B1.1 Logic
Lecture 11

Martin Bays

Oxford, MT 2023

11. Substitution

Discussion: Let $\mathcal{A}$ be an $\mathcal{L}$-structure, $\phi \in \text{Form}(\mathcal{L})$, and suppose $\mathcal{A} \models \forall x_i \phi$.
If c is a constant symbol in $\mathcal{L}$, then $\mathcal{A} \models \phi[c/x_i]$ where $\phi[c/x_i]$ is the result of replacing each free instance of x_i in ϕ with c.

We would like to say more generally that

$$\models \forall x_i \phi \to \phi[t/x_i]$$

for a term t, but we have to be careful:

11.1 Example
Let $\mathcal{L}$ contain a constant symbol c, and let $\phi := \exists x_0 \neg x_0 \doteq x_1$.

Then $\mathcal{A} \models \forall x_1 \phi$ for any $\mathcal{L}$-structure $\mathcal{A}$ with at least two elements,
and then also $\mathcal{A} \models \phi[c/x_1] = \exists x_0 \neg x_0 \doteq c$.

However, if were to define $\phi[x_0/x_1]$ in the same way, we would obtain $\exists x_0 \neg x_0 \doteq x_0$, which does not hold in any $\mathcal{A}$.

Problem: the variable x_0 has become bound in the substitution.

11.2 Definition
For $\phi \in \text{Form}(\mathcal{L})$, a variable x_i, and a term $t \in \text{Term}(\mathcal{L})$, the result of **substituting** t **for** x_i **in** ϕ is the formula

$$(\phi)[t/x_i]$$

which is obtained by replacing each *free* occurrence of x_i in ϕ with the string t, *as long as* this does not lead to new bound occurrences of variables being introduced; if it does, we say that $(\phi)[t/x_i]$ is **undefined**.

We can restate this as a recursive definition:

(i) If ϕ is atomic, $(\phi)[t/x_i]$ is the result of replacing each instance of x_i in ϕ with t.

(ii) $(\neg\psi)[t/x_i] := \neg(\psi)[t/x_i]$
(undefined if $(\psi)[t/x_i]$ is).

(iii) $((\psi \to \chi))[t/x_i] := ((\psi)[t/x_i] \to (\chi)[t/x_i])$
(undefined if $(\psi)[t/x_i]$ or $(\chi)[t/x_i]$ is).

(iv) $(\forall x_i \psi)[t/x_i] := \forall x_i \psi$.

(v) If $j \neq i$, $(\forall x_j \psi)[t/x_i] := \forall x_j (\psi)[t/x_i]$ unless x_j occurs in t and x_i occurs free in ψ, in which case $(\forall x_j \psi)[t/x_i]$ is undefined.

Notation: When no ambiguity could result, we often write $\phi[t/x_i]$ for $(\phi)[t/x_i]$.

Let $\mathcal{L}$ be a first-order language, $\mathcal{A}$ an $\mathcal{L}$-structure.

11.3 Definition
For v an assignment in $\mathcal{A}$ and $t \in \text{Term}(\mathcal{L})$, define

$$v_{t/x_i}(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \widetilde{v}(t) & \text{if } j = i \end{cases}$$

11.4 Substitution Lemma
Let v be an assignment in an $\mathcal{L}$-structure $\mathcal{A}$. Let $\phi \in \text{Form}(\mathcal{L})$, $t \in \text{Term}(\mathcal{L})$, and suppose $\phi[t/x_i]$ is defined.

Then $\mathcal{A} \models \phi[t/x_i][v]$ iff $\mathcal{A} \models \phi[v_{t/x_i}]$.

Proof:

<u>**Case 1** ϕ atomic:</u>

First, for $u \in \text{Term}(\mathcal{L})$ define:

$u[t/x_i]$:= the term obtained by replacing each occurrence of x_i in u by t.

Then $\widetilde{v_{t/x_i}}(u) = \widetilde{v}(u[t/x_i])$.
(Exercise)

Now if $\phi = P(t_1, \ldots, t_k)$ for a k-ary relation symbol P in $\mathcal{L}$, then:

$$\begin{array}{ll} & \mathcal{A} \models \phi[v_{t/x_i}] \\ \text{iff} & (\widetilde{v_{t/x_i}}(t_1), \ldots, \widetilde{v_{t/x_i}}(t_k)) \in P^{\mathcal{A}} \\ \text{iff} & (\widetilde{v}(t_1[t/x_i]), \ldots, \widetilde{v}(t_k[t/x_i])) \in P^{\mathcal{A}} \\ \text{iff} & \mathcal{A} \models P(t_1[t/x_i], \ldots, t_k[t/x_i])[v] \\ \text{iff} & \mathcal{A} \models \phi[t/x_i][v] \end{array}$$

If $\phi = t_1 \doteq t_2$, a similar argument applies.

IH: Lemma holds for shorter formulas.

Case 2 $\phi = \neg\psi$ or $\phi = (\xi \to \rho)$:
Follows directly from IH.

Case 3 $\phi = \forall x_i \psi$:
Then $\phi[t/x_i] = \phi$.

$x_i \notin$ Free(ϕ),
so v and v_{t/x_i} agree on all $x \in$ Free(ϕ),
so by Lemma 10.3,

$$\mathcal{A} \models \phi[v_{t/x_i}] \text{ iff } \mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[t/x_i][v]$$

as required.

Case 4 $\phi = \forall x_j \psi$, $j \neq i$:
Then $\phi[t/x_i] = \forall x_j(\psi)[t/x_i]$.
If x_i does not occur free in ψ, then $\phi[t/x_i] = \phi$, and we conclude exactly as in the previous case.
So suppose x_i occurs free in ψ.
Then since $\phi[t/x_i]$ is defined, x_j does not occur in t. Hence:

Claim: *If v^* agrees with v except maybe at x_j, then $\widetilde{v^*}(t) = \tilde{v}(t)$, so v^*_{t/x_i} agrees with v_{t/x_i} except maybe at x_j.*
Conversely, if v' agrees with v_{t/x_i} except maybe at x_j then $v' = v^*_{t/x_i}$ for some such v^*.

Now: $\mathcal{A} \models \phi[t/x_i][v]$
$\Leftrightarrow \mathcal{A} \models \forall x_j(\psi)[t/x_i][v]$
$\Leftrightarrow \mathcal{A} \models \psi[t/x_i][v^*]$ for all v^* agreeing with v except maybe at x_j,
$\Leftrightarrow \mathcal{A} \models \psi[v^*_{t/x_i}]$ for all v^* agreeing with v except maybe at x_j (by IH),
$\Leftrightarrow \mathcal{A} \models \psi[v']$ for all v' agreeing with v_{t/x_i} except maybe at x_j (by the Claim),
$\Leftrightarrow \mathcal{A} \models \phi[v_{t/x_i}]$.

11.5 Corollary

For any $\phi \in \text{Form}(\mathcal{L})$ *and* $t \in$ *Term*$(\mathcal{L})$ *such that* $\phi[t/x_i]$ *is defined,*

$$\models (\forall x_i \phi \to \phi[t/x_i]).$$

Proof: Let v be an assignment in an $\mathcal{L}$-structure $\mathcal{A}$.

Suppose $\mathcal{A} \models \forall x_i \phi[v]$.
Then $\mathcal{A} \models \phi[v_{t/x_i}]$, since v_{t/x_i} agrees with v except maybe at x_i.
Hence $\mathcal{A} \models \phi[t/x_i][v]$ by the Substitution Lemma (11.4).

□