Stuff that I covered in the weeks 1–4 of MT 2023:

- Definition of singular homology with coefficients in an abelian group R.
  $\Delta^n := \{(x_0,\dots,x_n) \in \mathbb{R}_+^n : \sum x_i = 1\}$
  $C_n(X;R) = \oplus_{\sigma:\Delta^n\to X} R$
  $C_.(X)$, $\partial_n : C_n(X) \to C_n(X)$.
  $\partial_n(\sigma) = \sum (-1)^i \sigma|_{[0,\dots,\hat{\imath},\dots n]}$ where $\sigma|_{[0,\dots,\hat{\imath},\dots n]} (x_0,\dots,x_{n-1}) := \sigma(x_0,\dots,x_{i-1},0,x_i,\dots,x_{n-1})$
  Equivalently $\sigma|_{[0,\dots,\hat{\imath},\dots n]} := \sigma \circ d_i$ where $d_i(x_0,\dots,x_{n-1}) := (x_0,\dots,x_{i-1},0,x_i,\dots,x_{n-1})$.
  $\partial_n \partial_{n-1} = 0$
  $Z_n(X) = \ker(\partial_n)$, $B_n(X) = \mathrm{im}(\partial_{n+1})$, $H_n = Z_n/B_n$.
- Definition of singular cohomology.
  $C^n(X;R) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X;\mathbb{Z}), R)$
  $C^n(X;R) = \prod_{\sigma:\Delta^n\to X} R$
  $\delta^n : C^n(X) \to C^n(X)$
  $\delta^n \phi(\sigma) = \sum (-1)^i \phi(\sigma|_{[0,\dots,\hat{\imath},\dots n]})$
- $H_0(X) = \mathbb{Z}^{\#\text{ of connected components}}$
  $H_1(X) = \pi_1(X)_{ab}$ [statement without proof (it's one of the exercises)]
  $H_.(\sqcup X_i) = \oplus H_.(X_i)$, $H^*(\sqcup X_i) = \prod H^*(X_i)$.
- General definition of (co)chain complex.
  (co)chain maps.
- a chain homotopy between $f_., g_. : C_. \to D_.$ is $h_. : C_. \to D_{.+1}$ satisfying $h\partial + \partial h = g - f$.
  chain homotopic maps induce the same map on H_* (or H^*).
  Homotopic maps $f,g: X \to Y$ induce chain homotopic maps $f_*, g_* : C_.(X) \to C_.(Y)$ and chain homotopic maps $f^*, g^* : C^*(Y) \to C^*(X)$.
  homotopic maps $f,g: X \to Y$ induce same map on H_* and H^*.
  X homotopy equivalent to Y $\Rightarrow$ $H_*(X) \cong H_*(Y)$ and $H^*(X) \cong H^*(Y)$.
- Δ-complex, defined as a bunch of sets I_n and maps $d^i : I_n \to I_{n-1}$ satisfying $d^j d^i = d^{i-1} d^j$ if $j<i$.
  Geometric realisation X of a Δ-complex: $\sqcup_{n\in\mathbb{N}} I_n \times \Delta^n / (d^i\alpha, x) \sim (\alpha, d_i x)$.
  simplicial homology and cohomology:
  $C_.^{simpl}(X,R) := \oplus_{\alpha \in I\_n} R$, with differential $\partial_n(\alpha) := \sum (-1)^i d^i \alpha$.
- short/long exact sequences.
  The LES in homology associated to a SES of chain complexes.
  Relative homology and cohomology, for a pair $A \subset X$.
  $C_.(X,A) = C_.(X)/C_.(A)$, $C^n(X,A;R) = \mathrm{Hom}_{\mathbb{Z}}(C_n(X,A;\mathbb{Z}), R)$
  $H_.(X,A) = \ker(\partial)/\mathrm{im}(\partial)$. $H^*(X,A) = \ker(\delta)/\mathrm{im}(\delta)$
  The SES of chain complexes $0 \to C_.(A) \to C_.(X) \to C_.(X,A) \to 0$
  The long exact sequences in H_* and H^* associated to a pair $A \subset X$.
  Reduced (co)homology, defined as $H_*(X,\{pt\})$ and $H^*(X,\{pt\})$.
- Statement of excision: $H_.(X,A) = H_.(X\setminus E, A\setminus E)$ if $E \subset A \subset X$ and the closure of E is contained in the interior of A.

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Week 5, MT 2023:
Given a space X, and an open cover U={U_i} of X, write $C_n^U(X;R) = \oplus_{\sigma:\Delta^n\to X}R$ where the sum is indexed over those singular simplices whose image lands in one of the U_i.

Theorem(**small simplices theorem**): The inclusion $C_.^U(X;R) \to C_.(X;R)$ induces an isomorphism at the level of homology.

[postpone the proof until later] We first show some consequences:

Consequence #1:
Theorem(**excision**): If $E \subset A \subset X$ and the closure of E is contained in the interior of A, then the natural map $H_.(X\backslash E,A\backslash E) \to H_.(X,A)$ is an isomorphism.

Proof: consider the open cover U = {interior of A, complement of closure of E}.
A singular simplex $\sigma:\Delta^n \to X$ whose image lands in one of the two elements of U is either disjoint from E, or entirely contained in A. Therefore $C_.^U(X,A) = C_.^U(X\backslash E,A\backslash E)$.
We get two SES connected by inclusion maps:

$$
\begin{array}{ccccccccc}
0 \to & C_.(A) & \to & C_.(X) & \to & C_.(X,A) & \to 0 & & \\
 & \uparrow & & \uparrow & & \uparrow & & & \\
0 \to & C_.^U(A) & \to & C_.^U(X) & \to & C_.^U(X,A) & = & C_.^U(X\backslash E,A\backslash E) & \to 0.
\end{array}
$$

Passing to homology, we get two LES, and comparison maps
$H_.^U(A) \to H_.^U(A)$ (1)
$H_.^U(X) \to H_.^U(X)$ (2)
$H_.^U(X,A) \to H_.(X,A)$.
By an application of the 5-lemma [state the 5-lemma], since (1) and (2) are isomoprhisms, we get that the third map is also an isomorphism
Therefore $H_.(X\backslash E,A\backslash E) = H_.^U(X\backslash E,A\backslash E) = H_.^U(X,A) = H_.(X,A)$. QED

State and prove the 5-lemma.

Consequence #2:
Corollary:
If $A \subset X$ is an NDR pair (explain what NDR means), then $H_.(X,A)$ = reduced $H_.(X/A)$.

Proof:
Let V be the neighbourhood of A from the definition of NDR.
Compare the LES associated to $A \subset X$ and the LES associated to $A \subset X$.
By using the fact that $A \hookrightarrow V$ induces an isomorphism in $H_.$, we see that we can once again apply the 5-lemma, to get $H_.(X,A) \cong H_.(X,V)$. Therefore
$H_.(X,A) \cong H_.(X,V) \cong^{\text{excision}} H_.(X\backslash A,V\backslash A) \cong^{\text{excision}} H_.(X/A,V/A) \cong H_.(X/A,pt)$.
(The last isomorphism is again by the same argument as above, this time comparing the LES of $V/A \subset X/A$ to the LES of $pt \subset X/A$)
QED

Consequence #3:

Theorem(**Mayer-Vietoris**): Whenever $A \cup B = X$ and A,B are open (or whenever we have a situation which is homotopy equivalent to the above e.g. the two closed hemispheres of a sphere), then we have a LES

$... \to H_\bullet(A \cap B) \to H_\bullet(A) \oplus H_\bullet(B) \to H_\bullet(X) \to H_{\bullet-1}(A \cap B) \to ...$

Proof:

Letting $\mathcal{U}=\{A,B\}$, we have a SES of chain complexes

$0 \to C_\bullet(A \cap B) \to C_\bullet(A) \oplus C_\bullet(B) \to C_\bullet^{\mathcal{U}}(X) \to 0$

where the maps are the ones you expect, except for a pesky little minus sign.

Therefore, we get a a LES

$... \to H_\bullet(A \cap B) \to H_\bullet(A) \oplus H_\bullet(B) \to H_\bullet^{\mathcal{U}}(X) \to H_{\bullet-1}(A \cap B) \to ...$

But $H_\bullet^{\mathcal{U}}(X) = H_\bullet(X)$.

QED

Do some examples of Mayer-Vietoris:

– wedge of two (well-pointed) connected spaces: $H_\bullet(X \vee Y) = H_\bullet(X) \oplus H_\bullet(Y)$ in positive degrees.

– sphere covered by two hemispheres.

– genus 2 Riemann surfaces cut along a separating curve.

(uses that $T^2 \setminus D^2$ is homotopy equivalent to $S^1 \vee S^1$; explain why that's the case.

Compute the map $S^1 \hookrightarrow T^2 \setminus D^2$ at the level of homology by means of $H_1 = (\pi_1)_{ab}$).

Proof of **small simplices theorem:**

Recall the statement: $C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$ induces an isomorphism at the level of homology.

Strategy of proof:

• Define $S : C_\bullet(X) \to C_\bullet(X)$, where S stands for "subdivide".

[Draw some examples of what S does on some 1-chains: it replaces each singular 1-simplex by two singular 1-simplices going in opposite direction, one of which has a coefficient (-1). Then draw some examples of what S does on some 2-chains: it replaces each singular 2-simplex by six singular 2-simplices, again with various signs.]

• Prove that S is chain homotopic to the identity map $C_\bullet(X) \to C_\bullet(X)$.

• Prove that $\forall\ c \in C_\bullet(X)\ \exists\ N \in \mathbb{N}$ such that $S^N(c) \in C_\bullet^{\mathcal{U}}(X)$.

Assuming the above, let us prove the surjectivity of $H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$:

Pick $[c] \in H_\bullet(X)$.

Then $\forall N$, $[S^N(c)] = [c]$ by virtue of S (hence S^N) being chain homotopic to the identity.

But $[S^N(c)] \in H_\bullet^{\mathcal{U}}(X)$ for N large enough. ✓

...and injectivity of $H_\bullet^{\mathcal{U}}(X) \to H_\bullet(X)$:

Pick $[c] \in H_\bullet^{\mathcal{U}}(X)$ and assume that its image in $H_\bullet(X)$ is zero.

We want to show that $c \in \mathrm{im}(\partial: C_{\bullet+1}^{\mathcal{U}}(X) \to C_\bullet^{\mathcal{U}}(X))$.

Pick $C \in C_\bullet(X)$ such that $\partial C = c$, and $N \in \mathbb{N}$ large enough so that $S^N C \in C_\bullet^{\mathcal{U}}(X)$.

Let h be the chain homotopy between 1 and S^N, so that $h\partial C + \partial h C = C - S^N C$.

That is:

$$hc + \partial hC = C - S^N C.$$

Applying ∂ to the above:

$$\partial hc = c - \partial S^N C.$$

Thus $c = \partial(hc - S^N C)$ as desired, provided that h maps $C_\bullet^{U}(X) \to C_{\bullet+1}^{U}(X)$.

So, when we construct h, we'll have to be careful that it doesn't increase the size of the simplices. But this will be obvious from the construction.

Next task:

Define $S : C_\bullet(X) \to C_\bullet(X)$ and $h : C_\bullet(X) \to C_{\bullet+1}(X)$, and check that $h\partial + \partial h = id - S$.

We will construct S and h in a way which is natural in X, meaning that if $f:X \to Y$ is any map, we will construct S and h in such a way that the following diagrams commutes:

$$\begin{array}{ccc} C_\bullet(X) & \xrightarrow{S} & C_\bullet(X) \\ \downarrow f_* & & \downarrow f_* \\ C_\bullet(Y) & \xrightarrow{S} & C_\bullet(Y) \end{array} \qquad (*)$$

and

$$\begin{array}{ccc} C_\bullet(X) & \xrightarrow{h} & C_{\bullet+1}(X) \\ \downarrow f_* & & \downarrow f_* \\ C_\bullet(Y) & \xrightarrow{h} & C_{\bullet+1}(Y) \end{array} \qquad (**)$$

In formulas: $S(f_*(\sigma)) = f_*(S(\sigma))$. and $h(f_*(\sigma)) = f_*(h(\sigma))$.

If we know S and h on the singular simplex $\imath \in C_n(\Delta^n)$ given by the identity map $\imath := id_{\Delta^n} : \Delta^n \to \Delta^n$, then we can use (*) and (**) to deduce what they do on an arbitrary singular simplex $\sigma : \Delta^n \to X$. Indeed, we must have $S(\sigma) = S(\sigma_*(\imath)) = \sigma_*(S(\imath))$ and $h(\sigma) = h(\sigma_*(\imath)) = \sigma_*(h(\imath))$. So **it's enough to define $S(\imath)$ and $h(\imath)$**.

By a similar argument to above, in order to check the relation $h\partial + \partial h = id - S$, it's enough to check it when applied to $\imath := id_{\Delta^n} : \Delta^n \to \Delta^n$. Indeed:

$$\begin{aligned} h\partial\sigma + \partial h\sigma &= h\partial\sigma_*(\imath) + \partial h\sigma_*(\imath) \\ &= \sigma_*(h\partial(\imath) + \partial h(\imath)) \\ &= \sigma_*(\imath - S\imath) \\ &= \sigma_*(\imath) - S\sigma_*(\imath) \\ &= \sigma - S\sigma \end{aligned}$$

So **it's enough to check $h\partial\imath + \partial h\imath = \imath - S\imath$**.

Let Cone: $C_\bullet(\Delta^n) \to C_{\bullet+1}(\Delta^n)$ be the operation which sends a singular k-simplex $\sigma : \Delta^k \to \Delta^n$ to the singular (k+1)-simplex $Cone(\sigma) : \Delta^{k+1} \to \Delta^n$ defined by

$$Cone(\sigma)(x_0,\ldots,x_{k+1}) := x_0 \cdot b + (1-x_0)\cdot\sigma(x_1/(1-x_0),\ldots,x_{k+1}/(1-x_0)),$$

where $b := 1/(n+1)\cdot(1,\ldots,1)$ = barycenter of Δ^n.

[draw an example of $\sigma : \Delta^k \to \Delta^n$, and then draw $Cone(\sigma) : \Delta^{k+1} \to \Delta^n$]

<u>Lemma:</u> the above operation satisfies $\partial \circ Cone = id - Cone \circ \partial$.

[Draw a picture to show why this looks plausible, and tell the students that the proof is left as an exercise.]

Inductive definition of S:

- For n=0, we define $S : C_0(X) \to C_0(X)$ to be the identity map.
- For n≥1, we define $S(\imath)$ for $\imath := id_{\Delta^n} : \Delta^n \to \Delta^n$ by the formula $S(\imath) := Cone(S(\partial\imath))$.

The RHS makes reference to $S : C_{n-1}(X) \to C_{n-1}(X)$, which is assumed to be already defined by induction.

[draw some examples in dimensions 0, 1, and 2 to unpack the above inductive definition.]

Inductive definition of h:

- For n=0, we define $h : C_0(X) \to C_1(X)$ to be the zero map.
- For n≥1, we define $h(\imath)$ for $\imath := id_{\Delta^n} : \Delta^n \to \Delta^n$ by the formula $h(\imath) := Cone(\imath - h(\partial\imath))$.

The RHS makes reference to $h : C_{n-1}(X) \to C_n(X)$, which is assumed to be already defined by induction.

Finally, we check that the equation $h\partial\sigma + \partial h\sigma = \sigma - S\sigma$ holds true.
We may assume by induction that the above equation holds true for all chains σ of degree <n (it's easy to check for σ of degree 0).
As explained above, to prove the above equation for all chains of degree n, it's enough to argue that it holds true for $\imath = id_{\Delta^n}$.
And here we go:

$$\begin{aligned} \partial h\imath &\overset{\text{def of h}}{=} \partial(Cone(\imath - h\partial\imath)) \\ &\overset{\text{Lemma}}{=} \imath - h\partial\imath - Cone(\partial\imath - \partial h\partial\imath) \\ &\overset{\text{induction}}{=} \imath - h\partial\imath - Cone(S\partial\imath + h\partial\partial\imath) \\ &\overset{\text{def of S}}{=} \imath - h\partial\imath - S\imath \end{aligned}$$

Final task:
Prove that $\forall\ c \in C_\bullet(X)\ \exists\ N \in \mathbb{N}$ such that $S^N(c) \in C_\bullet^{\mathcal{U}}(X)$.
It's enough to show this when c consists of a single singular n-simplex $\sigma:\Delta^n \to X$.

Pulling back the open cover $\mathcal{U}$ along the map $\sigma:\Delta^n \to X$ to an open cover $\mathcal{U}'$ of Δ^n, it's enough to show that $\exists\ N \in \mathbb{N}$ such that $S^N(\imath) \in C_\bullet^{\mathcal{U}'}(\Delta^n)$.

[draw iterated barycentric subdivisions of an interval, and of a triangle.
Explain that our task is to show that the simplices become smaller and smaller.]

So, if we can prove the following lemma, we're good:

<u>Lemma:</u>

If $\sigma \subset \mathbb{R}^n$ is a straight-line simplex (the convex hull of n+1 points in $\mathbb{R}^n$) with diameter D, then each of the (n+1)! straight-line simplices which occur in the barycenric subdivision on σ has diameter ≤ n/(n+1)·D.

Proof:
We first note that if $\sigma = \text{conv}\{v_0,...,v_n\}$ is a straight-line simplex in $\mathbb{R}^n$, and $w \in \mathbb{R}^n$ is any point, then $\max_{v\in\sigma} \text{dist}(v,w) = \max_i \text{dist}(v_i,w)$. I.e. the maximal distance to a point in σ is achieved at some vertex of σ:

$$\text{dist}(v,w) = \| \sum x_i v_i - w \| = \| \sum x_i(v_i - w) \| \le \sum x_i \|v_i - w\| \le \max_i \|v_i - w\| \quad \text{because } \sum x_i=1.$$

The diameter of a simplex is therefore given by $\text{diam}(\sigma) = \max_{i,j} \|v_i - v_j\|$.

Let $\sigma = \text{conv}\{v_0,...,v_n\}$ be a straight-line simplex with diameter D, and let $\tau = \text{conv}\{w_0,...,w_n\}$ be a simplex which occurs in the barycenric subdivision on σ. We need to show:
$\forall i,j \ \|w_i - w_j\| \le n/(n+1)\cdot D$.

If neither w_i nor w_j is the barycenter of σ, then w_i and w_j are contained in some face of σ, and we're done by induction (with a better constant).

So we may asume that $w_j = b := 1/(n+1) \cdot (v_0 +...+ v_n)$.
We need to show: $\forall i \ \|w_i - b\| \le n/(n+1)\cdot D$.

we've seen ∃ a vertex v_k of σ such that $\|w_i - b\| \le \|v_k - b\|$.
So it's enough to show: $\forall k \ \|v_k - b\| \le n/(n+1)\cdot D$.

The straight line through v_k and b intersects σ into a segment of length L, and the ratio of lengths is always $\|v_k - b\| / L = n/(n+1)$, independently of σ.

Therefore $\|v_k - b\| = n/(n+1) \cdot L \le n/(n+1) \cdot D$. QED (lemma)

This finishes the proof of the small simplices theorem. QED

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<u>Tuesday week 6, MT 2023</u>

Universal coefficient theorem.
Basic questions that the UCT tries to answer:
- Is $H_*(X,R)$ determined by $H_*(X,\mathbb{Z})$?
- Is $H^*(X,R)$ determined by $H_*(X,\mathbb{Z})$? (And, if yes, how?)

input = $H_*(X,\mathbb{Z})$

Certainly, $C_*(X,R)$ and $C^*(X,R)$ are determined by $C_*(X,\mathbb{Z})$, via the formulas

$C_*(X,R) = C_*(X,\mathbb{Z}) \otimes R$ and $C^*(X,R) = \mathrm{Hom}_{\mathbb{Z}}(C_*(X,\mathbb{Z}), R)$

Recall $\mathbf{Hom}_{\mathbb{Z}}$ just means homomorphisms of abelian groups.
The subscript $_{\mathbb{Z}}$ means '$\mathbb{Z}$-module', but a $\mathbb{Z}$-module is the same thing as an abelian group.

And $\mathbf{A}\otimes\mathbf{B}$ (also denoted $A\otimes_{\mathbb{Z}}B$) is the ab group whose elements are formal sums $\sum_i a_i\otimes b_i$ with $a_i \in A$ and $b_i \in B$,
modulo the equivalence relation generated by $(a+a')\otimes b \sim a\otimes b + a'\otimes b$ and by $a\otimes(b+b') \sim a\otimes b + a\otimes b'$.
Alternatively, $A\otimes B$ is the quotient of $\oplus_A B$ by the subgroup generated by $(a+a')\otimes b - a\otimes b + a'\otimes b$,
or the quotient of $\oplus_B A$ by the subgroup generated by $a\otimes(b+b') \sim a\otimes b + a\otimes b'$.

In order to formulate the UCT, one needs **Ext** and **Tor** which, just like Hom and $\otimes$, are bifunctors.
They take two abelian groups as input, and produce a new abelian group.

<u>Definition of Tor and Ext:</u>
For any abelian group A, using that evey subgroup of a free abelian group is free, one can find a short exact sequence
$0 \longrightarrow \mathbb{Z}^{\oplus J} \xrightarrow{f} \mathbb{Z}^{\oplus I} \longrightarrow A \longrightarrow 0$.
(The chain complex ... $0 \to 0 \to \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I}$ is called a free *resolution* of A.)

One then defines
$\mathbf{Ext(A,B)} := \mathrm{coker}\,(f^*: \prod_I B \to \prod_J B)$
and
$\mathbf{Tor(A,B)} := \ker\,(f_*: \oplus_J B \to \oplus_I B)$.
where we've applied the functors Hom(-,B) and $-\otimes B$ to the map $f : \mathbb{Z}^{\oplus J} \to \mathbb{Z}^{\oplus I}$, respectively.

<u>Facts</u> (I won't prove this):
Ext(A,B) is a contravariant functor of the variable A, and covariant of the variable B (just like Hom is).
Tor(A,B) is a covariant functor of each variable, and satisfies Tor(A,B) = Tor(B,A) (just like $- \otimes -$).

<u>Example:</u>

$\mathrm{Ext}(\mathbb{Z}/2,B) = B / \{ 2b : b \in B\}$

$\mathrm{Tor}(\mathbb{Z}/2,B) = \{ b \in B : 2b=0 \}$

(can be seen by taking the free resolution of $\mathbb{Z}/2$ given by $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$.)

Note that
$\mathbf{Hom(A,B)} = \ker\,(f^*: \prod_I B \to \prod_J B)$ because that's $\ker(\mathrm{Hom}(\mathbb{Z}^{\oplus I},B) \to \mathrm{Hom}(\mathbb{Z}^{\oplus J},B))$
and
$\mathbf{A} \otimes \mathbf{B} = \mathrm{coker}\,(f_*: \oplus_J B \to \oplus_I B)$ because that's $\mathrm{coker}(\mathbb{Z}^{\oplus J}\otimes B \to \mathbb{Z}^{\oplus I}\otimes B)$.

The second is harder to check:

Proof:
The map coker($\mathbb{Z}^{\oplus J}\otimes B \to \mathbb{Z}^{\oplus I}\otimes B$) ⟶ $A\otimes B$ is visibly surjective.
Because for a typical element $\sum_i a_i\otimes b_i \in A\otimes B$, one can lift each a_i to $\mathbb{Z}^{\oplus I}$.

We need to see that if $\sum_i x_i\otimes b_i \in \mathbb{Z}^{\oplus I}\otimes B \mapsto 0 \in A\otimes B$, then it comes from $\mathbb{Z}^{\oplus J}\otimes B$.
The expression $\sum_i x_i\otimes b_i$ represents an element of $\oplus_B(\mathbb{Z}^{\oplus I})$.
Since its image in $\oplus_B A$ represents zero in $A\otimes B$, it can be written as $\sum_k a_k\otimes(b'_k+b''_k) - a_k\otimes b'_k - a_k\otimes b''_k \in \oplus_B A$.
Lift each $a_k\in A$ to some $x'_k\in\mathbb{Z}^{\oplus I}$ and consider the corresponding sum $\sum_k x'_k\otimes(b'_k+b''_k) - x'_k\otimes b'_k - x'_k\otimes b''_k \in \oplus_B(\mathbb{Z}^{\oplus I})$.
That new element of $\oplus_B(\mathbb{Z}^{\oplus I})$ differs from our original $\sum_i x_i\otimes b_i$ by something in $\oplus_B\ker(\mathbb{Z}^{\oplus I} \to A) = \oplus_B(\mathbb{Z}^{\oplus J})$.
We have written $\sum_i x_i\otimes b_i \in \oplus_B(\mathbb{Z}^{\oplus I})$ as a sum of something in $\oplus_B(\mathbb{Z}^{\oplus J})$ and something that represents 0 in $\mathbb{Z}^{\oplus I}\otimes B$.
⇒ we have written our $\sum_i x_i\otimes b_i \in \mathbb{Z}^{\oplus I}\otimes B$ as something in $\mathbb{Z}^{\oplus J}\otimes B$.   QED

Theorem(**universal coefficient theorem**):
There exist natural, split short exact sequences

$0 \longrightarrow H_n(X,\mathbb{Z}) \otimes R \longrightarrow \mathbf{H_n(X,R)} \longrightarrow \mathrm{Tor}(H_{n-1}(X,\mathbb{Z}), R) \longrightarrow 0$

$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X,\mathbb{Z}), R) \longrightarrow \mathbf{H^n(X,R)} \longrightarrow \mathrm{Hom}(H_n(X,\mathbb{Z}), R) \longrightarrow 0$

Proof:
The proof relies on the following observation:
The short exact sequence

$0 \to Z_n(X) \longrightarrow C_n(X) \xrightarrow{\partial} B_{n-1}(X) \to 0$ (∗)

can be interpreted as a short exact sequence of chain complexes

$\mathbf{0 \to Z_\bullet(X) \to C_\bullet(X) \to B_{\bullet-1}(X) \to 0}$

where the the 1st and 3rd terms are viewed as chain complexes with zero differential.

(Look at associated LES? In the associated LES of homology groups,
the connecting homomorphism $B_{\bullet-1}(X) \to Z_{\bullet-1}(X)$ is just the usual inclusion.

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z_n(X) & \to & \mathbf{C_n(X)} & \to & \mathbf{B_{n-1}(X)} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathbf{Z_{n-1}(X)} & \to & \mathbf{C_{n-1}(X)} & \to & B_{n-2}(X) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$
)

Applying the functors – ⊗ R and Hom(– , R) to get
two new short exact sequences of chain complexes

$0 \longrightarrow Z_\bullet(X) \otimes R \longrightarrow C_\bullet(X,R) \longrightarrow B_{\bullet-1}(X) \otimes R \longrightarrow 0$

and

$0 \longrightarrow \mathrm{Hom}(B_{\bullet-1}(X),R) \longrightarrow C^*(X,R) \longrightarrow \mathrm{Hom}(Z_\bullet(X),R) \longrightarrow 0.$

(*Note:* These two functors do not, in general send SES to SES. But ($*$) is a split SES, because $B_{n-1}(X)$ is a free abelian group. Recall, every subgroup of a free abelian group is free.)

We get corresponding LES in (co)homology:

$\ldots \longrightarrow B_n(X) \otimes R \longrightarrow Z_n(X) \otimes R \longrightarrow H_n(X,R) \longrightarrow B_{n-1}(X) \otimes R \longrightarrow Z_{n-1}(X) \otimes R \longrightarrow \ldots$

and

$\ldots \longrightarrow \mathrm{Hom}(Z_{n-1}(X),R) \longrightarrow \mathrm{Hom}(B_{n-1}(X),R) \longrightarrow H^*(X,R) \longrightarrow \mathrm{Hom}(Z_n(X),R) \longrightarrow \mathrm{Hom}(B_n(X),R) \longrightarrow \ldots$

(Like above, the maps $B_n(X) \otimes R \to Z_n(X) \otimes R$ and $\mathrm{Hom}(Z_{n-1}(X),R) \to \mathrm{Hom}(B_{n-1}(X),R)$ are induced by the inclusion $B_n(X) \otimes R \hookrightarrow Z_n(X)$.)

We rewrite this as short exact sequences:

$0 \longrightarrow \mathrm{coker}(B_n(X) \otimes R \to Z_n(X) \otimes R) \longrightarrow H_n(X,R) \longrightarrow \ker(B_{n-1}(X) \otimes R \to Z_{n-1}(X) \otimes R) \longrightarrow 0$

and

$0 \longrightarrow \mathrm{coker}(\mathrm{Hom}(Z_{n-1}(X),R) \to \mathrm{Hom}(B_{n-1}(X),R)) \longrightarrow H^*(X,R) \longrightarrow \ker(\mathrm{Hom}(Z_n(X),R) \to \mathrm{Hom}(B_n(X),R)) \longrightarrow 0$

which we then recognise as

$0 \longrightarrow H_n(X,\mathbb{Z}) \otimes R \longrightarrow H_n(X,R) \longrightarrow \mathrm{Tor}(H_{n-1}(X,\mathbb{Z}), R) \longrightarrow 0$

and

$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X,\mathbb{Z}), R) \longrightarrow H^*(X,R) \longrightarrow \mathrm{Hom}(H_n(X,\mathbb{Z}), R) \longrightarrow 0$

in view of the fact that $(\ldots \to 0 \to B_n(X) \to Z_n(X))$ is a free resolution of $H_n(X)$.
Here, we've used that if $\ldots \to 0 \to \mathbb{Z}^{\oplus J} \xrightarrow{f} \mathbb{Z}^{\oplus I}$ is a free resolution of A, then
A⊗R = coker (f_*: $\oplus_J R \to \oplus_I R$)
Tor(A,R) = ker (f_*: $\oplus_J R \to \oplus_I R$)
Ext(A,R) = coker (f^*: $\prod_I R \to \prod_J R$)
Hom(A,R) = ker (f^*: $\prod_I R \to \prod_J R$)

Proof that these SES are split:
Recall that $0 \to Z_n(X) \longrightarrow C_n(X) \xrightarrow{\partial} B_{n-1}(X) \to 0$ is split.
Pick a splitting, which gives us a retraction $Z_n(X) \xleftarrow{p} C_n(X)$ of the natural inclusion.
The operation $- \circ p$ induces a splitting $\mathrm{Hom}(C_n(X),R) \leftarrow \mathrm{Hom}(Z_n(X),R)$ of the natural map.

Applying this to some $f \in \ker(\mathrm{Hom}(Z_n(X),R) \to \mathrm{Hom}(B_n(X),R)) = \mathrm{Hom}(H_n(X,\mathbb{Z}), R)$
we get a map $f \circ p : C_n(X) \to R$ that vanishes on $B_n(X)$.
That's the same as a map $C_n(X) \to R$ that vanishes when precomposed with $\partial{:}C_{n+1}(X) \to C_{n+1}(X)$,
i.e. an element of $C^n(X,R)$ in the kernel of $\delta{:}C^n(X,R) \to C^{n+1}(X,R)$, i.e.,
an element of $Z^n(X,R)$.

We may then compose with the quotient map $Z^n(X,R) \to H^n(X,R)$ to get a map
$H^*(X,R) \leftarrow \ker(\mathrm{Hom}(Z_n(X),R) \to \mathrm{Hom}(B_n(X),R)) = \mathrm{Hom}(H_n(X,\mathbb{Z}), R)$.

This construction provides a splitting of the natural map $H^*(X,R) \longrightarrow \mathrm{Hom}(H_n(X,\mathbb{Z}), R)$.

The splitting is *not natural* because the retraction $Z_n(X) \xleftarrow{p} C_n(X)$ is not natural.
It cannot be picked simultaneously for all spaces X in such a way that $\forall\ X \to Y$, the diagram

$$\begin{array}{ccc} Z_n(X) & \longleftarrow & C_n(X) \\ \downarrow & & \downarrow \\ Z_n(Y) & \longleftarrow & C_n(Y) \end{array}$$

commutes.

(See Hatcher p.264 for why the UCT homology short exact sequence is split.)
QED

$$0 \longrightarrow H_n(X,\mathbb{Z}) \otimes R \longrightarrow H_n(X,R) \longrightarrow \mathrm{Tor}(H_{n-1}(X,\mathbb{Z}), R) \longrightarrow 0$$

Work out examples of UCT:
– (co)homology of $\mathbb{R}P^2$.

$H_*(\mathbb{R}P^2, \mathbb{Z}) = [\mathbb{Z}, \mathbb{Z}/2, 0, 0, 0, ...]$
$H_*(\mathbb{R}P^2, \mathbb{Z}/2) = [\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, 0, ...]$
$H^*(\mathbb{R}P^2, \mathbb{Z}) = [\mathbb{Z}, 0, \mathbb{Z}/2, 0, 0, ...]$
$H^*(\mathbb{R}P^2, \mathbb{Z}/2) = [\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2, 0, 0, ...]$

– (co)homology of Klein Bottle. (exercise)

<u>Corollary</u>(**excision for H***):
If $E \subset A \subset X$ and the closure of E is contained in the interior of A, then the natural map $H^*(X,A) \to H^*(X \backslash E, A \backslash E)$ is an isomorphism.

Proof:
The universal coefficient theorem for $H^*(X,A)$ and for $H^*(X \backslash E,A \backslash E)$ are short exact sequences

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X,A;\mathbb{Z}), R) \longrightarrow H^*(X,A;R) \longrightarrow \mathrm{Hom}(H_n(X,A;\mathbb{Z}), R) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X\backslash E,A\backslash E;\mathbb{Z}), R) \longrightarrow H^*(X\backslash E,A\backslash E;R) \longrightarrow \mathrm{Hom}(H_n(X\backslash E,A\backslash E;\mathbb{Z}), R) \longrightarrow 0.$$

By the naturality of the UCT, the inclusion $C.(X \backslash E,A \backslash E) \to C.(X,A)$ induces comparison maps that fit into a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X,A;\mathbb{Z}), R) & \longrightarrow & H^*(X,A;R) & \longrightarrow & \mathrm{Hom}(H_n(X,A;\mathbb{Z}), R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_{n-1}(X\backslash E,A\backslash E;\mathbb{Z}),R) & \longrightarrow & H^*(X\backslash E,A\backslash E;R) & \longrightarrow & \mathrm{Hom}(H_n(X\backslash E,A\backslash E;\mathbb{Z}), R) & \longrightarrow & 0 \end{array}$$

The 1st and 3rd vertical arrows induce isomorphisms by the excision theorem for homology (using that Ext(- , R) and Hom(- , R) are functors). So we're done by the 5 lemma. QED