Infinite Groups

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Comparison between solvable and polycyclic

Proposition

A solvable group is polycyclic if and only if it is noetherian.

Remarks

1. *There are noetherian groups that are not virtually polycyclic, e.g. Tarski monsters: finitely generated groups such that every proper subgroup is cyclic (Al. Olshanskii, technique used to disprove the von Neumann-Day conjecture).*
2. *A class $\mathcal{C}$ of groups is noetherian $\Rightarrow$ given any property $(*)$ satisfied by the trivial group $\{1\}$, a group $G \in \mathcal{C}$ contains a maximal subgroup with property $(*)$.*

Noetherian induction for polycyclic groups

We introduce a third type of inductive argument for polycyclic groups: the noetherian induction.

Assume that we have to prove that every polycyclic group has a certain property P. It suffices to check that:

- the trivial group $\{1\}$ has property P (initial case);
- a group G such that all its proper quotients G/N have P must have property P (inductive step).

Indeed, assume that, once all the above was checked, one finds a group G that does not have property P.

Let $(*)$ be the property "K is a normal subgroup such that G/K does not have property P", and let N be a maximal subgroup satisfying $(*)$.

Then G/N is polycyclic, without property P, such that all its proper quotients have property P, contradicting the inductive step.

The Noetherian induction works for any class of Noetherian groups stable by taking quotients.

Example of f.g. solvable non-polycyclic group

Example

Recall that the lamplighter group is the wreath product $G = \mathbb{Z}_2 \wr \mathbb{Z}$, *and that it is finitely generated (Ex. Sheet 1).*

The commutator subgroup G' *coincides with the following subgroup of* $\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}_2$:

$$C = \{f : \mathbb{Z} \to \mathbb{Z}_2 \mid \mathrm{Supp}(f) \text{ has even cardinality}\}, \quad (1)$$

where $\mathrm{Supp}(f) = \{n \in \mathbb{Z} \mid f(n) = 1\}$.

[NB. The notation here is additive, the identity element is 0.*]*

In particular, G' *is not finitely generated.*

The group G *is metabelian (since* G' *abelian).*

The lamplighter group continued

- Not all the subgroups in the lamplighter group G are finitely generated: G' is not, $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$ is not.
- G is not virtually torsion-free: For any finite-index subgroup $H \leqslant G$, $H \cap \bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$ has finite index in $\bigoplus_{n\in\mathbb{Z}}\mathbb{Z}_2$; in particular this intersection is infinite and contains elements of order 2.
- G is not finitely presented.

  The last three statements imply that the lamplighter group is not polycyclic.

The Baumslag-Solitar group

An example of solvable (even metabelian) finitely presented group that is not polycyclic is the Baumslag–Solitar group.

$$G = BS(1,p) = \langle a, b | aba^{-1} = b^p \rangle \text{ for } |p| \geqslant 2.$$

The matrices

$$a = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate a subgroup of $SL(2, \mathbb{R})$ isomorphic to $BS(1,p)$.

Baumslag-Solitar group continued

- The derived subgroup G' of G is (isomorphic to)

$$G' = \left\{ \begin{pmatrix} 1 & mp^k \\ 0 & 1 \end{pmatrix} \; ; \; m, k \in \mathbb{Z} \right\}.$$

- Therefore $G = BS(1, p)$ is metabelian.
- The derived subgroup G' is not finitely generated.

  Hence G is not polycyclic.

Nilpotency class and derived length

Every nilpotent group is solvable.

Question: find a relationship between nilpotency class and derived length.

Proposition

1. *For every group G and every $i \geqslant 0$,*

$$G^{(i)} \leqslant C^{2^i} G.$$

2. *If G is k-step nilpotent then its derived length is at most*

$$[\log_2 k] + 1 .$$

Proof (1) by induction on $i \geqslant 0$.

The statement is obviously true for $i = 0$. Assume that it is true for i.

Then

$$G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right] \leqslant \left[C^{2^i}G, C^{2^i}G\right] \leqslant C^{2^{i+1}}G.$$

(2) follows immediately from (1). □

Remark

The derived length can be much smaller than the nilpotency class:

the dihedral group D_{2n} with $n = 2^k$ is k-step nilpotent and metabelian.

In particular we do not have $\ell_{\rm der}(G) \geq f(k)$, with $\lim_{k\to\infty} f(k) = \infty$.

Linear groups

In what follows $\mathbb{K}$ is an algebraically closed field (e.g. $\mathbb{C}$), V is a finite-dimensional vector space over $\mathbb{K}$.

$End(V)$ is the algebra of (linear) endomorphisms of V.

$GL(V)$ is the group of invertible endomorphisms of V.

A linear action of a group G on V is called representation of G on V.

It amounts to the existence of a group homomorphism $\rho : G \to GL(V)$.

The representation may not be faithful (i.e. ρ might have a kernel.)

A group G that is isomorphic to a subgroup of $GL(V)$, for some V, is called a matrix group or a linear group.

The subalgebra of $End(V)$ generated by a linear group G will be denoted by $\mathbb{K}[G]$; this is just the linear span of G over $\mathbb{K}$.

Trace, $GL(V)$ and $End(V)$

Lemma

The map $\tau : End(V) \times End(V) \to \mathbb{K}$, $\tau(A, B) = \mathrm{trace}(AB)$ is a non-degenerate bi-linear form.

Fixing a basis for V determines:

- an isomorphism of groups $GL(V) \simeq GL_n(\mathbb{K})$, where $GL_n(\mathbb{K})$ is the group of invertible $n \times n$ matrices over $\mathbb{K}$;
- an isomorphism of algebras $End(V) \simeq M_n(\mathbb{K})$, where the latter is the algebra of all $n \times n$ matrices over $\mathbb{K}$.

Irreducible, reducible and triangularizable actions

If V is a vector space and $A \leqslant End(V)$ is a subgroup, then A is said to act irreducibly on V if V contains no proper subspace $\{0\} \subsetneq V' \subsetneq V$ such that $aV' \subset V'$ for all $a \in A$.

We say that the action of A on V is completely reducible if V decomposes as a direct sum of irreducible subspaces.

A linear group $G \leq GL(V)$ is called triangularizable if there exists a basis of V with respect to which G is represented by upper-triangular matrices.

Actions of abelian groups

Lemma

If A is an abelian group acting irreducibly on V then V has dimension 1.

Proof. $\mathbb{K}$ algebraically closed $\Rightarrow$ every $a \in A$ has at least an eigenvalue. A abelian $\Rightarrow$ the corresponding space of eigenvectors is b-invariant for every $b \in A$, hence it must coincide with V.
Thus, every $a \in A$ is a multiple of the identity map on V, hence by irreducibility V must have dimension 1. $\square$

Proposition

If A is an abelian group acting on V then there exists a basis of V with respect to which A becomes upper triangular.

Actions of abelian groups, continued

Proof. By induction on the dimension of V. Obvious in dimension 1.
Assume true for dimension $< n$, take V of dimension n.
If A acts irreducibly apply previous Lemma.
Assume A acts reducibly, and preserves a proper subspace $V' < V$.
We obtain two induced actions of A: on V' (by restriction) and on $V'' = V/V'$ (by projection).
Both actions become actions by triangular matrices with the right choice of basis.
The combination of the two bases yields a basis in V with respect to which A becomes upper triangular. □

Our goal in this lecture is to generalize this last result to solvable groups.

Burnside Theorem and applications

Theorem (Burnside's Theorem)

If $A \subset End(V)$ is a subalgebra which acts absolutely irreducibly on a finite-dimensional vector space V, then $A = End(V)$. In particular, if $G \leqslant End(V)$ is a subsemigroup acting irreducibly, then G spans $End(V)$ as a vector space, i.e. $\mathbb{K}[G] = End(V)$.

Theorem

Suppose that $G \leq \mathrm{GL}_n(\mathbb{K})$ is irreducible and that

$$|\{\mathrm{tr}(g) \mid g \in G\}| = q < \infty.$$

Then $|G| \leq q^{n^2}$.

Proof of first application to Burnside

Proof. By Burnside's Theorem, G contains $m = n^2$ linearly independent matrices $w(1), \ldots, w(m)$.
For $\underline{\mu} \in k^m$ let

$$G(\underline{\mu}) = \{g \in G \mid \mathrm{tr}(w(s)g) = \mu_s \ (s = 1, \ldots, m)\}.$$

Observe that $g = (g_{ij}) \in G(\underline{\mu})$ if and only if it satisfies the equations

$$\sum_{i=1}^{n} \sum_{l=1}^{n} w(s)_{il} g_{li} = \mu_s \quad (s = 1, \ldots, m).$$

This is a system of $m = n^2$ linearly independent equations, so it has at most one solution (g_{ij}). The result follows as there are just q^{n^2} possibilites for $\underline{\mu}$. □