Infinite Groups

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Brief incursion into residual finiteness

The idea when introducing this concept is to approximate an infinite group by its finite quotients.

So one needs to have enough finite quotients.

Proposition

Let G be a group. The following are equivalent:

1. $$\bigcap_{i \in I} H_i = \{1\},$$
   where $\{H_i : i \in I\}$ is the set of all finite-index subgroups in G;
2. *for every $g \in G \setminus \{1\}$, there exists a finite group Φ and a homomorphism $\varphi : G \to \Phi$, such that $\varphi(g) \neq 1$.*

Definition

A group satisfying the above is called residually finite.

Residual finiteness, equivalence

Proof. The key remark is that

$$\bigcap_{i\in I} H_i = \bigcap_{j\in J} N_j,$$

where $\{N_j : j \in J\}$ is the set of all finite-index normal subgroups in G. This is because: for every $H \leqslant G$ of finite index there exists $N \lhd G$ of finite index, $N \leqslant H$.

(1) $\Rightarrow$ (2) $\forall g \neq 1, \exists N \lhd G$ of finite index, $g \notin N$. Take $\varphi : G \to G/N$.

(2) $\Rightarrow$ (1) $\forall g \neq 1, \exists \varphi : G \to F$ finite, such that $g \notin \ker \varphi$. Therefore $g \notin \bigcap_{j\in J} N_j$ □

Examples of RF groups

Example

The group $\Gamma = GL(n, \mathbb{Z})$ *is residually finite.*

Indeed, we take subgroups $\Gamma(p) \leqslant \Gamma$, $\Gamma(p) = \ker(\varphi_p)$, where $\varphi_p : \Gamma \to GL(n, \mathbb{Z}_p)$ is the reduction modulo p.

Assume $g \in \Gamma$ is a non-trivial element.

If g has a non-zero off-diagonal entry $g_{ij} \neq 0$, then $g_{ij} \neq 0 \mod p$, whenever $p > |g_{ij}|$. Thus, $\varphi_p(g) \neq 1$.

If $g \in \Gamma$ has only zero entries off-diagonal then it is a diagonal matrix with only ± 1 on the diagonal, and at least one entry -1. Then $\varphi_3(g)$ has at least one 2 on the diagonal, hence $\varphi_3(g) \neq 1$.

Thus Γ is residually finite.

A Theorem of Mal'cev. A Lemma of Selberg

Theorem (A. I. Mal'cev)

Let Γ be a finitely generated subgroup of $GL(n, R)$, where R is a commutative ring with unity. Then Γ is residually finite.

Mal'cev's theorem is complemented by the following result:

Theorem (Selberg's Lemma)

Let Γ be a finitely generated subgroup of $GL(n, F)$, where F is a field of characteristic zero. Then Γ contains a torsion-free subgroup of finite index.

Properties of RF

Proposition

1. *G, H residually finite (RF) $\Rightarrow G \times H$ RF;*
2. *G RF and $H \leqslant G \Rightarrow H$ RF;*
3. *$H \leqslant G$ of finite index and H RF $\Rightarrow G$ RF;*
4. *H finitely generated RF and Q RF $\Rightarrow H \rtimes Q$ RF.*

Remark

There exist short exact sequences

$$\{1\} \longrightarrow \mathbb{Z}_2 \xrightarrow{i} G \xrightarrow{\pi} Q \longrightarrow \{1\},$$

with Q finitely generated RF and G not RF (J. Millson 1979).

Corollary

The free group F_2 of rank 2 is residually finite. Every free group of (at most) countable rank is residually finite.

Remark

This in particular shows that G RF does not imply G/N RF, for $N \lhd G$.

Remark

Given a short exact sequence

$$\{1\} \longrightarrow H \xrightarrow{i} G \xrightarrow{\pi} F(X) \longrightarrow \{1\},$$

with H finitely generated RF and X finite or countable, G is residually finite.

Back to polycyclic groups

Proposition

Polycyclic groups are finitely presented and residually finite.

Proof by induction on the length $\ell(G)$.

For $\ell(G) = 1$, G is cyclic.

Assume that the statement is true for polycyclic groups of length n, let G be polycyclic with $\ell(G) = n + 1$.

Let N_1 be the first (sub)normal subgroup in a cyclic series of minimal length $n + 1$.

Then N_1 is polycyclic of length n, hence finitely presented (respectively residually finite) by the induction hypothesis.

Induction proving polycyclic groups are FP and RF

We have the short exact sequence

$$\{1\} \longrightarrow N_1 \xrightarrow{i} G \xrightarrow{\pi} C \longrightarrow \{1\},$$

where C is cyclic.

This implies G finitely presented.

When C finite, N has finite index, hence G RF.

When $C = \mathbb{Z}$, $G = N_1 \rtimes \mathbb{Z}$, hence RF.

$\square$

Normal poly-C_∞ subgroup

Proposition

A polycyclic group contains a normal subgroup of finite index which is poly-C_∞.

Proof By induction on the length $\ell(G) = n$.
For $n = 1$ the group G is cyclic and the statement true.
Assume the assertion is true for n and consider a polycyclic group G with a cyclic series of length $n + 1$.
The induction hypothesis implies that N_1 (the first group in the series) contains a normal subgroup S of finite index which is poly-C_∞.
Proposition 2.8, (2), in Revision Notes implies that S contains S_1 characteristic subgroup of N_1 of finite index.
Since $N_1 \lhd G$, S_1 is normal in G.
$S_1 \leqslant S \Rightarrow S_1$ is poly-C_∞.
If G/N_1 is finite then S_1 has finite index in G.

Normal poly-C_∞ subgroup 2

Assume G/N_1 is infinite cyclic.
Then the group $K = G/S_1$ contains the finite normal subgroup $F = N_1/S_1$ such that K/F is isomorphic to $\mathbb{Z}$.
In other words, we have the short exact sequence

$$\{1\} \longrightarrow F \xrightarrow{\varphi} K \xrightarrow{\psi} \mathbb{Z} \longrightarrow \{1\}.$$

Then K is a semidirect product of F and an infinite cyclic subgroup $\langle x \rangle$.
The conjugation by x defines an automorphism of F and since $\mathrm{Aut}(F)$ is finite, there exists r such that the conjugation by x^r is the identity on F.
We conclude that $\langle x^r \rangle$ is a finite index normal subgroup of K.
We have that $\langle x^r \rangle = G_1/S_1$, where G_1 is a finite index normal subgroup in G, and G_1 is poly-C_∞ since S_1 is poly-C_∞. □