B1.1 Logic
Lecture 14

Martin Bays

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To finish the proof of completeness, it remains to prove:

13.5 Lemma
Every maximal consistent witnessing set $\Sigma \subseteq \text{Sent}(\mathcal{L})$ *has a model.*

Proof:
A term is **closed** if no variable appears in it. Let T be the set of closed $\mathcal{L}$-terms.

Define an equivalence relation E on T by

$$t_1 E t_2 \text{ iff } \Sigma \vdash t_1 \doteq t_2$$

(This *is* an equivalence relation – see Sheet 4 Question 1(b).)

Let T/E be the set of equivalence classes t/E for $t \in T$.

Define an $\mathcal{L}$-structure $\mathcal{A}$ with domain T/E by

$$c^{\mathcal{A}} := c/E$$
$$f^{\mathcal{A}}(t_1/E, \ldots, t_k/E) := f(t_1, \ldots, t_k)/E$$
$$P^{\mathcal{A}} := \{(t_1/E, ..., t_k/E) \mid \Sigma \vdash P(t_1, \ldots, t_k)\}$$

(for c a constant symbol, f a k-ary function symbol, and P a k-ary predicate symbol).

Note: $t^{\mathcal{A}} = t/E$ for any $t \in T$.

Exercise: The definitions above do not depend on representatives, i.e. if $t_i/E = t'_i/E$ for $i = 1, \ldots k$ then:

- $f(t_1, \ldots, t_k)/E = f(t'_1, \ldots, t'_k)/E$
- $\Sigma \vdash P(t_1, \ldots, t_k) \Leftrightarrow \Sigma \vdash P(t'_1, \ldots, t'_k)$

This follows from **A7** and **A4**; see Sheet 4 Question 1(c).

We conclude by showing: $\mathcal{A} \models \Sigma$.

We show more generally that for any $\sigma \in \text{Sent}(\mathcal{L})$,

$$\mathcal{A} \models \sigma \text{ iff } \Sigma \vdash \sigma.$$

We prove this by induction on the number of symbols among $\{\neg, \rightarrow, \forall\}$ in σ.

- $\sigma = P(t_1, ..., t_k)$. Then:

$$\begin{aligned} \mathcal{A} \models \sigma &\Leftrightarrow (t_1^{\mathcal{A}}, ..., t_k^{\mathcal{A}}) \in P^{\mathcal{A}} \\ &\Leftrightarrow (t_1/E, ..., t_k/E) \in P^{\mathcal{A}} \\ &\Leftrightarrow \Sigma \vdash \sigma. \end{aligned}$$

- $\sigma = t_1 \doteq t_2$. Then:

$$\begin{aligned} \mathcal{A} \models \sigma &\Leftrightarrow t_1^{\mathcal{A}} = t_2^{\mathcal{A}} \\ &\Leftrightarrow t_1/E = t_2/E \\ &\Leftrightarrow \Sigma \vdash \sigma. \end{aligned}$$

- $\sigma = \neg\tau$:

$$
\begin{array}{lll}
 & \mathcal{A} \models \neg\tau & \\
\text{iff} & \mathcal{A} \not\models \tau & \text{[def. of `}\models\text{']} \\
\text{iff} & \Sigma \nvdash \tau & \text{[IH]} \\
\text{iff} & \Sigma \vdash \neg\tau & [\Sigma \text{ max. cons.}]
\end{array}
$$

- $\sigma = (\tau \to \rho)$:

$$
\begin{array}{lll}
 & \mathcal{A} \models (\tau \to \rho) & \\
\text{iff} & \mathcal{A} \not\models \tau \text{ or } \mathcal{A} \models \rho & \text{[def. `}\models\text{']} \\
\text{iff} & \Sigma \nvdash \tau \text{ or } \Sigma \vdash \rho & \text{[IH]} \\
\text{iff} & \text{not } (\Sigma \vdash \tau \text{ and } \Sigma \nvdash \rho) & \\
\text{iff} & \text{not } (\Sigma \vdash \tau \text{ and } \Sigma \vdash \neg\rho) & [\Sigma \text{ max. cons.}] \\
\text{iff} & \Sigma \nvdash \neg(\tau \to \rho) & \text{[taut. (see below)]} \\
\text{iff} & \Sigma \vdash (\tau \to \rho) & [\Sigma \text{ max. cons.}]
\end{array}
$$

where the penultimate line uses the following tautologies:

$$(\tau \to (\neg\rho \to \neg(\tau \to \rho)))$$
$$(\neg(\tau \to \rho) \to \tau)$$
$$(\neg(\tau \to \rho) \to \neg\rho).$$

- $\sigma = \forall x_i \phi$:
  By the Substitution Lemma 11.4,
  $\mathcal{A} \models \phi[t/x_i] \Leftrightarrow \mathcal{A} \models \phi[v_t]$ where v_t is any assignment with $v_t(x_i) = t^{\mathcal{A}} = t/E$.

  So since the domain of $\mathcal{A}$ is T/E,
  $\mathcal{A} \models \forall x_i \phi$ iff for all $t \in T$, $\mathcal{A} \models \phi[t/x_i]$.

  Now for $t \in T$: $\phi[t/x_i] \in \text{Sent}(\mathcal{L})$, so by IH,
  $\mathcal{A} \models \phi[t/x_i]$ iff $\Sigma \vdash \phi[t/x_i]$.

  So to show $\Sigma \vdash \forall x_i \phi$ iff $\mathcal{A} \models \forall x_i \phi$, it suffices to show:
  $\Sigma \vdash \forall x_i \phi$ iff for all $t \in T$, $\Sigma \vdash \phi[t/x_i]$.

We prove:
$\Sigma \vdash \forall x_i \phi$ iff for all $t \in T$, $\Sigma \vdash \phi[t/x_i]$.

$\Rightarrow$: **A4** + **MP**.

For the converse, first note:

$$\{\forall x_i \neg\neg\phi\} \vdash \forall x_i \phi; \qquad (\star)$$

indeed, by **A4** we have $\{\forall x_i \neg\neg\phi\} \vdash \neg\neg\phi$; conclude using the tautology $(\neg\neg\phi \to \phi)$ and **Gen**.

Now suppose $\Sigma \nvdash \forall x_i \phi$.
Then $\Sigma \nvdash \forall x_i \neg\neg\phi$, by $(\star)$.
So by maximality, $\Sigma \vdash \neg\forall x_i \neg\neg\phi$,
i.e. $\Sigma \vdash \exists x_i \neg\phi$.
Since Σ is witnessing, $\Sigma \vdash (\neg\phi)[c/x_i]$ for some constant symbol c.
Then since Σ is consistent, $\Sigma \nvdash \phi[c/x_i]$.
But $c \in T$, so it is not the case that for all $t \in T$, $\Sigma \vdash \phi[t/x_i]$. $\square_{13.5}$

This concludes our proof of the Completeness Theorem 13.1.

In fact, our proof of completeness yields a stronger result.

13.9 Definition: A structure is **countable** if its domain is countable (i.e. finite or countably infinite).

The model constructed in Lemma 13.5 is countable, because the set T of closed terms is, so we have actually proven the following strengthening of Lemma 13.2:

13.10 Weak downwards Löwenheim-Skolem Theorem
Every consistent set of sentences has a countable model.

Exactly as in the propositional case, we deduce compactness from completeness and soundness.

13.11 Compactness Theorem:
A set of sentences $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$ *has a model if and only if every finite subset* $\Sigma_0 \subseteq_{fin} \Sigma$ *has a model.*