B1.1 Logic
Lecture 12

Martin Bays

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12. A formal system for Predicate Calculus

12.1 Definition
Associate to each first-order language $\mathcal{L}$ the formal system $K(\mathcal{L})$ with the following axioms and rules:

Axioms
For any $\alpha, \beta, \gamma \in$ Form($\mathcal{L}$), $t \in$ Term($\mathcal{L}$), and $i, j \in \mathbb{N}$, the following are axioms:

A1 $(\alpha \to (\beta \to \alpha))$.
A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$.
A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$.
A4 $(\forall x_i \alpha \to \alpha[t/x_i])$ <u>if</u> $\alpha[t/x_i]$ is defined.
A5 $(\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta))$ <u>if</u> $x_i \notin$ Free(α).
A6 $\forall x_i \, x_i \doteq x_i$.
A7 $(x_i \doteq x_j \to (\phi \to \phi'))$, where ϕ is <u>atomic</u> and ϕ' is obtained from ϕ by replacing <u>some</u> (i.e. one or more) occurrences of x_i in ϕ by x_j.

Rules

MP (Modus Ponens): From α and $(\alpha \to \beta)$ infer β.

Gen (Generalisation): For any variable x_i, from α infer $\forall x_i \alpha$.

Let Sent($\mathcal{L}$) be the set of $\mathcal{L}$-sentences.

If $\Sigma \subseteq$ Sent($\mathcal{L}$), a formula $\phi \in$ Form($\mathcal{L}$) is **provable** from hypotheses Σ, written

$$\Sigma \vdash \phi,$$

if there is a sequence of $\mathcal{L}$-formulas (a **derivation** or **proof**) $\phi_1, \ldots, \phi_n$ with $\phi_n = \phi$ such that for each $i \leq n$:

- (A1-A7) ϕ_i is an axiom, or
- (Hyp) $\phi_i \in \Sigma$, or
- (MP) $\phi_k = (\phi_j \to \phi_i)$ for some $j, k < i$, or
- (Gen) $\phi_i = \forall x_k \phi_j$ for some $j < i$ and some $k \in \mathbb{N}$.

$\vdash \phi$ abbreviates $\emptyset \vdash \phi$.

12.3 Example *Swapping variables*

Suppose Free(ϕ) $= \{x_i\}$.
Then $\{\forall x_i \phi\} \vdash \forall x_j \phi[x_j/x_i]$

1	$\forall x_i \phi$	[$\in \Sigma$]
2	$(\forall x_i \phi \rightarrow \phi[x_j/x_i])$	[A4]
3	$\phi[x_j/x_i]$	[MP 1,2]
4	$\forall x_j \phi[x_j/x_i]$	[(Gen)]

12.4 Soundness Theorem for Pred. Calc.

If $\Sigma \vdash \phi$ *then* $\Sigma \models \phi$.

Proof: By induction on length of a proof.

First we show that **A1**-**A7** are logically valid.
For **A1**, **A2**, and **A3**, this is immediate.
A4 and **A5**: Cor 11.5 resp. Cor 10.4.
A6: easy exercise.

A7: Suppose ϕ is atomic, and ϕ' results from replacing some instances of x_i with x_j.
Let $\mathcal{A}$ be an $\mathcal{L}$-structure and v an assignment in $\mathcal{A}$ such that

$$\mathcal{A} \models x_i \doteq x_j[v] \text{ and } \mathcal{A} \models \phi[v].$$

We want to show that $\mathcal{A} \models \phi'[v]$.

Now $v(x_i) = v(x_j)$,
so $\widetilde{v}(t') = \widetilde{v}(t)$ for any term t' obtained from t by replacing zero or more occurrences of x_i by x_j
(easy induction on terms).

If $\phi = P(t_1, \ldots, t_k)$ then say $\phi' = P(t'_1, \ldots, t'_k)$.

$$\begin{array}{rll} \mathcal{A} \models \phi[v] & \text{iff} & (\tilde{v}(t_1), \ldots, \tilde{v}(t_k)) \in P^{\mathcal{A}} \\ & \text{iff} & (\tilde{v}(t'_1), \ldots, \tilde{v}(t'_k)) \in P^{\mathcal{A}} \\ & \text{iff} & \mathcal{A} \models P(t'_1, \ldots, t'_k)[v] \\ & \text{iff} & \mathcal{A} \models \phi'[v] \text{ as required} \end{array}$$

Similarly if ϕ is $t_1 \doteq t_2$.

MP: For any $\mathcal{A}$ and v:

if $\mathcal{A} \models \alpha[v]$ and $\mathcal{A} \models (\alpha \to \beta)[v]$ then $\mathcal{A} \models \beta[v]$;

so: if $\Sigma \models \alpha$ and $\Sigma \models (\alpha \to \beta)$ then $\Sigma \models \beta$.

Generalisation:
Suppose $\Sigma \models \psi$;
we want to show $\Sigma \models \forall x_i \psi$.

So let $\mathcal{A}$ be such that $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$, and let v be an arbitrary assignment on $\mathcal{A}$.
We must show $\mathcal{A} \models \forall x_i \psi[v]$.
So let $v^\star$ agree with v except maybe at x_i.
We must show $\mathcal{A} \models \psi[v^\star]$.
But since $\Sigma \models \psi$, we have $\mathcal{A} \models \psi[v']$ for *any* assignment v', in particular for v^*. $\square$

12.5 Deduction Theorem for Pred. Calc.
Let $\Sigma \subseteq \text{Sent}(\mathcal{L})$, and $\psi \in \text{Sent}(\mathcal{L})$, and $\phi \in \text{Form}(\mathcal{L})$.

$$\textit{If } \Sigma \cup \{\psi\} \vdash \phi \textit{ then } \Sigma \vdash (\psi \to \phi).$$

Proof: Same as for prop. calc. (Theorem 6.6); induction on the length of a proof, with one more case:

IH: $\Sigma \vdash (\psi \to \phi_j)$
to show: $\Sigma \vdash (\psi \to \forall x_i \phi_j)$,
where generalisation (**Gen**) has been used to infer $\forall x_i \phi_j$ from ϕ_j.

By IH and **Gen**: $\Sigma \vdash \forall x_i (\psi \to \phi_j)$
A5 $\vdash (\forall x_i (\psi \to \phi_j) \to (\psi \to \forall x_i \phi_j))$, since $x_i \notin \text{Free}(\psi) = \emptyset$.
So by **MP**, $\Sigma \vdash (\psi \to \forall x_i \phi_j)$ as required.

$\square$

12.6 Lemma
Let α *be a tautology of the* Propositional Calculus *with propositional variables among* $p_0, \ldots, p_n$*, let* $\psi_0, \ldots, \psi_n \in \text{Form}(\mathcal{L})$*,*
and let α' *be the* $\mathcal{L}$*-formula obtained from* α *by replacing each occurrence of* p_i *by* ψ_i*.*
Then $\vdash \alpha'$*.*

Proof:
By completeness of L_0, there is a proof $\alpha_1, ..., \alpha_{n-1}, \alpha$ in L_0.

Since **A1**, **A2**, **A3** and **MP** are in $K(\mathcal{L})$, substituting ψ_i for p_i in each α_i yields a proof $\alpha'_1, ..., \alpha'_{n-1}, \alpha'$ in $K(\mathcal{L})$. $\square$

A formula α' as in Lemma 12.6 is called a **tautology** of $\mathcal{L}$. (Note that all tautologies are logical validities, but not vice versa.)

By the lemma, we may freely introduce tautologies in our proofs in $K(\mathcal{L})$.

12.7 Example Suppose $(\exists x_i \phi \to \psi) \in$ Sent($\mathcal{L}$). Then

$$\{(\exists x_i \phi \to \psi)\} \vdash \forall x_i(\phi \to \psi)$$

Proof: Let $\Sigma = \{(\exists x_i \phi \to \psi), \neg\psi\}$

1	$(\neg\forall x_i \neg\phi \to \psi)$	$[\in \Sigma]$
2	$((\neg\forall x_i \neg\phi \to \psi) \to (\neg\psi \to \forall x_i \neg\phi))$	[taut.]
3	$(\neg\psi \to \forall x_i \neg\phi)$	[MP 1,2]
4	$\neg\psi$	$[\in \Sigma]$
5	$\forall x_i \neg\phi$	[MP 3,4]
6	$(\forall x_i \neg\phi \to \neg\phi)$	[A4]
7	$\neg\phi$	[MP 5,6]

(In line 6, we used that $(\neg\phi)[x_i/x_i] = \neg\phi$.)

Hence $\Sigma \vdash \neg\phi$. So

$$\begin{array}{ll} (\exists x_i \phi \to \psi) \vdash (\neg\psi \to \neg\phi) & \text{[DT]} \\ (\exists x_i \phi \to \psi) \vdash (\phi \to \psi) & \text{[A3, MP]} \\ (\exists x_i \phi \to \psi) \vdash \forall x_i(\phi \to \psi) & \text{[Gen]} \end{array}$$

$\square$