C^k G$.

- For every $k \geqslant 1$, $C^k G / C^{k+1} G$ is finitely generated abelian, hence there exists a finite subnormal descending series

$$C^k G = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_n \geqslant A_{n+1} = C^{k+1} G$$

such that every quotient A_i / A_{i+1} is cyclic.

- By inserting all these finite descending series into the one defined by the $C^k G$'s, we obtain a finite subnormal cyclic series for G. □

Proposition

Given any homomorphism $\varphi : \mathbb{Z}^n \to \mathrm{Aut}(\mathbb{Z}^m)$, *the semidirect product* $\mathbb{Z}^m \rtimes_\varphi \mathbb{Z}^n$ *is poly-*C_∞.

Two key properties of polycyclic groups

Proposition

Polycyclic groups are finitely presented and residually finite.

Finite presentation is proved using a general property:

Proposition

Let $N \lhd G$. If both N and G/N are finitely presented then G is finitely presented.

Proof Let $N = \langle X \mid r_1, \dots, r_k \rangle$, and $G/N = \langle \overline{Y} \mid \rho_1, \dots, \rho_m \rangle$ be finite presentations, where Y is a finite subset of G s. t. $\overline{Y} = \{yN \mid y \in Y\}$. G is generated by $S = X \cup Y$. S satisfies the following relations:

$$r_i(X) = 1\,,\ 1 \leqslant i \leqslant k\,,\ \rho_j(Y) = u_j(X)\,,\ 1 \leqslant j \leqslant m\,, \tag{1}$$

$$x^y = v_{xy}(X)\,,\ x^{y^{-1}} = w_{xy}(X)\,. \tag{2}$$

We denote the above finite set of relations by T.

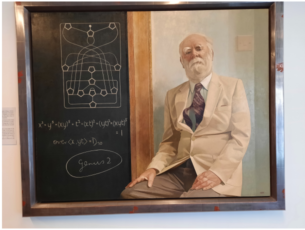
Proof continued

We claim that $G = \langle S \mid T \rangle$. Let $K = \langle\langle T \rangle\rangle$ in $F(S)$.
The epimorphism $\pi_S : F(S) \to G$ defines an epimorphism $\varphi : F(S)/K \to G$. Goal: to prove φ is an isomorphism.
Let wK be an element in $\ker(\varphi)$, w word in S.
Relations (2) imply that there exist a word $w_1(X)$ in X and a word $w_2(Y)$ in Y, such that $wK = w_1(X)w_2(Y)K$.
Applying the projection $\pi : G \to G/N$, we see that $\pi(\varphi(wK)) = 1$, i.e. $\pi(\varphi(w_2(Y)K)) = 1$.
Therefore $w_2(Y)$ is a product of finitely many conjugates of $\rho_i(Y)$, hence $w_2(Y)K$ is a product of finitely many conjugates of $u_j(X)K$, by (1).
This and the relations (2) imply that $w_1(X)w_2(Y)K = v(X)K$ for some word $v(X)$ in X.
Then the image $\varphi(wK) = \varphi(v(X)K)$ is in N; therefore, $\varphi(v(X)K) = 1 \Leftrightarrow v(X)$ is a product of finitely many conjugates of $r_i(X)$.
This implies that $v(X)K = K$. □

Graham Higman

Remark

G finitely presented does not imply $H \leqslant G$ finitely presented or G/N finitely presented, for $N \lhd G$.



Theorem

Every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group.

Finite presentation continued

Proposition

Let G be a group, and $H \leq G$ such that $|G : H|$ is finite. Then G is FP if and only if H is FP.

Proof Suppose $G = \langle X \mid R \rangle$ with X and R finite.
We have an epimorphism $\pi : F = F(X) \to G$ with $K = \ker \pi = \langle\langle R \rangle\rangle$.
Let $E = \pi^{-1}(H)$. Then $|F : E| = |G : H|$ is finite, so $E = F(Y)$ for some finite Y.
Since $K \leqslant E$, each $r \in R$ satisfies $r = s_r(Y)$ for some word s_r on Y. Put $S = \{s_r(Y) \mid r \in R\}$. Then $\pi_1 = \pi_{|E} : E \to H$ is an epimorphism and

$$\ker \pi_1 = K = \langle\langle S \rangle\rangle = \left\langle S^F \right\rangle .$$

FP of finite index subgroups continued

Say $F = a_1 E \cup \ldots \cup a_n E$.

Then $S^F = (S^{a_1} \cup \ldots \cup S^{a_n})^E$ and

$$\left\langle S^F \right\rangle = \left\langle (S^{a_1} \cup \ldots \cup S^{a_n})^E \right\rangle = \langle\langle S^{a_1} \cup \ldots \cup S^{a_n} \rangle\rangle_E$$

Thus $\langle Y; S^{a_1} \cup \ldots \cup S^{a_n} \rangle$ is a presentation for H.

Conversely, suppose that H is FP.

Let $N \leq H$ be a normal subgroup of finite index in G (see Revision notes).

Then $|H : N|$ is finite, so N is FP by the first part.

Also G/N is FP because finite. Therefore G is FP. □