Exercise sheet 4. All lectures.

Part A.

Question 4.1. Let $f : X \to Y$ be a regular map between varieties. Suppose that X is quasi-projective. Let $\sigma : Y \to X$ be a regular map such that $f \circ \sigma = \mathrm{Id}_Y$ (such a map is called a *section* of f). Show that $\sigma(Y)$ is closed in X.

Solution. Note that we have by construction

$$\sigma(Y) = \{x \in X \,|\, (\sigma \circ f)(x) = x\}.$$

Let $\delta_X : X \to X \times X$ (resp. $\Delta_X \subseteq X \times X$) be the diagonal map (resp. the diagonal) of X. Let $\Gamma_{\sigma\circ f} \subseteq X \times X$ be the graph of $\sigma \circ f$ (see Prop.-Def. 12.6). By the above, we have

$$\sigma(Y) = \delta_X^{-1}(\Delta_X \cap \Gamma_{\sigma\circ f})$$

and this set is closed because Δ_X and $\Gamma_{\sigma\circ f}$ are closed (because X is separated).

Part B.

Question 4.2. Suppose in this exercise that $\mathrm{char}(k) = 0$. Find the singularities of the following curves C in k^2. For each singular point $P \in C$ compute the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space. Here $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_{C,P}$.

(1) $\mathrm{Z}(x^6 + y^6 - xy)$

(2) $\mathrm{Z}(y^2 + x^4 + y^4 - x^3)$

You may assume that the polynomials $x^6 + y^6 - xy$ and $y^2 + x^4 + y^4 - x^3$ are irreducible.

Solution. (1) Note that $\dim(C) = 1$ by Krull's theorem and by Theorem 8.7. Thus we need to find the points of C where the gradient of $x^6 + y^6 - xy$ vanishes. The gradient of $x^6 + y^6 - xy$ is $\langle 6x^5 - y, 6y^5 - x\rangle$. Hence we need to solve the equations $x^6 + y^6 - xy = 6x^5 - y = 6y^5 - x = 0$. We have

$$(x/6)(6x^5 - y) - (y/6)(6y^5 - x) + 2y^6 - xy = x^6 + y^6 - xy$$

and thus these equations are equivalent to

$$2y(y^5 - x) = 6x^5 - y = 6y^5 - x = 0.$$

Now if $y = 0$ then $x = 0$. If $y \neq 0$ then $y^5 = x = x/6$ so $y = 0$, which is a contradiction. So we must have $x = y = 0$. So $\langle 0, 0\rangle$ is the only singular point of C.

For $P = \langle 0, 0\rangle$ the dimension of $\mathfrak{m}_P/\mathfrak{m}_P^2$ as a k-vector space cannot be 1, since otherwise the ring $\mathcal{O}_{C,P}$ would be regular (apply Proposition 13.3). Since $\mathfrak{m}_P$ is generated as a $k[x, y]$-module by the elements x and y, we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension at most 2. Hence $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

(2) The reasoning is similar. Solve $y^2 + x^4 + y^4 - x^3 = 4x^3 - 3x^2 = 2y + 4y^3 = 0$. Combining, we obtain

$$4(y^2 + x^4 + y^4 - x^3) + (1/4 - x)(4x^3 - 3x^2) - y(2y + 4y^3) = (-3/4)x^2 + 2y^2 = 0.$$

Now if $x \neq 0$ then $x = 3/4$ since $4x^3 - 3x^2 = 0$ and so $y^2 = 27/128$. But then $y(2y + 4y^3) = 6503409/67108864$ which is a contradiction. So we have $x = 0$ and also $y = 0$. We conclude again that the origin is the only singular point of C. By the same reasoning as above, we see that $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension 2.

Question 4.3. Let C be the plane curve considered in (1) of question 4.2. Consider the blow-up B of C at each of its singular points in turn. How many irreducible components does the exceptional divisor of B have? Is B nonsingular?

Solution. Consider the curve $Z(x_1x_2 - x_1^6 - x_2^6) \subseteq k^2$ of (1) of question 4.2. Use the terminology of Propositions 14.1 and 14.2, letting $n = 2$ and $X = Z(x_1x_2 - x_1^6 - x_2^6) = Y$ (note that the point to blow-up is the origin by the solution of question 4.2 (1) so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = Z(x_1y_2 - x_2y_1, x_1x_2 - x_1^6 - x_2^6)$$

Let $U_1 := \{[1, Y_2] \,|\, Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} & \phi^{-1}(X) \cap (k^2 \times U_1) = Z(x_1y_2 - x_2, x_1x_2 - x_1^6 - x_2^6) = Z(x_1y_2 - x_2, x_1^2y_2 - x_1^6 - x_1^6y_2^6) \\ = \; & Z(x_1y_2 - x_2, x_1^2(y_2 - x_1^4 - x_1^4y_2^6)) = Z(x_1y_2 - x_2, x_1) \cup Z(x_1y_2 - x_2, y_2 - x_1^4 - x_1^4y_2^6)) \\ = \; & \{0\} \times U_1 \cup Z(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2) \end{aligned}$$

Now $Z(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2)$ does not contain $\{0\} \times U_1$ (since setting $x_1 = x_2 = 0$ implies that $y_2 = 0$) so we have $\mathrm{Bl}(X, 0) \cap (k^2 \times U_1) = Z(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2)$ by question **??** (2). Finally, note that $Z(x_1y_2 - x_2, y_2 - x_1^4 - x_2^4y_2^2) \cap (\{0\} \times U_1)$ contains only the point $\{0\} \times \{[1, 0]\}$. In other words, the intersection of the exceptional divisor of $\mathrm{Bl}(X, 0)$ with $\{0\} \times U_1$ is the point $\{0\} \times \{[1, 0]\}$.

Let now $U_2 := \{[Y_1, 1] \,|\, Y_1 \in k\} \subseteq \mathbb{P}^1(k)$. We compute as before

$$\begin{aligned} & \phi^{-1}(X) \cap (k^2 \times U_1) = Z(x_1 - x_2y_1, x_1x_2 - x_1^6 - x_2^6) = Z(x_1 - x_2y_1, y_1x_2^2 - x_2^6y_1^6 - x_2^6) \\ = \; & Z(x_1 - x_2y_1, x_2) \cup Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) = \{0\} \times U_2 \cup Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \end{aligned}$$

We conclude as before that

$$\mathrm{Bl}(X, 0) \cap (k^2 \times U_2) = Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4)$$

We compute $Z(x_1 - x_2y_1, y_1 - x_2^4y_1^6 - x_2^4) \cap (\{0\} \times U_2) = \{0\} \times \{[0, 1]\}$. So the intersection of the exceptional divisor of $\mathrm{Bl}(X, 0)$ with $\{0\} \times U_2$ is the point $\{0\} \times [0, 1]$.

Putting everything together, we see that the exceptional divisor of $\mathrm{Bl}(X, 0)$ consists of the points $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$. In particular, the exceptional divisor of $\mathrm{Bl}(X, 0)$ has two irreducible components.

We now check nonsingularity. We only have to check the nonsingularity of $\mathrm{Bl}(X, 0)$ at $\{0\} \times \{[1, 0]\}$ and $\{0\} \times \{[0, 1]\}$ since $\mathrm{Bl}(X, 0) \backslash \{\{0\} \times \{[1, 0]\} \cup \{0\} \times \{[0, 1]\}\}$ is isomorphic to $X \backslash \{0\}$ and $X \backslash \{0\}$ is nonsingular by the solution of question 4.2 (1).

We first check nonsingularity at $\{0\} \times \{[1, 0]\}$. Let $Q_1 := x_1y_2 - x_2$ and $Q_2 := y_2 - x_1^4 - x_2^4y_2^2$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1}Q_1 & \frac{\partial}{\partial x_2}Q_1 & \frac{\partial}{\partial y_2}Q_1 \\ \frac{\partial}{\partial x_1}Q_2 & \frac{\partial}{\partial x_2}Q_2 & \frac{\partial}{\partial y_2}Q_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -4x_1^3 & -4x_2^3y_2^2 & 1 - 2x_2^4y_2 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has rank 2. Using Lemma 13.5 we see that $\mathrm{Bl}(X, 0)$ is nonsingular at $\{0\} \times \{[1, 0]\}$.

We now check nonsingularity at $\{0\} \times \{[0,1]\}$. Let $Q_1 := x_1 - x_2 y_1$ and $Q_2 := y_1 - x_2^4 y_1^6 - x_2^4$. We have

$$\begin{pmatrix} \frac{\partial}{\partial x_1} Q_1 & \frac{\partial}{\partial x_2} Q_1 & \frac{\partial}{\partial y_2} Q_1 \\ \frac{\partial}{\partial x_1} Q_2 & \frac{\partial}{\partial x_2} Q_2 & \frac{\partial}{\partial y_2} Q_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4x_2^3 - 4x_2^3 y_1^6 & 1 - 6x_2^4 y_1^5 \end{pmatrix}$$

and evaluating at 0 we get the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which again has rank 2. Again using Lemma 13.5 we see that $\mathrm{Bl}(X, 0)$ is nonsingular at $\{0\} \times \{[0,1]\}$.

So all in all $\mathrm{Bl}(X, 0)$ is nonsingular and its exceptional divisor has two irreducible components (which are points).

Question 4.4. Let $V \subseteq k^2$ be the algebraic set defined by the equation $x_1 x_2 = 0$. Show that $\mathrm{Bl}(V, 0)$ has two disjoint irreducible components and that each of these components is isomorphic to k.

Solution. Use the terminology of questions Proposition 14.1 and 14.2, letting $n = 2$ and $X = \mathrm{Z}(x_1 x_2) = Y$ (note that the point to blow-up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1 y_2 - x_2 y_1, x_1 x_2)$$

Let $U_1 := \{[1, Y_2] \mid Y_2 \in k\} \subseteq \mathbb{P}^1(k)$. In $k^2 \times U_1$, we have

$$\begin{aligned} & \phi^{-1}(X) \cap (k^2 \times U_1) = \mathrm{Z}(x_1 y_2 - x_2, x_1 x_2) = \mathrm{Z}(x_1 y_2 - x_2, x_1) \cup \mathrm{Z}(x_1 y_2 - x_2, x_2) \\ = \quad & \{0\} \times U_1 \cup \mathrm{Z}(x_1 y_2, x_2) = \{0\} \times U_1 \cup \mathrm{Z}(x_1, x_2) \cup \mathrm{Z}(y_2, x_2) = \{0\} \times U_1 \cup \mathrm{Z}(y_2, x_2) \end{aligned}$$

Now note that by definition $\mathrm{Bl}(X, 0)$ is the closure of $\phi^{-1}(X \backslash 0)$. In particular, $\mathrm{Bl}(X, 0)$ is the union of the closures of $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0)$ and $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0)$, ie the blow-ups of $\mathrm{Z}(x_1)$ and of $\mathrm{Z}(x_2)$, respectively. Now note that $\phi^{-1}(\mathrm{Z}(x_1)\backslash 0) \cap (k^2 \times U_1) = \emptyset$ (see the solution to Q2 (3)). Noting also that $\mathrm{Z}(y_2, x_2)$ is irreducible, we see that $\mathrm{Bl}(X, 0) \cap (k^2 \times U_1)) = \mathrm{Z}(y_2, x_2)$.

A completely similar reasoning with U_2 in place of U_1 shows that $\mathrm{Bl}(X, 0) \cap (k^2 \times U_2) = \mathrm{Z}(y_1, x_1)$. Hence $\mathrm{Bl}(X, 0) \subseteq \mathrm{Z}(y_2, x_2) \cup \mathrm{Z}(y_1, x_1) \subseteq k^2 \times \mathbb{P}^1(k)$, where we view the polynomials x_1, x_2, y_1, y_2 as homogenous polynomials in the y-variables. On the other hand we have $\mathrm{Z}(y_2, x_2) \cap \mathrm{Z}(y_1, x_1) = \mathrm{Z}(x_1, x_2, y_1, y_2) = \emptyset$ and $\mathrm{Z}(y_2, x_2) \simeq \mathrm{Z}(y_1, x_1) \simeq k$. Since $\mathrm{Bl}(X, 0)$ has two irreducible components of dimension 1 by the above, we thus have $\mathrm{Bl}(X, 0) = \mathrm{Z}(y_2, x_2) \cup \mathrm{Z}(y_1, x_1)$.

Question 4.5. Let $C \subseteq k^2$ be defined by the equation $P(x_1, x_2) = 0$, where $P(x_1, x_2)$ is an irreducible polynomial. Suppose that C goes through the origin 0 of k^2 and is non singular there. Show that the natural morphism $\mathrm{Bl}(C, 0) \to C$ is an isomorphism. *[Hint: construct an inverse map directly, without looking at coordinate charts]*

Solution. Use the terminology of questions Proposition 14.1 and 14.2, letting $n = 2$ and

$$X = \mathrm{Z}(P(x_1, x_2)) = Y$$

(note that the point to blow up is the origin so we do not have to translate X). We first compute $\phi^{-1}(X)$. Let $\pi : k^n \times \mathbb{P}^1(k) \to k^n$ be the natural projection. By definition

$$\phi^{-1}(X) = \pi^{-1}(X) \cap Z = \mathrm{Z}(x_1 y_2 - x_2 y_1, P(x_1, x_2)).$$

By assumption, we have either have $P_{x_1}(0,0) \neq 0$ or $P_{x_2}(0,0) \neq 0$. We suppose that $P_{x_1}(0,0) \neq 0$, the reasoning being similar if $P_{x_2}(0,0) \neq 0$. Also, note that by assumption, we have $P(0,0) = 0$. This implies that we can write

$$P(x_1, x_2) = A(x_2) + x_1 Q(x_1, x_2)$$

where $Q(0,0) \neq 0$ and $A(0) = 0$. In particular, $x_2 | A(x_2)$. Write $B(x_2) = A(x_2)/x_2$. We define a map $\sigma : X \to k^n \times \mathbb{P}^1(k)$ by the formulae

$$\sigma(X_1, X_2) = \langle X_1, X_2 \rangle \times [B(X_2), -Q(X_1, X_2)] \quad (*)$$

if $Q(X_1, X_2) \neq 0$ and

$$\sigma(X_1, X_2) := \langle X_1, X_2 \rangle \times [X_1, X_2] \quad (**)$$

otherwise. We have

$$B(X_2)X_2 - (-Q(X_1, X_2))X_1 = 0$$

so that whenever $\langle X_1, X_2 \rangle \neq \langle 0, 0 \rangle$ and $\langle X_1, X_2 \rangle \in X$, the formulae $(*)$ and $(**)$ give the same point in $k^n \times \mathbb{P}^1(k)$. Also, by construction, $\sigma(X) \subseteq \phi^{-1}(X)$ and $\phi \circ \sigma = \mathrm{Id}_X$. Now by Question 4.1, the set $\sigma(X)$ is closed. The set $\sigma(X)$ is thus closed and irreducible and it is isomorphic to X as a closed subvariety of $\phi^{-1}(X)$ (since $\phi|_{\sigma(X)}$ provides an inverse to $\sigma : X \to \sigma(X)$). Finally, $\sigma(X)$ is not contained in $\{0\} \times \mathbb{P}^1(k)$. Hence it is contained in $\mathrm{Bl}(X, 0)$ (see Proposition 14.2) and it thus coincides with it since $\mathrm{Bl}(X, 0)$ is birational to X and thus has the same dimension as X. The map σ thus provides an isomorphism between $\mathrm{Bl}(X, 0)$ and X.

Part C.

Question 4.6. (1) Let $f : X \to Y$ be a dominant morphism of varieties. Suppose that Y is irreducible. Show that $\dim(X) \geqslant \dim(Y)$.

(2) Let $f : X \to Y$ be a dominant morphism of irreducible varieties. Suppose that the field extension $\kappa(X)|\kappa(Y)$ is algebraic. Show that there are affine open subvarieties $U \subseteq X$ and $W \subseteq Y$ such that $f(U) = W$ and such that the map of rings $\mathcal{O}_X(U) \to \mathcal{O}_Y(V)$ is injective and finite.

(3) Let $f : X \to Y$ be a dominant morphism of irreducible quasi-projective varieties. Show that there is a $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$. [*Hint. Reduce to the situation where Y is affine and apply Noether's normalisation lemma to show that you may assume wlog that $Y = k^n$ for some n. Now use the existence of transcendence bases and (2) to show that there is an open subvariety $U \subseteq X$ and an open subvariety W of $k^{\dim(X)-\dim(Y)} \times k^n$ such that $f|_U$ factors as a finite and surjective morphism $U \to W$, followed by the projection to k^n. Now deduce the result from (1) and a computation of the dimension of the fibres of the projection $k^{\dim(X)-\dim(Y)} \times k^n \to k^n$.*]

(4) Deduce that in the situation of (3), the set of $y \in Y$ such that we have $\dim(f^{-1}(\{y\})) \geqslant \dim(X) - \dim(Y)$ is dense in Y.

Solution. (1) Let $\{X_i\}$ be the irreducible components of X. Then $f(X_i)$ is irreducible for all i and hence the closure $\overline{f(X_i)}$ is also irreducible for all i (by question 2.5 (1)). Hence we must have $\cup_i \overline{f(X_i)} = Y$, otherwise f is not dominant. Now if $\overline{f(X_i)} \neq Y$ for all i then Y is not irreducible, which is impossible. So there is an index i_0 such that $\overline{f(X_{i_0})} = Y$. In that case we have a field extension $\kappa(X_{i_0})|\kappa(Y)$ and thus $\dim(X_{i_0}) \geqslant \dim(Y)$ by Proposition 9.2. In now follows from the definition of dimension that $\dim(X) \geqslant \dim(Y)$.

(2) We first prove the following statement of commutative algebra. Let $\phi : A \to B$ be a homomorphism of finitely generated integral k-algebras. Suppose that $\mathrm{Spm}(\phi)(\mathrm{Spm}(B))$ is dense in $\mathrm{Spm}(A)$ and suppose that the induced map $\mathrm{Frac}(\phi) : \mathrm{Frac}(A) \to \mathrm{Frac}(B)$ is an algebraic extension of fields. Then there is an element $f \in A$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective and finite.

To prove this assertion, note that by question 1.5 we already know that under the given assumptions, ϕ must be injective. Note also that since we have a commutative diagram

$$\begin{tikzcd} \mathrm{Frac}(A) \arrow[r, "\mathrm{Frac}(\phi)"] & \mathrm{Frac}(B) \\ A \arrow[u] \arrow[r, "\phi"] & B \arrow[u] \end{tikzcd}$$

all whose maps are injective, the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective for any choice of $f \in A\backslash\{0\}$ (remember that A and B are integral domains). Thus we only have to show that there is $f \in A\backslash\{0\}$ such that the induced map $A[f^{-1}] \to B[\phi(f)^{-1}]$ is finite. Now let $b_1, \dots, b_l$ be generators of B as a k-algebra. By assumption, each $b_i/1 \in \mathrm{Frac}(B)$ satisfies a monic polynomial equation with coefficients in $\mathrm{Frac}(A)$. Let $f \in A$ be the product of the denominators of all the coefficients of all these equations. Note that $B[\phi(f)^{-1}]$ is generated as a k-algebra by $1/\phi(f)$ and by the elements $b_i/1$ (use Lemma 5.3 in CA). In particular, $B[\phi(f)^{-1}]$ is generated by the $b_i/1$ as a $A[f^{-1}]$-algebra. On the other hand, by construction, the elements $b_i/1$ all satisfy integral equations over $A[f^{-1}]$. Hence $A[f^{-1}] \to B[\phi(f)^{-1}]$ is a finite map of rings (see section 8 in CA).

Note that the fact that $A[f^{-1}] \to B[\phi(f)^{-1}]$ is injective and finite implies that the induced map

$$\mathrm{Spm}(B[\phi(f)^{-1}]) \to \mathrm{Spm}(A[f^{-1}])$$

is surjective (use Th. 8.8 and Cor. 8.10 in CA).

Returning to the problem at hand, note that we may wlog assume that X and Y are affine (take an affine open Y' in Y and an affine open X' in $f^{-1}(Y')$ and replace X by X' (resp. Y by Y'). Applying the result of commutative algebra that we just proved to $A = \mathcal{O}_X(X)$ and $B = \mathcal{O}_Y(Y)$ we obtain the desired result.

(3) Note that Th. 9.1 (Noether's normalisation lemma), Prop. 8.12, Th. 8.8 and Cor. 8.10 in CA imply that for some $n \geqslant 0$ there is a surjective morphism $h : Y \to k^{\dim(Y)}$, such that the fibre $h^{-1}(\bar{v})$ of h over $\bar{v}$ is finite for all $\bar{v} \in k^n$. Since the fibres of the composed morphism $h \circ f$ are finite disjoint unions of fibres of f, we may thus replace f by $h \circ f$ and suppose that $Y = k^n$ for some $n \geqslant 0$.

Now consider the field extension $\kappa(X)|\kappa(Y)$. Choose a transcendence basis $b_1, \dots, b_\delta \in \kappa(X)$ of $\kappa(X)$ over $\kappa(Y)$. Write $\kappa(Y) = \kappa(k^n) = k(x_1, \dots, x_n)$. The set $x_1, \dots, x_n, b_1, \dots, b_\delta$ is then by construction a transcendence basis for $\kappa(X)$ over k. Since we know that $\dim(k^n) = n$ (see Theorem 8.4), we deduce from Proposition 9.2 that $\delta = \dim(X) - \dim(Y)$. Now the subfield $\kappa(Y)(b_1, \dots, b_\delta)$ of $\kappa(X)$ is isomorphic as a k-algebra to $k(x_1, \dots, x_n, y_1, \dots, y_\delta)$, which is the function field of $k^{n+\delta}$. The inclusion $k(x_1, \dots, x_n) \hookrightarrow k(x_1, \dots, x_n, y_1, \dots, y_\delta)$ is induced by the natural projection morphism $\pi : k^{n+\delta} \to k^n$ (unroll the definitions). Hence we have a rational dominant map $a : X \to k^{n+\delta}$ such that the rational dominant map associated with the morphism $f : X \to Y$ is the composition of a with the rational dominant map associated with π (apply Proposition 9.4 and question 3.4). Applying (2) we obtain open affine subvarietes $U \subseteq X$ and $W \subseteq k^{n+\delta}$ and a surjective morphism $g : U \to W$, which represents a. Let now now $f' = \pi \circ g$.

Note that by question 3.4 again, we have $f' = f|_U$. Let $y \in \pi(W) = f'(U) = f(U)$. We compute

$$\begin{aligned} & \dim(f^{-1}(y)) \geqslant \dim(f^{-1}(y) \cap U) = \dim((f')^{-1}(y)) \\ = \quad & \dim(g^{-1}(\pi^{-1}(y) \cap W)) \geqslant \dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta = \dim(X) - \dim(Y) \end{aligned}$$

Here we used question 2.7 for the first inequality and we used (1) for the inequality

$$\dim(g^{-1}(\pi^{-1}(y) \cap W) \geqslant \dim(\pi^{-1}(y) \cap W)$$

(remember that g is surjective). To justify the equality

$$\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y)) = \delta$$

note that $\pi^{-1}(y) \simeq k^\delta$. We thus have $\dim(\pi^{-1}(y) \cap W) = \dim(\pi^{-1}(y))$ by Proposition 9.2 and we have $\dim(\pi^{-1}(y)) = \delta$ by Theorem 8.4.

(4) Let $U \subseteq Y$ be an open subvariety. Applying (3) to the morphism $f^{-1}(U) \to U$, we see that there is a point $y \in U$ such that $\dim(f^{-1}(y)) \geqslant \dim(f^{-1}(U)) - \dim(U) = \dim(X) - \dim(Y)$. Since U was arbitrary, this shows what we want.

Question 4.7. (1) Show that all the morphisms from $\mathbb{P}^2(k)$ to $\mathbb{P}^1(k)$ are constant. *[Hint: Use question 4.6 and the projective dimension theorem.]*

(2) Using (1) or using another method, show that the morphisms from $\mathbb{P}^n(k)$ to $\mathbb{P}^1(k)$ are constant.

Solution. (1) Let $f : \mathbb{P}^2(k) \to \mathbb{P}^1(k)$ is a morphism. Suppose for contradiction that f is not constant. By Corollary 12.10, the image $f(\mathbb{P}^2(k))$ is closed, and it is also irreducible, since $\mathbb{P}^2(k)$ is irreducible. Hence $f(\mathbb{P}^2(k)) = \mathbb{P}^1(k)$ (because $\dim(\mathbb{P}^1(k)) = 1$). Now let $y_1, y_2 \in \mathbb{P}^1(k)$ be such that $y_1 \neq y_2$ and $\dim(f^{-1}(y_1)), \dim(f^{-1}(y_1)) \geqslant \dim(\mathbb{P}^2(k)) - \dim(\mathbb{P}^1(k)) = 1$. This exists by question 4.6. Since $\dim(\mathbb{P}^2(k)) = 2$ we then actually have $\dim(f^{-1}(y_1)) = \dim(f^{-1}(y_1)) = 1$. Let C_1 (resp. C_2) be an irreducible component of $\dim(f^{-1}(y_1))$ (resp. $\dim(f^{-1}(y_2))$) such that $\dim(C_1) = \dim(C_2) = 1$. We have $\dim(C_1) + \dim(C_2) - 2 = 0$ and so by Proposition 11.2 we have $C_1 \cap C_2 \neq \emptyset$. This is a contradiction.

(2) Let $n \geqslant 2$. First note that $\mathbb{P}^2(k)$ is isomorphic to the closed subvariety $\mathrm{Z}(x_3, x_4, \ldots, x_n)$ of $\mathbb{P}^n(k)$. To see this note that the image of the morphism $\iota : \mathbb{P}^2(k) \to \mathbb{P}^n(k)$ given by the formula

$$[X_0, X_1, X_2] \mapsto [X_0, X_1, X_2, 0 \ldots ((n-2)\text{-times}) \ldots, 0]$$

is $\mathrm{Z}(x_3, x_4, \ldots, x_n)$. This morphism is an isomorphism onto $\mathrm{Z}(x_3, x_4, \ldots, x_n)$ because the morphism

$$\mathbb{P}^n(k) \backslash \mathrm{Z}(x_0, x_1, x_2) \to \mathbb{P}^2(k)$$

given by the formula

$$[X_0, X_1, X_2, \ldots, X_n] \mapsto [X_0, X_1, X_2]$$

gives an inverse to ι when restricted to $\mathrm{Z}(x_3, x_4, \ldots, x_n)$.

Let now $f : \mathbb{P}^n(k) \to \mathbb{P}^1(k)$ be a morphism. Suppose for contradiction that f is not constant. Let $\bar{v}_1, \bar{v}_2 \in \mathbb{P}^n(k)$ be two points such that $f(\bar{v}_1) \neq f(\bar{v}_2)$. Let M be an invertible $(n+1) \times (n+1)$-matrix such that $M([1, 0, 0, \ldots, 0]) = \bar{v}_1$ and $M([0, 1, 0, 0, \ldots, 0]) = \bar{v}_2$. Let $\phi_M : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$ be the automorphism defined by M (see question 2.8). The morphism $f \circ \phi_M \circ \iota : \mathbb{P}^2(k) \to \mathbb{P}^1(k)$ is then not constant, which is a contradiction by (1).