# B1.1 Logic
# Lecture 13

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# 13. The Completeness Theorem for Predicate Calculus

Let $\mathcal{L}$ be a countable first-order language.

**13.1 Theorem** (Gödel)
*Let* $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$ *and* $\phi \in \mathrm{Form}(\mathcal{L})$*.*

*If* $\Sigma \models \phi$ *then* $\Sigma \vdash \phi$*.*

Here, $\Sigma \vdash \phi$ means that $\phi$ is provable from hypotheses $\Sigma$ in the proof system $K(\mathcal{L})$.

In outline, our proof strategy is much as in the propositional case:

- Reduce to: consistent $\Rightarrow$ satisfiable.
- Show: any consistent $\Sigma$ extends to "maximal consistent witnessing" $\Sigma'$.
- Show: maximal consistent witnessing $\Rightarrow$ satisfiable.

Call $\Sigma \subseteq \text{Sent}(\mathcal{L})$ **consistent** (in $K(\mathcal{L})$) if for no $\tau \in \text{Sent}(\mathcal{L})$ do we have both
$\Sigma \vdash \tau$ and $\Sigma \vdash \neg\tau$.

**Remark**
If $\Sigma$ is inconsistent, then $\Sigma \vdash \chi$ for *any* $\chi \in \text{Sent}(\mathcal{L})$, since $(\tau \to (\neg\tau \to \chi))$ is a tautology.

**13.2 Lemma**
*Every consistent set of sentences has a model.*

i.e. if $\Sigma \subseteq \text{Sent}(\mathcal{L})$ is consistent then for some $\mathcal{L}$-structure $\mathcal{A}$,
$\mathcal{A} \models \sigma$ for every $\sigma \in \Sigma$.
c.f. Lemma 7.8.

**Proof of Theorem 13.1 from Lemma 13.2**

First we treat the case of a sentence $\sigma \in \text{Sent}(\mathcal{L})$.

$\Sigma \models \sigma \Rightarrow \Sigma \cup \{\neg\sigma\}$ has no model
$\Rightarrow_{(13.2)} \Sigma \cup \{\neg\sigma\}$ is not consistent
$\Rightarrow \Sigma \cup \{\neg\sigma\} \vdash \tau$ and $\Sigma \cup \{\neg\sigma\} \vdash \neg\tau$ for some $\tau$
$\Rightarrow_{\text{DT}} \Sigma \vdash (\neg\sigma \to \tau)$ and $\Sigma \vdash (\neg\sigma \to \neg\tau)$.
But $\Sigma \vdash ((\neg\sigma \to \tau) \to ((\neg\sigma \to \neg\tau) \to \sigma))$ [taut]
$\Rightarrow \Sigma \vdash \sigma$ [MP twice]

Now let $\phi \in \text{Form}(\mathcal{L})$, and say $\text{Free}(\phi) = \{x_{i_1}, ..., x_{i_n}\}$.
Let $\sigma := \forall x_{i_1} ... \forall x_{i_n} \phi$.

If $\Sigma \models \phi$ then $\Sigma \models \sigma$, so $\Sigma \vdash \sigma$ by the above. But then by repeatedly applying (A4) and (MP), we obtain $\Sigma \vdash \phi$, as required.

$\square_{13.2 \Rightarrow 13.1}$

To prove Lemma 13.2, we want to introduce an additional assumption.

**13.2' Lemma**:
*Suppose* $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *is consistent and* $\mathcal{L}$ *contains infinitely many constant symbols not appearing in* $\Sigma$. *Then* $\Sigma$ *has a model.*

We deduce Lemma 13.2 for arbitrary $\mathcal{L}$ and $\Sigma$ from Lemma 13.2' as follows.

Let $C = \{c_0, c_1, ...\}$ be a set of distinct symbols disjoint from $\mathcal{L}$, and define the extended language $\mathcal{L}' := \mathcal{L} \cup C$ in which each $c_i$ is a constant symbol.

**13.3 Lemma**

*If* $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *and* $\tau \in \text{Sent}(\mathcal{L})$ *is provable from* $\Sigma$ *in* $K(\mathcal{L}')$*, then* $\tau$ *is provable from* $\Sigma$ *in* $K(\mathcal{L})$*.*

**Proof**

Exercise sheet 4, Question 3(b). □

**Proof of Lemma 13.2 from Lemma 13.2'**:

By Lemma 13.3, since $\Sigma \subseteq \text{Sent}(\mathcal{L})$ is consistent in $K(\mathcal{L})$, it is also consistent in $K(\mathcal{L}')$;

indeed, otherwise (via the tautology $(\tau \to (\neg\tau \to \chi)))$ any $\chi \in \text{Sent}(\mathcal{L})$ is provable from $\Sigma$ in $K(\mathcal{L}')$ and hence in $K(\mathcal{L})$, contradicting consistency in $K(\mathcal{L})$.

By Lemma 13.2' applied with $\mathcal{L}'$ in place of $\mathcal{L}$, there is an $\mathcal{L}'$-structure $\mathcal{A}'$ satisfying $\Sigma$.

Let $\mathcal{A}$ be the $\mathcal{L}$-structure obtained from $\mathcal{A}'$ by "forgetting" the new constants $C$.

Then $\mathcal{A}$ satisfies $\Sigma$, as required. $\square_{13.2' \Rightarrow 13.2}$

## 13.4 Definition

- $\Sigma \subseteq \text{Sent}(\mathcal{L})$ is called **maximal consistent** if $\Sigma$ is consistent, and for any $\psi \in \text{Sent}(\mathcal{L})$: $\Sigma \vdash \psi$ or $\Sigma \vdash \neg\psi$.

- $\Sigma \subseteq \text{Sent}(\mathcal{L})$ is called **witnessing** if for all $\psi \in \text{Form}(\mathcal{L})$ with $\text{Free}(\psi) \subseteq \{x_i\}$ and such that $\Sigma \vdash \exists x_i \psi$, there is some constant symbol $c \in \mathcal{L}$ such that $\Sigma \vdash \psi[c/x_i]$

To prove Lemma 13.2', it suffices to prove the following two lemmas:

## 13.5 Lemma

*Every maximal consistent witnessing set* $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *has a model.*

## 13.6 Lemma

*If* $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *is consistent and* $\mathcal{L}$ *contains infinitely many constant symbols not appearing in* $\Sigma$*, then* $\Sigma$ *extends to a maximal consistent witnessing set* $\Sigma' \subseteq \text{Sent}(\mathcal{L})$.

For the proof of 13.6 we need two further lemmas.

**13.7 Lemma**

*If* $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$ *is consistent, then for any sentence* $\psi$*, either* $\Sigma \cup \{\psi\}$ *or* $\Sigma \cup \{\neg\psi\}$ *is consistent.*

*Proof:* Exercise – as in the proof of Theorem 7.5. $\square$.

**13.8 Lemma**
*Assume* $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *is consistent, and* $\Sigma \vdash \exists x_i \psi \in \text{Sent}(\mathcal{L})$, *and* $c$ *is a constant symbol of* $\mathcal{L}$ *which does* not *occur in* $\psi$ *nor in any* $\sigma \in \Sigma$.
*Then* $\Sigma \cup \{\psi[c/x_i]\}$ *is consistent.*

**Proof:**
It suffices to show that if $c$ does not occur in $\chi \in \text{Sent}(\mathcal{L})$ and $\Sigma \cup \{\psi[c/x_i]\} \vdash \chi$,
then already $\Sigma \vdash \chi$. Indeed:
If $\Sigma \cup \{\psi[c/x_i]\}$ were inconsistent then (via the tautology $(\alpha \to (\neg\alpha \to \beta))$) we would have for *any* $\chi$ that $\Sigma \cup \{\psi[c/x_i]\} \vdash \chi$ and $\Sigma \cup \{\psi[c/x_i]\} \vdash \neg\chi$;
picking $\chi$ in which $c$ does not occur, it would follow that $\Sigma \vdash \chi$ and $\Sigma \vdash \neg\chi$, contradicting consistency of $\Sigma$.

So suppose $\Sigma \cup \{\psi[c/x_i]\} \vdash \chi \in \mathrm{Sent}(\mathcal{L})$ and $c$ does not occur in $\chi$. Recall we also assumed that $c$ does not occur in $\Sigma$ or $\psi$.

By DT, $\Sigma \vdash (\psi[c/x_i] \to \chi)$
It follows that $\Sigma \vdash (\psi \to \chi)$
(Exercise Sheet 4 Question 3(a)).

By Gen, $\Sigma \vdash \forall x_i(\psi \to \chi)$.
It follows that $\Sigma \vdash (\exists x_i \psi \to \chi)$
(Exercise Sheet 4 Question 2).

But we assumed $\Sigma \vdash \exists x_i \psi$,
so by MP, $\Sigma \vdash \chi$, as required.

$\square_{13.8}$

**Proof of 13.6:**
Let $\Sigma \subseteq \text{Sent}(\mathcal{L})$ be consistent, and suppose $\mathcal{L}$ contains infinitely many constant symbols not appearing in $\Sigma$.

We show that $\Sigma$ extends to a maximal consistent witnessing set.

Sent($\mathcal{L}$) is countable; say
$\text{Sent}(\mathcal{L}) = \{\tau_1, \tau_2, \tau_3, \ldots\}$.

Construct finite sets $\Delta_i \subseteq \text{Sent}(\mathcal{L})$

$$\Delta_0 \subseteq \Delta_1 \subseteq \Delta_2 \subseteq \ldots$$

such that $\Sigma \cup \Delta_n$ is consistent for each $n \geq 0$, as follows:

Let $\Delta_0 := \emptyset$. Then $\Sigma \cup \Delta_0 = \Sigma$ is consistent.

If $\Delta_n$ has been constructed let

$$\Delta'_n := \begin{cases} \Delta_n \cup \{\tau_{n+1}\} & \text{if } \Sigma \cup \Delta_n \cup \{\tau_{n+1}\} \\ & \text{ is consistent} \\ \Delta_n \cup \{\neg\tau_{n+1}\} & \text{otherwise.} \end{cases}$$

Then $\Sigma \cup \Delta'_n$ is consistent by Lemma 13.7.

If $\neg\tau_{n+1} \in \Delta'_n$ or if $\tau_{n+1}$ is not of the form $\exists x_i \psi$, let $\Delta_{n+1} := \Delta'_n$.

Otherwise, i.e. if $\tau_{n+1} = \exists x_i \psi \in \Delta'_n$:
Choose a constant symbol $c \in \mathcal{L}$ which occurs in no formula in $\Sigma \cup \Delta'_n \cup \{\psi\}$
(possible since $\Delta'_n \cup \{\psi\}$ is finite).
Let $\Delta_{n+1} := \Delta'_n \cup \{\psi[c/x_i]\}$.
By Lemma 13.8, $\Sigma \cup \Delta_{n+1}$ is consistent.

Let $\Sigma' := \Sigma \cup \bigcup_{n \geq 0} \Delta_n$.
Then $\Sigma'$ is maximal consistent (as in 7.5), and $\Sigma'$ is witnessing by construction.

$\square_{13.6}$