7. RIEMANN SURFACES

7.1 Examples

Definition 7.1 *A Riemann surface is a connected topological surface S with a holomorphic atlas. A holomorphic atlas is a collection of charts $\{\varphi_i : U_i \to V_i\}$ where $V_i \subseteq \mathbb{C}$ and the transition maps are biholomorphic – that is holomorphic bijections with holomorphic inverses. Note that not all authors assume Riemann surfaces to be connected.*

The definition of holomorphic maps between Riemann surfaces can then be made in a like manner to how we defined smooth maps between smooth surfaces. Recall that holomorphic maps are nuch more rigid than smooth functions – as, for example, becomes apparent with the identity theorem. The issue of classifying Riemann surfaces up to biholomorphism (the correct notion of isomorphism here) is much more subtle than in the smooth case, with a great variety in the possible complex structures that a certain topological type can be endowed with. On these points we note:

Proposition 7.2 *Riemann surfaces are orientable.*

Proof. The transition maps are holomorphic, with non-zero derivatives, and so are orientation-preserving. ■

And the following is also true – left to Sheet 4, Exercise 3.

- A holomorphic function on a compact Riemann surface is constant.

Example 7.3 ***(The complex plane)*** *$\mathbb{C}$ is a Riemann surface. The identity map $\iota : \mathbb{C} \to \mathbb{C}$ forms a holomorphic atlas by itself.*

Example 7.4 ***(Riemann mapping theorem)*** *Every simply connected, non-empty proper open subset $U \subseteq \mathbb{C}$ is biholomorphic to an open half-plane.*

Example 7.5 ***(Annuli)*** *An annulus*

$$A = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$$

is not homeomorphic to an open half-plane – as it is not simply connected – and so is not biholomorphic to it. All such annuli are diffeomorphic to one another, but there is a famous theorem of complex analysis which shows two such annuli are biholomorphic if they have the same ratio of radii r_2/r_1.

Example 7.6 ***(The Riemann sphere)*** *The Riemann sphere can also be thought of as the complex projective line or the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. We can provide a holomorphic atlas with two charts*

$$\begin{aligned} U_1 &= \mathbb{C}_\infty \setminus \{\infty\}, \qquad V_1 = \mathbb{C}, \qquad \varphi_1(z) = z; \\ U_2 &= \mathbb{C}_\infty \setminus \{0\}, \qquad V_2 = \mathbb{C}, \qquad \varphi_2(z) = z^{-1}. \end{aligned}$$

Then $\varphi_1(U_1 \cap U_2) = \mathbb{C}\backslash\{0\}$ *and*

$$\varphi_2 \circ \varphi_1^{-1} = \frac{1}{z}$$

which is biholomorphism between $\varphi_1(U_1 \cap U_2)$ *and* $\varphi_2(U_1 \cap U_2)$.

Example 7.7 *(**Meromorphic maps**) In light of the previous example, a meromorphic map on a Riemann surface can be considered as a holomorphic map to the Riemann sphere. Say that* $f\colon \mathbb{C} \to \mathbb{C}_\infty$ *is a meromorphic function – so holomorphic except for finitely many poles. When* $f(z)$ *is finite then* $\varphi_1 \circ f(z) = f(z)$ *is holomorphic and when* $f(z) = \infty$ *then* $\varphi_2 \circ f(z) = 1/f(z)$ *which is holomorphic with a zero of the same order as the order of the pole of* $f(z)$.

Example 7.8 *(**Uniformization theorem**) Every simply-connected Riemann surface is biholomorphic to one of (a) the Riemann sphere, (b) the complex plane, (c) the open, upper half-plane.*

Example 7.9 *(**Complex structures on the torus**) Consider the lattice*

$$\Lambda = \mathbb{Z} \oplus \omega\mathbb{Z}$$

where $\omega \in \mathbb{C}\backslash\mathbb{R}$. *Then* $\mathbb{C}/\Lambda$ *is homeomorphic to a torus (Figure 7.1) and naturally inherits the structure of a Riemann surface from* $\mathbb{C}$.

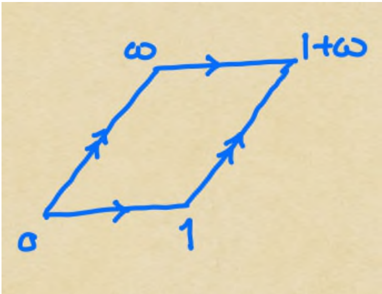

Figure 7.1 – parallelogram in lattice

But in general these complex tori are not biholomorphic to one another. It turns out (Kirwan p.141) that two complex tori $\mathbb{C}/\Lambda$ *and* $\mathbb{C}/\tilde{\Lambda}$ *are biholomorphic if and only if* $\Lambda = a\tilde{\Lambda}$ *or equally if* $J(\Lambda) = J(\tilde{\Lambda})$ *where*

$$J(\Lambda) = \frac{g_2^3}{g_2^3 - 27g_3^2}, \qquad g_2 = 60 \sum_{w \in \Lambda\backslash\{0\}} \frac{1}{w^4}, \qquad g_3 = 140 \sum_{w \in \Lambda\backslash\{0\}} \frac{1}{w^6}.$$

In particular, there are uncountably many biholomorphism classes of complex structures on a torus.

Example 7.10 *(**The Riemann surface of** $\sqrt{z}$) The affine surface is*

$$\Sigma = \left\{(z, w) \in \mathbb{C}^2 \mid w^2 = z\right\}.$$

Topologically this is not complicated. The map $w \mapsto (w^2, w)$ *is a homeomorphism from* $\mathbb{C}$ *to* Σ *and so* Σ *is topologically a plane. When we include a point at infinity it is topologically a sphere.*

However if we wish to understand Σ as the Riemann surface of the multifunction of $\sqrt{z}$ then we need to define $\sqrt{z}$ on a cut plane. This was already discussed in section 0.4 with regard to the topology of Σ. Here we focus on Σ as a Riemann surface. For $(z, w) \in \Sigma$ where $z \neq 0$ then we can use either z or w as a local holomorphic co-ordinate, but around $(0,0)$ we need to use w as the local co-ordinate. This is because 0 is a branch point of $\sqrt{z}$, something we will discuss in more detail later. There is a similar issue if we want to include the point at infinity which is again a branch point of $\sqrt{z}$.

Example 7.11 ***(A non-singular cubic)*** *The cubic*

$$y^2 = x(x-1)(x-\lambda), \qquad \lambda \neq 0, 1,$$

is a non-singular (affine) cubic Σ in $\mathbb{C}^2$. It has a projectivized version $\bar{\Sigma}$ with equation

$$y^2 z = x(x-z)(x-\lambda z)$$

which has a single point at infinity $[z \colon x \colon y] = [0 \colon 0 \colon 1]$. The above equations are known as Legendre form.

For $x \neq 0, 1, \lambda, \infty$ there are two values of y. We make cuts between 0 and 1 and between λ and ∞. We can then define two holomorphic branches on this cut plane (Figure 7.2)

$$\pm\sqrt{x(x-1)(x-\lambda)}$$

and most points of Σ take the form $\left(x, \sqrt{x(x-1)(x-\lambda)}\right)$ or $(x, -\sqrt{x(x-1)(x-\lambda)})$. Let

$$\begin{aligned} \Sigma_+ &= \left\{ \left(x, \sqrt{x(x-1)(x-\lambda)}\right) \mid x \in \text{ cut plane} \right\} \subseteq \mathbb{C}^2; \\ \Sigma_- &= \left\{ \left(x, -\sqrt{x(x-1)(x-\lambda)}\right) \mid x \in \text{ cut plane} \right\} \subseteq \mathbb{C}^2. \end{aligned}$$

The points that are missing from $\bar{\Sigma}$ are the point at infinity and those points associated with a value of x on the cuts.

Near the point $x = 0$ then $y^2 \approx \lambda x$. So as we move around the point $x = 0$ there is a sign change in the branches, just as there is with the standard holomorphic branches of the square root. This explains why the tabs A and B are so aligned as in Figure 7.3. The same argument can be made for the second cut. Including the point at infinity, we see that $\bar{\Sigma}$ is homeomorphic to a torus. Such an algebraic curve is called an ***elliptic curve****.*

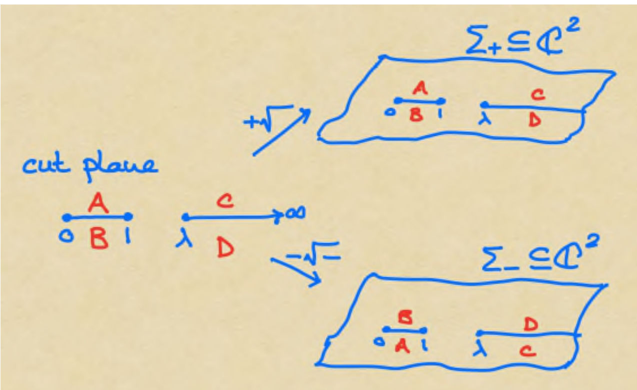

Figure 7.2 – branches on cut plane

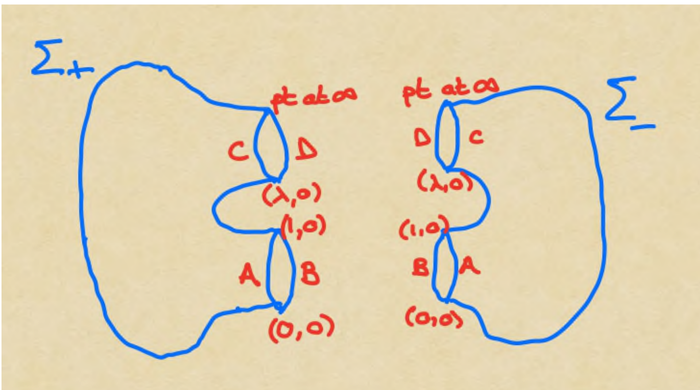

Figure 7.3 – gluing Σ_+ and Σ_-

It can be shown that the values

$$\lambda, \quad \lambda^{-1}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1}, \quad \frac{\lambda-1}{\lambda},$$

lead to biholomorphic complex tori.

Example 7.12 ***(Hyperelliptic curves)*** *We can extend the analysis of the previous Riemann surface to curves with equations*

$$y^2 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$$

where $n \geqslant 2$ and the α_i are distinct. In section 0.4 we saw that when $n = 2$ the Riemann surface is a sphere and in the previous example obtain a torus when $n = 3$. When we increase n by 2 then we need to introduce a further cut in the plane and add a further handle to the surface. So the surface has genus $(n-1)/2$ when n is odd and genus $(n-2)/2$ when n is even.

However the projective curve is singular at its point at infinity – see the following remark. So we cannot just assign a complex structure on the surface, inherited from complex projective plane. Away from the branch points α_i we can use either x or y as the local holomorphic co-ordinate. At the branch points we need to use y as in the previous example with the square root. For now, assume $n = 2k$ is even and we introduce at ∞ the following co-ordinates

$$X = \frac{1}{x}, \qquad Y = \frac{y}{x^k}.$$

The defining equation now reads as

$$\left(\frac{1}{X}-\alpha_1\right)\left(\frac{1}{X}-\alpha_2\right)\cdots\left(\frac{1}{X}-\alpha_{2k}\right) = \left(\frac{Y}{X^k}\right)^2$$

which rearranges to

$$(1-\alpha_1 X)(1-\alpha_2 X)\cdots(1-\alpha_{2k} X) = Y^2.$$

Near infinity, when $X \approx 0$, we have $Y^2 \approx 1$ and so we can compactify Σ with two points at infinity associated with $(X,Y) = (0,1)$ and $(X,Y) = (0,-1)$. Near these two points X is a local holomorphic co-ordinate. A similar approach can be taken when n is odd, with just one point being needed at infinity.

This shows that complex structures can be assigned to a torus of any genus $g \geqslant 2$; the above Riemann surfaces are called hyperelliptic curves. All complex structures (up to biholomorphism) on the torus of a given genus can be studied via a classifying space known as a 'moduli space'. All complex structures for genus 2 arise as hyperelliptic curves but for $g > 2$ the hyperelliptic curves are not generic within the moduli space.

Remark 7.13 *(Off-syllabus) The above non-singular cubic is the zero set in the complex projective plane of the function*

$$F(x,y,z) = y^2 z - x(x-z)(x-\lambda z).$$

A singular point of the cubic is any point satisfying $F = \nabla F = 0$ and a quick check shows none exist on the cubic. The cubic's complex structure is inherited from the ambient projective space.

On Sheet 0, Exercise 5, we met the cubic

$$y^2 - x(x-1)^2 = 0$$

and a check shows this curve to be singular at $(x, y) = (1, 0)$ *. This singularity is called a node. The complex projective curve is topologically a pinched torus.*

When $n > 3$, *the hyperelliptic curve*

$$G(x, y, z) = y^2 z^{n-2} - (x - \alpha_1 z)(x - \alpha_2 z) \cdots (x - \alpha_n z) = 0$$

can be checked to have a singularity at its point at infinity, where $x = 0, y = 1, z, = 0$. *This is why we complete the complex structure of the hyperelliptic curves in a different manner.*

7.2 The Riemann-Hurwitz formula

Proposition 7.14 *(**Local form of a holomorphic map**) For any holomorphic map* $f\colon S \to R$ *between Riemann surfaces, with* $f(s) = r$, *we can choose local complex co-ordinates around* $s \in S, r \in R$ *in terms of which* f *is the map*

$$f\colon D \to D \quad \text{given by} \quad f(z) = z^n.$$

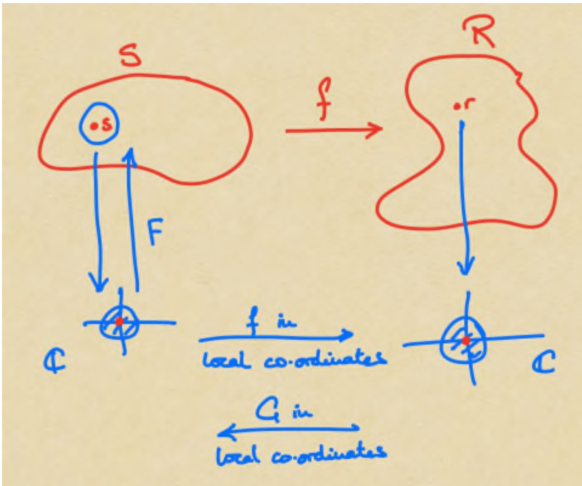

Figure 7.4 – local form of a holomorphic map

Proof. We can assume that, by translating if necessary, the local co-ordinates are chosen with s, r corresponding to $0 \in \mathbb{C}$, so that $f(0) = 0$ in local co-ordinates. The Taylor series for $f(z)$ therefore begins

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots$$

where $n \geqslant 1$ and $a_n \neq 0$ is the first non-zero coefficient. A holomorphic nth root of f is then defined near 0 with

$$f(z)^{1/n} = a_n^{1/n} z + \cdots .$$

The derivative of this nth root at 0 is $a_n^{1/n} \neq 0$ and so, by the inverse function theorem, there is a local holomorphic inverse G defined near 0. Then $G(0) = 0$ and

$$f(G(z))^{1/n} = z.$$

We can now change co-ordinates in the domain using the local biholomorphism G. To be explicit, if F is the original parameterization, defined locally near $s \in S$, then the new parameterization is $F \circ G$, defined near 0. The new local expression for f becomes

$$z \mapsto f(G(z)) = z^n.$$

■

Corollary 7.15 ***(Open Mapping Theorem)*** *A non-constant holomorphic map between Riemann surfaces is an open map. That is the image of an open set is open.*

Proof. This is left to Sheet 4, Exercise 3. ■

Definition 7.16 *Let $f: S \to R$ be a holomorphic map between Riemann surfaces and let $s \in S$. Then there are local co-ordinates around s and $f(s)$ such that f has the form $z \mapsto z^n$. The number n is called the* ***valency*** *of f at s and is written $v_f(s)$. Geometrically this is the number of solutions to the equation $f(z) = w$ for small $w \neq 0$. Thus the valency does not depend on the choice of co-ordinates.*

If $n > 1$ then we say that $f(s)$ is a ***branch point*** *and s is a* ***ramification point****. Note that s is a ramification point if and only if $f'(s) = 0$ in local co-ordinates.*

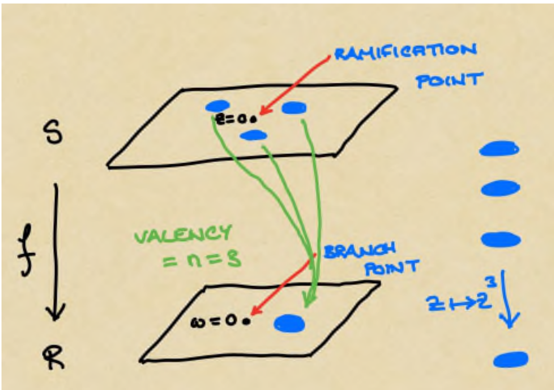

Figure 7.5 – local picture at a ramification point

Lemma 7.17 *A holomorphic function on a compact Riemann surface has finitely many ramification points.*

Proof. Locally $f(z) = z^n$ and so $f'(z) = nz^{n-1} \neq 0$ for $z \neq 0$. Hence the ramification points of f form a discrete set and hence a finite set, as S is (sequentially) compact. ■

Example 7.18 *Consider the map $f(z) = z^2$ from the Riemann sphere to itself. The ramification points of f are 0 and ∞ with the valency equalling 2 at each point.*

Proposition 7.19 ***(Degree of a map)*** *Given a non-constant holomorphic map $f: S \to R$ between compact Riemann surfaces, the* ***degree*** *of f is defined to be*

$$\deg(f) = \sum_{s \in f^{-1}(r)} v_f(s),$$

for any $r \in R$. This definition is independent of the choice of r.

Proof. Since S is compact, and f is not constant, then $f^{-1}(r)$ is finite. Choose local co-ordinates around each point p of $f^{-1}(r)$ such that around each point f is given by $z \mapsto z^{v_f(p)}$. Without loss of generality we may assume that the domains D_p of these local co-ordinates are disjoint, that their images are the same open neighbourhood V of r and that

$$f^{-1}(V) = \bigcup_{p \in f^{-1}(r)} D_p.$$

It follows that

$$\sum_{p \in f^{-1}(r)} v_f(p)$$

is the number of distinct solutions to the equation $f(z) = w$ for $w \in V \backslash \{r\}$. Thus this sum is locally constant and since R is connected then this sum is constant on R. ∎

Corollary 7.20 *For all points $r \in R$, except branch points, there are precisely* $\deg(f)$ *points in S which map to r.*

Example 7.21 *For the earlier example of $f(z) = z^2$ on the Riemann sphere, we have* $\deg(f) = 2$. *For $r \neq 0, \infty$ then $f^{-1}(r)$ consists of the two squares roots of r, each of which has valency* 1. *For $r = 0, \infty$ then $f^{-1}(r)$ is a singleton with valency* 2.

Theorem 7.22 *(**Riemann-Hurwitz Formula**) For any non-constant holomorphic function $f\colon S \to R$ between compact Riemann surfaces*

$$\chi(S) = \deg(f)\chi(R) - \sum_{\substack{\text{ramification} \\ \text{points } p}} (v_f(p) - 1)\,.$$

*The sum on the RHS is referred to as the **branching index** of f.*

Example 7.23 *For our earlier map $f(z) = z^2$ on the Riemann sphere, the above equation holds as $\chi(R) = \chi(S) = 2 = \deg(f)$ and $v_f(0) = v_f(\infty) = 2$. So we arrive at*

$$2 = 2 \times 2 - (2 - 1) - (2 - 1)\,,$$

which is true.

Proof. Pick a triangulation for R so that the branch points belong to the vertices of the triangulation. We want the pre-image to yield a triangulation of S. So we subdivide the triangles into smaller triangles if necessary, so that each triangle $T \subseteq R$ lies inside an open set $V \subseteq R$ small enough so that $f^{-1}(V) \to V$ can be written in the usual local form on each connected component $U \subseteq S$ of $f^{-1}(V)$. If the local form of $f\colon U \to V$ is $z \mapsto z$, then the pre-image of T is a triangle. But if the local form is $z \mapsto z^n$ where $n > 1$, then $f^{-1}(T)$ consists of n triangles.

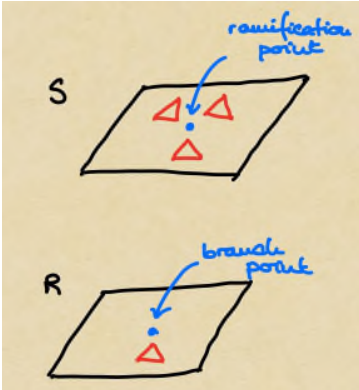

Figure 7.6a

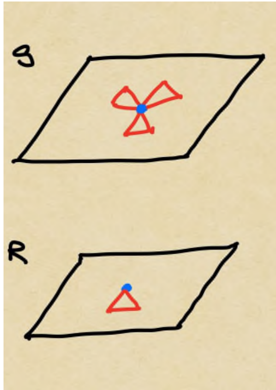

Figure 7.6b

If the vertex is not a branch point then these n triangles have n times as many vertices, edges and faces as T does (Figure 7.6a). Thus they contribute n times to $\chi(S)$. However if the vertex is a branch point, then the n triangles meet at the corresponding ramification point. We have lost $v_f(p) - 1$ vertices (Figure 7.6b). So the subdivision of S satisfies

$$V(S) = \deg(f)V(R) - \sum_{\substack{\text{ramification} \\ \text{points } p}} (v_f(p) - 1)$$

and $E(S) = \deg(f)E(R)$ and $F(S) = \deg(f)F(R)$. The result follows. ■

Example 7.24 *Suppose that $g(S) < g(R)$. Then any holomorphic map $f\colon S \to R$ is constant.*

Solution. Assume for a contradiction that f is not constant. We have

$$\chi(S) = \deg(f)\chi(R) - B$$

where $B \geqslant 0$ is the branching index. We then have

$$2 - 2g(S) = \deg(f)\,(2 - 2g(R)) - B$$

Rearranging gives

$$2g(R) - 2g(S) = (\deg(f) - 1)\,(2 - 2g(R)) - B.$$

The LHS is positive but, as $g(R) \geqslant 1$, the RHS is at most zero. ■

7.3 Meromorphic and Elliptic Functions

Recall that we can identify a meromorphic function f on S with a holomorphic map $f\colon S \to \mathbb{C}_\infty$ provided that f is not identically ∞. We can note here that f has equal number of poles and zeros on S, counting multiplicities, as that number is just $\deg(f)$. The following result may come as a surprise, but is a first intimation of connections with algebraic geometry.

Theorem 7.25 *(a) The meromorphic functions $\mathbb{C}_\infty \to \mathbb{C}_\infty$ are rational functions.*

(b) The biholomorphisms of $\mathbb{C}_\infty$ are the Möbius maps.

Remark 7.26 *Note that as a consequence of (a) the meromorphic functions on $\mathbb{C}_\infty$ form a field, the function field of $\mathbb{C}_\infty$.*

Proof. (a) As a consequence of the identity theorem, zeros of holomorphic functions are isolated. As $\mathbb{C}_\infty$ is compact, this means that $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ has finitely many zeros $z_1, \ldots, z_n$ and finitely many poles $p_1, \ldots, p_m$ in $\mathbb{C}$ – we will attend to a possible zero or pole at ∞ in a moment. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be the orders of the zeros and the poles and set

$$g(z) = \prod_{j=1}^{n} (z - z_j)^{a_j} \prod_{k=1}^{m} (z - p_k)^{-b_k}.$$

Then f/g is meromorphic and it no longer has any zeros or poles in $\mathbb{C}$.

By the earlier comment, f/g has an equal number of zeros and poles and cannot have both at ∞. This means that $f/g\colon \mathbb{C}_\infty \to \mathbb{C}$ is holomorphic and so, by Sheet 4, Exercise 3(ii), is constant. Hence $f =$ constant $\times\, g(z)$ is rational.

(b) By part (a) a biholomorphism is a rational function. As a biholomorphism is bijective, then there can be at most one zero and one pole. If that zero and pole have any multiplicity then then rational function will not be injective locally and so the numerator and denominator must have degree one and be independent of one another. That is, the biholomorphism must be a Möbius map. ■

Recall now that we have met complex tori both as the Riemann surface of the multifunction

$$\sqrt{(z - e_1)(z - e_2)(z - e_3)} \qquad e_1, e_2, e_3 \text{ are distinct},$$

and also as the quotient

$$\frac{\mathbb{C}}{\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2} \qquad \frac{\omega_2}{\omega_1} \notin \mathbb{R}.$$

We have not, thus far, made any connection between these two definitions.

Definition 7.27 *An* ***elliptic function*** *is a meromorphic function f on $\mathbb{C}$ which is doubly periodic – that is f is periodic in two independent directions ω_1 and ω_2.*

Definition 7.28 *The* ***Weierstrass $\wp$-function*** *(or* ***Weierstrass elliptic function****) associated with the lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ equals*

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Remark 7.29 *It is impossible to find a meromorphic function on $\mathbb{C}/\Lambda$ with only one simple pole as this would mean $\mathbb{C}/\Lambda$, a torus, is homeomorphic to $\mathbb{C}_\infty$, a sphere. If instead we take a meromorphic function with a double pole, then we can WLOG assume it to be at 0 with Laurent coefficient $c_{-2} = 1$. In order to make the function doubly periodic then we might expect to include the sum*

$$\sum_{0 \neq \omega \in \Lambda} \frac{1}{(z-\omega)^2}$$

but this is unfortunately divergent. However the inclusion of second term (which is itself divergent) makes the infinite sum convergent.

I list below some important facts about $\wp(z)$ but they are, in the main, turgid to prove. The important role of $\wp(z)$ is in providing a link between our two descriptions of complex tori.

- On a domain bounded away from the poles, $\wp(z)$ converges to an elliptic function.
- In the fundamental parallelogram $\{\alpha\omega_1 + \beta\omega_2 \mid 0 \leqslant \alpha, \beta \leqslant 1\}$, $\wp$ has a pole of order 2 at $z = 0$.
- $\wp\colon \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ has degree 2.
- $\wp' = 0$ at $\omega_1/2, \omega_2/2$ and $(\omega_1 + \omega_2)/2$.
- In the fundamental parallelogram $\wp$ has ramification points at $0, \omega_1/2, \omega_2/2$ and $(\omega_1 + \omega_2)/2$.
- The valencies at the ramification points are each 2.
- The branch points of $\wp$ are denoted

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \qquad e_2 = \wp\left(\frac{\omega_2}{2}\right), \qquad e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right), \qquad \infty = \wp(0).$$

- $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\left(\wp(z) - e_1\right)\left(\wp(z) - e_2\right)\left(\wp(z) - e_3\right).$$

Finally we have the following theorem:

Theorem 7.30 *(a) The following is a biholomorphism*

$$\begin{array}{rcl} \mathbb{C}/\Lambda & \to & \left\{(z,w) \in \mathbb{C}^2 \mid w^2 = 4(z-e_1)(z-e_2)(z-e_3)\right\} \cup \{\infty\}; \\ z & \mapsto & (\wp(z), \wp'(z)). \end{array}$$

(a) It can be shown that the function field of meromorphic functions on the complex torus $\mathbb{C}/\Lambda$ is $\mathbb{C}(\wp, \wp')$.