An introduction to open sets and closed sets in $\mathbb{R}$

Alexander F. Ritter
Mathematical Institute, Oxford

November 19, 2023.

An example to illustrate limit points and isolated points

Consider the set

$$S := \{\tfrac{1}{n} : n \geq 1 \in \mathbb{N}\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}.$$

Then S admits 0 as a **limit point**, meaning there exists a sequence $\frac{1}{n}$ of points in S *distinct from* 0, which converge to 0.

$\frac{1}{2}$ is not a limit point of S: the sequences in S that converge to $\frac{1}{2}$ are sequences which eventually become equal to the constant sequence $\frac{1}{2}$. It is called an **isolated point**.

The idea is that points of S are accumulating near 0, or clustering around 0: arbitrarily close to 0 we find points of S distinct from 0. Whereas the point $\frac{1}{2}$ is isolated in S: a small enough neighbourhood around $\frac{1}{2}$ only intersects S in the point $\frac{1}{2}$.

Question. What do you think are the limit points of the interval $(0, 1)$?

Limit points and isolated points

Let $S \subseteq \mathbb{R}$, and $p \in \mathbb{R}$ (we do not require p to be in S).
p is a **limit point** of S (or **accumulation point** or **cluster point**), if

$$\forall \varepsilon > 0, \exists s \in S \setminus \{p\} : |s - p| < \varepsilon.$$

Remark

(p is a limit point) $\Leftrightarrow$ (any open interval around p intersects $S \setminus \{p\}$)

Lemma

(p is a limit point) $\Leftrightarrow$ ($p = \lim s_n$ for some sequence $s_n \in S \setminus \{p\}$)

Proof.

($\Rightarrow$) Pick $\varepsilon = \frac{1}{n}$ to get $s_n \in S \setminus \{p\}$ with $|s_n - p| < \frac{1}{n}$, so $s_n \to p$.
($\Leftarrow$) $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$: $|s_n - p| < \varepsilon \; \forall n \geq N$. So $s := s_N \in S \setminus \{p\}$ has $|s - p| < \varepsilon$. □

$s \in S$ is an **isolated point** of S if it is not a limit point of S. So in some open interval around s there are no other points of S beyond s.
In symbols: $\exists \varepsilon > 0$ such that $\{x \in S : |x - s| < \varepsilon\} = \{s\}$.

Closures

Define $S' :=$ {limit points of S}.
The **closure** of S is $\overline{S} := S \cup S'$.

Example: $S = (0,1) \cup \{2\}$. Then $S' = [0,1]$ and $\overline{S} = [0,1] \cup \{2\}$ (e.g. observe, for $n \geq 2 \in \mathbb{N}$, $0 = \lim \frac{1}{n}$, $\frac{1}{n} \in S \setminus \{0\}$, and $1 = \lim(1 - \frac{1}{n})$, $1 - \frac{1}{n} \in S \setminus \{1\}$). The isolated points of S are $S \setminus S' = \{2\}$.

Challenge Exercise: Find $S \subseteq \mathbb{R}$ with $S \supseteq S' \supseteq S'' \neq \emptyset$ and $S''' = \emptyset$.

Corollary

$\overline{S} = \{$*limits of all convergent sequences in* $S\}$.

Proof.

($\subseteq$) By the Lemma, any limit point p arises as a limit of a convergent sequence $s_n \in S \setminus \{p\}$. Any $s \in S$ is the limit of the constant sequence s.
($\supseteq$) If $p = \lim s_n$ and $s_n \in S$, build $\widetilde{s}_n$ from s_n by removing all s_n which equal p. If there are only finitely many $\widetilde{s}_n$ left, then $s_n \in S$ is eventually the constant sequence p, so $p \in S$. Otherwise, $\widetilde{s}_n \in S \setminus \{p\}$ is a sequence converging to p, so $p \in S'$. □

Closed sets

$S \subseteq \mathbb{R}$ is a **closed set** if $S = \overline{S}$.

Corollary

$(p \in \overline{S}) \Leftrightarrow$ (any open interval around p intersects S)
(S closed) $\Leftrightarrow$ (S contains limits of convergent sequences $s_n \in S$)

Exercise: C_i closed $\Rightarrow \cap_{i \in I} C_i$ closed. For finite I also $\cup_{i \in I} C_i$ is closed.

Remark: $S \subseteq \mathbb{R}$ is **dense** if $\overline{S} = \mathbb{R}$. E.g. $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Lemma

$\overline{\overline{S}} = \overline{S}$. *("Closures are closed")*

Proof.

It suffices to show $(\overline{S})' \subseteq S'$. Let $p \in (\overline{S})'$. So $\exists s_n \in \overline{S} \setminus \{p\}$: $s_n \to p$.
Passing to a subsequence, $|s_n - p| < \frac{1}{n}$. If $s_n \in S \setminus \{p\}$, let $\widetilde{s}_n := s_n$.
Otherwise $s_n \in S' \setminus \{p\}$, so pick $\widetilde{s}_n \in S$ with $|\widetilde{s}_n - s_n| < \min(\frac{1}{n}, |p - s_n|)$.
$\Rightarrow \widetilde{s}_n \in S \setminus \{p\}$, $|\widetilde{s}_n - p| = |\widetilde{s}_n - s_n + s_n - p| \leq |\widetilde{s}_n - s_n| + |s_n - p| < \frac{2}{n}$,
thus $\widetilde{s}_n \to p$. So $p \in S'$. □

Open sets

$U \subseteq \mathbb{R}$ is an **open set** if the complement $S := \mathbb{R} \setminus U$ is closed.

Note: $\emptyset, \mathbb{R}$ are both open and closed, $(0,1]$ is neither open nor closed.

Corollary

(U is open) $\Leftrightarrow$ (any $p \in U$ lies in an open interval $J_p \subseteq U$)

Proof.

Let $S := \mathbb{R} \setminus U$. Recall $p \in \overline{S} \Leftrightarrow$ any open interval around p intersects S.
So $p \notin \overline{S} \Leftrightarrow$ ($\exists$open interval J_p around p with $J_p \cap S = \emptyset$), so $J_p \subseteq \mathbb{R} \setminus S$.
Finally use: (U open)$\Leftrightarrow$ (S is closed) $\Leftrightarrow$ ($S = \overline{S}$), and $\mathbb{R} \setminus S = U$. □

Corollary

($U \subseteq \mathbb{R}$ open) $\Leftrightarrow$ (U is a union of open intervals).

Proof.

$U = \bigcup_{p \in U} J_p$, using that $p \in J_p \subseteq U$. □

Exercise: U_i open $\Rightarrow \cup_{i \in I} U_i$ open. For finite I also $\cap_{i \in I} U_i$ is open.

Final remarks: neighbourhoods, boundary, and interior

A **neighbourhood** $V \subseteq \mathbb{R}$ of $p \in \mathbb{R}$ means there is an open set $U \subseteq \mathbb{R}$ with $p \in U \subseteq V \subseteq \mathbb{R}$. **Open neighbourhood** means V is open.

The **boundary** of S is

$$\partial S = \overline{S} \cap \overline{\mathbb{R} \setminus S}.$$

The **interior** of S is

$$\mathrm{Int}(S) = \bigcup \{\text{open } U \subseteq S\},$$

so it is the largest open set contained in S.

Example: $S = (0, 1] \cup \{2\}$

- $\partial S = \{0, 1, 2\}$,
- $\mathrm{Int} S = (0, 1)$.

Exercises:

- $p \in \partial S \Leftrightarrow$ (every neighbourhood of p intersects both S and $\mathbb{R} \setminus S$)
- $\mathrm{Int}(S) = \mathbb{R} \setminus \overline{\mathbb{R} \setminus S}$,
- $\partial S = \overline{S} \setminus \mathrm{Int}(S)$.