B4.3 Distribution Theory

Sheet 2 — MT23

Differentiation, homogeneous distributions and example of distribution defined by principal value integral

Only work on the questions from Section B should be handed in.

Section A

1. Let f, $g \in \mathrm{C}^1(\mathbb{R})$ and define

$$u(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geq 0. \end{cases}$$

Explain why

$$\langle T_u, \phi \rangle = \int_{\mathbb{R}} u(x)\phi(x)\,\mathrm{d}x, \quad \phi \in \mathscr{D}'(\mathbb{R}),$$

is a distribution on $\mathbb{R}$ and calculate the distributional derivative T_u'. What can you say about T_v corresponding to the function

$$v(x) = \begin{cases} f(x) & \text{if } x < 0 \\ a & \text{if } x = 0 \\ g(x) & \text{if } x > 0, \end{cases}$$

where $a \in \mathbb{C}$ is a constant that is different from both $f(0)$ and $g(0)$?

Solution: Note that u is piecewise C^1 (meaning that exists an at most finite set of points $F \subset \mathbb{R}$, such that u is C^1 on $\mathbb{R} \setminus F$ and the one-sided limits $u(x_0+)$, $u(x_0-)$, $u'(x_0+)$, $u'(x_0-)$ exist in $\mathbb{C}$ for each $x_0 \in F$). In our case we of course have that $F = \{0\}$ and that $u(0+) = g(0)$, $u(0-) = f(0)$, $u'(0+) = g'(0)$, $u'(0-) = f'(0)$. In particular, u is therefore locally integrable and so T_u defines a distribution on $\mathbb{R}$. By definition of distributional derivative we find using integration by parts (that we may use on each of the intervals $(-\infty, 0)$ and $(0, \infty)$ because u is piecewise C^1):

$$\begin{aligned}
\langle T_u', \phi \rangle &= -\langle T_u, \phi' \rangle = -\int_{\mathbb{R}} u(x)\phi'(x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{0} f(x)\phi'(x)\,\mathrm{d}x - \int_{0}^{\infty} g(x)\phi'(x)\,\mathrm{d}x \\
&= -\Big[f(x)\phi(x)\Big]_{x\to-\infty}^{x=0} + \int_{-\infty}^{0} f'(x)\phi(x)\,\mathrm{d}x - \Big[g(x)\phi(x)\Big]_{x=0}^{x\to\infty} + \int_{0}^{\infty} g'(x)\phi(x)\,\mathrm{d}x \\
&= -f(0)\phi(0) + g(0)\phi(0) + \int_{\mathbb{R}} \big(f'\mathbf{1}_{(-\infty,0)} + g'\mathbf{1}_{(0,\infty)}\big)\phi\,\mathrm{d}x.
\end{aligned}$$

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Therefore $T_u' = \big(g(0) - f(0)\big)\delta_0 + T_{f'\mathbf{1}_{(-\infty,0)} + g'\mathbf{1}_{(0,\infty)}}$.

Since $u(x) = v(x)$ for all $x \in \mathbb{R} \setminus \{0\}$ we have in particular that they agree almost everywhere, so the corresponding distributions are the same: $T_u = T_v$. Consequently also $T_v' = T_u'$.

Comment on notation. If you have seen the fundamental lemma of the calculus of variations (covered in lectures during Week 3) you will know that a locally integrable function w is uniquely determined as an $\mathrm{L}^1_{\mathrm{loc}}$ function (so almost everywhere) by the corresponding distribution T_w, and that we therefore may directly identify w and T_w and simply write w also for the corresponding distribution. This is often convenient and we employ this convention in question 2.

2. (a) Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise continuous[1] and $k \in \mathbb{R}$, then the function $u(x,t) = f(x - kt)$, $(x,t) \in \mathbb{R}^2$, is locally integrable on $\mathbb{R}^2$. Conclude that it defines a distribution and show that it satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions on $\mathbb{R}^2$.

(b) Prove that $u(x,y) = \log\big(x^2 + y^2\big)$ is locally integrable on $\mathbb{R}^2$, and that we have

$$\Delta u = 4\pi\delta_0$$

in the sense of distributions on $\mathbb{R}^2$, where δ_0 is the Dirac delta function on $\mathbb{R}^2$ concentrated at the origin.

Solution: See Section 2.5 in *Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Scientific.*

[1] It means that there is a finite subset F of $\mathbb{R}$ such that f is continuous at each point of $\mathbb{R} \setminus F$ and that if $x_0 \in F$, then both the one-sided limits $f(x_0+)$ and $f(x_0-)$ of f at x_0 exist in $\mathbb{R}$.

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Section B

3. Let $a > 0$. For each $\phi \in \mathscr{D}(\mathbb{R})$ we let

$$\langle T_a, \phi \rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\phi(x)}{|x|} \,\mathrm{d}x + \int_{-a}^{a} \frac{\phi(x) - \phi(0)}{|x|} \,\mathrm{d}x.$$

   (a) Show that T_a hereby is well-defined and that it is a distribution on $\mathbb{R}$.

   (b) Now assume that $\phi \in \mathscr{D}(\mathbb{R})$ satisfies $\phi(0) = 0$. Show that then

$$\langle T_a, \phi \rangle = \int_{-\infty}^{\infty} \frac{\phi(x)}{|x|} \,\mathrm{d}x.$$

   What distribution is $T_a - T_b$ for $0 < b < a$?

Solution: (a) We first show that T_a is well-defined for all $\phi \in \mathscr{D}(\mathbb{R})$: Clearly, the first two integrals appearing in the definition are well-defined. For the integral over $[-a, a]$ we let

$$\Phi(x) = \begin{cases} \frac{\phi(x)-\phi(0)}{|x|} & \text{if } x \neq 0, \\ \phi'(0) & \text{if } x = 0. \end{cases}$$

Then Φ is piecewise continuous (and continuous from the right at 0), hence is (Riemann-)integrable over $[-a, a]$. Thus the expression $\langle T_a, \phi \rangle$ is well-defined as a complex number. Linearity of $T_a \colon \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ is now a consequence of linearity of the integral. To show that T_a is a distribution we must show that it is $\mathscr{D}$ continuous and we do that by establishing the boundedness property. Fix a compact set K in $\mathbb{R}$ and take $\ell > a$ such that $K \subset [-\ell, \ell]$. Then for $\phi \in \mathscr{D}(\mathbb{R})$ with support in K we estimate as follows:

$$\begin{aligned} \left| \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\phi(x)}{|x|} \,\mathrm{d}x \right| &\leq 2 \int_a^{\ell} \frac{\mathrm{d}x}{x} \sup |\phi| \\ &= 2 \log \frac{\ell}{a} \sup |\phi|, \end{aligned}$$

$$\begin{aligned} \left| \int_{-a}^{a} \frac{\phi(x) - \phi(0)}{|x|} \,\mathrm{d}x \right| &\overset{FTC}{=} \left| \int_{-a}^{a} \int_0^1 \phi'(tx) \,\mathrm{d}t \frac{x}{|x|} \,\mathrm{d}x \right| \\ &\leq \int_{-a}^{a} \int_0^1 \left| \phi'(tx) \right| \mathrm{d}t \,\mathrm{d}x \\ &\leq 2a \sup |\phi'|. \end{aligned}$$

Consequently we have shown that $\left| \langle T_a, \phi \rangle \right| \leq c \big(\sup |\phi| + \sup |\phi'| \big)$, where $c = 2 \max(a, \log \frac{\ell}{a})$, and since K was an arbitrary compact set in $\mathbb{R}$ this proves that T_a is a distribution on $\mathbb{R}$. Note also that the bound shows that T_a has order at most 1.

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(b) Let $\phi \in \mathscr{D}(\mathbb{R})$ and assume $\phi(0) = 0$. In this case we note that $\phi(x)/|x|$ is integrable on $\mathbb{R}$ (because it is $\Phi(x)$ for $x \neq 0$ and it vanishes outside the support of ϕ), hence

$$\langle T_a, \phi \rangle = \int_{\mathbb{R}} \frac{\phi(x)}{|x|}\,\mathrm{d}x.$$

If $0 < b < a$, then for $\phi \in \mathscr{D}(\mathbb{R})$ we calculate:

$$\begin{aligned}
\langle T_a - T_b, \phi \rangle &= \left[\left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) - \left(\int_{-\infty}^{-b} + \int_b^{\infty} \right) \right] \frac{\phi(x)}{|x|}\,\mathrm{d}x \\
&\quad + \left[\int_{-a}^{a} - \int_{-b}^{b} \right] \frac{\phi(x) - \phi(0)}{|x|}\,\mathrm{d}x \\
&= -\left(\int_b^a + \int_{-a}^{-b} \right) \frac{\phi(x)}{|x|}\,\mathrm{d}x + \left(\int_b^a + \int_{-a}^{-b} \right) \frac{\phi(x) - \phi(0)}{|x|}\,\mathrm{d}x \\
&= -\left(\int_b^a + \int_{-a}^{-b} \right) \frac{\phi(0)}{|x|}\,\mathrm{d}x \\
&= -2\phi(0) \log \frac{a}{b}.
\end{aligned}$$

Consequently, $T_a - T_b = -2 \log \frac{a}{b}\, \delta_0$.

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4. (Homogeneous distributions)

   (a) Let $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > -n$ and denote by $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ the usual euclidean norm on $\mathbb{R}^n$. Define

   $$\langle T_\alpha, \phi \rangle = \int_{\mathbb{R}^n} |x|^\alpha \phi(x) \, \mathrm{d}x, \quad \phi \in \mathscr{D}(\mathbb{R}^n).$$

   Show that T_α is a regular distribution on $\mathbb{R}^n$. (*Hint: Use polar coordinates. If you prefer, then it is ok to only do the calculation for $n = 1$ and $n = 2$.*)

   (b) For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

   $$(d_r\varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n.$$

   Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$.

   (c) A distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is said to be homogeneous of degree $\beta \in \mathbb{C}$ (or briefly: β-homogeneous) provided

   $$d_r u = r^\beta u$$

   holds for all $r > 0$. (*Note: $r^\beta \equiv \mathrm{e}^{\beta \log r}$ for $r > 0$.*)

   (i) Suppose $u \in \mathscr{D}'(\mathbb{R}^n)$ is β-homogeneous. Prove that for each $j \in \{1, \ldots, n\}$ the distributions $x_j u$ and $\partial_j u$ are $(\beta+1)$-homogeneous and $(\beta-1)$-homogeneous, respectively.

   (ii) Show that the distribution T_α defined in (a) is homogeneous of degree α.

   (iii) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.

   (d) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is β-homogeneous, then

   $$\sum_{j=1}^n x_j \partial_j u = \beta u$$

   holds in the sense of distributions on $\mathbb{R}^n$. This PDE is known as Euler's relation for β-homogeneous distributions.

Solution: (a) The function $|x|^\alpha$ is continuous on $\mathbb{R}^n \setminus \{0\}$, and if we define $u_\alpha(x) = |x|^\alpha$ for $x \neq 0$ and $u_\alpha(0) = 0$, then u_α is measurable. Since $|u_\alpha(x)| = |x|^{\mathrm{Re}(\alpha)}$ for $x \neq 0$ it is clear that u_α is locally integrable on $\mathbb{R}^n \setminus \{0\}$ regardless of what α is. Integrating the nonnegative function $|u_\alpha|$ over $B_1(0)$ we find using polar coordinates and that $\mathrm{Re}(\alpha) > -n$:

$$\begin{aligned}
\int_{B_1(0)} |u_\alpha(x)| \, \mathrm{d}x &= \int_0^1 \int_{|x|=r} |x|^{\mathrm{Re}(\alpha)} \, \mathrm{d}S_x \, \mathrm{d}r \\
&= \int_0^1 r^\alpha \omega_n r^{n-1} \, \mathrm{d}r \\
&= \frac{\omega_n}{n + \mathrm{Re}(\alpha)} < +\infty
\end{aligned}$$

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Thus $u_\alpha \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ and therefore T_α is a regular distribution.

(b) First we derive an adjoint identity. For ϕ, $\psi \in \mathscr{D}(\mathbb{R}^n)$ we have by the change of variables $y = rx$:

$$\int_{\mathbb{R}^n} d_r\phi\psi \,\mathrm{d}x = \int_{\mathbb{R}^n} \phi(y)\psi\left(\frac{y}{r}\right)\frac{1}{r^n}\,\mathrm{d}y = \int_{\mathbb{R}^n} \phi\psi_r \,\mathrm{d}x,$$

where we wrote ψ_r for the L^1-dilation of ψ by factor $r > 0$. Here we note that the operation of taking L^1-dilation by factor $r > 0$, so $r^{-n}d_{r^{-1}} \colon \mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)$, is a linear and $\mathscr{D}$ continuous map. Using the adjoint identity scheme we may then extend d_r to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$ by the rule

$$\langle d_r u, \phi\rangle = \langle u, \phi_r\rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^n).$$

(c) (i) Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $r > 0$. Note first that $x_j\phi_r = r\big(x_j\phi\big)_r$, and so using that u is β-homogeneous we get

$$\langle d_r\big(x_j u\big), \phi\rangle = \langle u, x_j\phi_r\rangle = r\langle d_r u, x_j\phi\rangle = r^{\beta+1}\langle u, \phi\rangle$$

proving that $d_r\big(x_j u\big) = r^{\beta+1}u$ as required. Next, note that $\partial_j\phi_r = r^{-1}\big(\partial_j\phi\big)_r$, so again using β-homogeneity of u we find

$$\langle d_r\big(\partial_j u\big), \phi\rangle = -\langle u, \partial_j\phi_r\rangle = -r^{-1}\langle d_r u, \partial_j\phi\rangle = r^{\beta-1}\langle u, \phi\rangle$$

proving that $d_r\big(\partial_j u\big) = r^{\beta-1}u$, as required.

(ii) For $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $r > 0$ we calculate:

$$\begin{aligned} \langle d_r T_\alpha, \phi\rangle &= \langle T_\alpha, \phi_r\rangle = \int_{\mathbb{R}^n} |x|^\alpha \phi_r(x)\,\mathrm{d}x \\ &\overset{y=x/r}{=} \int_{\mathbb{R}^n} |ry|^\alpha\phi(y)\,\mathrm{d}y = r^\alpha\int_{\mathbb{R}^n}|y|^\alpha\phi(y)\,\mathrm{d}y \\ &= r^\alpha\langle T_\alpha, \phi\rangle \end{aligned}$$

and so $d_r T_\alpha = r^\alpha T_\alpha$, as required.

(iii) For $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $r > 0$ we calculate: $\langle d_r\delta_0, \phi\rangle = \langle \delta_0, \phi_r\rangle = \frac{1}{r^n}\phi\big(\frac{0}{r}\big) = \frac{1}{r^n}\phi(0)$, so $d_r\delta_0 = r^{-n}\delta_0$, as required.

(d) If u had been a differentiable function we would obtain the Euler relation by differentiating the β-homogeneity condition. We do the same in the more general case considered here. Note that for $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have $\langle u, \phi_r\rangle = r^\beta\langle u, \phi\rangle$ for all $r > 0$. Clearly

$$\left.\frac{\mathrm{d}}{\mathrm{d}r}\right|_{r=1}\langle u, \phi_r\rangle = \beta\langle u, \phi\rangle$$

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and we would like to differentiate behind the distribution sign on the left-hand side. In order to do that we consider for $r \in (0,1)$ the difference-quotient:

$$\frac{\langle d_r u, \phi\rangle - \langle u, \phi\rangle}{r-1} = \left\langle u, \frac{\phi_r - \phi}{r-1}\right\rangle.$$

Note that for each $x \in \mathbb{R}^n$ and $t > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) = \frac{1}{t}\psi_t(x), \qquad \text{where } \psi(x) = -n\phi(x) - \nabla\phi(x)\cdot x,$$

and hence, as $t \to 1$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(x) \to \psi(x) \quad \text{uniformly in } x \in \mathbb{R}^n.$$

Therefore we get by FTC:

$$\Delta_r(x) \equiv \frac{\phi_r(x) - \phi(x)}{r-1} = \frac{1}{r-1}\int_1^r \frac{1}{t}\psi_t(x)\,\mathrm{d}t \to \psi(x)$$

uniformly in $x \in \mathbb{R}^n$ as $r \nearrow 1$. In order to exploit that u is a distribution we must improve this convergence to $\mathscr{D}$ convergence. To that end note that $\Delta_r \in \mathscr{D}(\mathbb{R}^n)$ and taking $R > 0$ so large that $\mathrm{supp}(\phi) \subset B_R(0)$ also $\mathrm{supp}(\Delta_r) \subset B_R(0)$ for all $r < 1$. Fix a multi-index $\alpha \in \mathbb{N}_0^n$. Then

$$\big(\partial^\alpha \Delta_r\big)(x) = \frac{1}{r-1}\int_1^r \frac{1}{t^{|\alpha|+1}}\big(\partial^\alpha\psi\big)_r(x)\,\mathrm{d}t \to \big(\partial^\alpha\psi\big)(x)$$

uniformly in $x \in \mathbb{R}^n$ as $r \nearrow 1$. Consequently, $\Delta_r \to \psi$ in $\mathscr{D}(\mathbb{R}^n)$ as $r \nearrow 1$ and so by $\mathscr{D}$ continuity of u, $\langle u, \psi\rangle = \beta\langle u, \phi\rangle$. Recalling the definition of ψ, using the definitions and the arbitrariness of the test function ϕ we are done.

5. Show that δ_a, the Dirac delta function concentrated at $a \in \mathbb{R}$, satisfies the equation

$$(x-a)u = 0. \tag{1}$$

Find the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (1).

(*Hint: See Corollary 1.11 in the Lecture Notes.*)

Solution: δ_a is a solution because we for $\phi \in \mathscr{D}(\mathbb{R})$ have $\langle (x-a)\delta_a, \phi\rangle = \langle \delta_a, (x-a)\phi\rangle = 0$. It follows that $c\delta_a$ is a solution for every $c \in \mathbb{C}$. To see that this is the general solution we assume that u is a solution. In order to show that u has the above form we construct suitable test functions. Fix $\eta \in \mathscr{D}(\mathbb{R})$ with $\eta(a) = 1$. Then for any $\phi \in \mathscr{D}(\mathbb{R})$ we get by FTC that $\phi(x) - \phi(a)\eta(x) = \psi(x)(x-a)$, where

$$\psi(x) = \int_0^1 \big(\phi'(a+t(x-a)) - \phi(0)\eta'(a+t(x-a))\big)\,\mathrm{d}t,\ x \in \mathbb{R}.$$

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It is not difficult to see that $\psi \in \mathrm{C}^\infty(\mathbb{R})$. Take $R > 0$ so large that $(-R, R)$ contains the supports of both ϕ and η. Then also ψ is supported in $(-R, R)$ and so $\psi \in \mathscr{D}(\mathbb{R})$. Now note that $\phi = \phi(a)\eta + (x-a)\psi$, so $\langle u, \phi \rangle = \langle u, \phi(a)\eta + (x-a)\psi \rangle = \phi(a)\langle u, \eta \rangle + \langle (x-a)u, \psi \rangle = \langle c\delta_a, \phi \rangle$, where $c = \langle u, \eta \rangle$ is a constant, as required.

6. (Distribution defined by principal value integral)

   Define for each $\phi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle = \lim_{a \searrow 0} \left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\phi(x)}{x} \, \mathrm{d}x.$$

   (a) Show that hereby $\mathrm{pv}\big(\frac{1}{x}\big) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (see Problem 4). Check that
$$\frac{\mathrm{d}}{\mathrm{d}x} \log|x| = \mathrm{pv}\big(\frac{1}{x}\big).$$

   (b) What is the order of $\mathrm{pv}\big(\frac{1}{x}\big)$?

   (c) Show that $u = \mathrm{pv}\big(\frac{1}{x}\big)$ solves the equation
$$xu = 1 \tag{2}$$
   in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (2)?

Solution: (a) Let $\phi \in \mathscr{D}(\mathbb{R})$ and take $A > 0$ so large that $\mathrm{supp}(\phi) \subset (-A, A)$. For each $a \in (0, A)$ we have because the function $x \mapsto \phi(0)/x$ is odd that

$$\left(\int_{-\infty}^{-a} + \int_a^\infty \right) \frac{\phi(x)}{x} \, \mathrm{d}x = \left(\int_{-A}^{-a} + \int_a^A \right) \frac{\phi(x) - \phi(0)}{x} \, \mathrm{d}x.$$

Here the function
$$\Phi(x) = \begin{cases} \frac{\phi(x) - \phi(0)}{x} & \text{if } 0 < |x| \leq A, \\ \phi'(0) & \text{if } x = 0, \end{cases}$$
is continuous, so the improper integral defining $\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \rangle$ is well-defined. It is then clear that $\mathrm{pv}\big(\frac{1}{x}\big) \colon \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ is well-defined and linear. We also record that for $\phi \in \mathscr{D}(\mathbb{R})$ with support in $(-A, A)$ we have

$$\left| \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle \right| \leq 2A \max|\phi'|$$

so that $\mathrm{pv}\big(\frac{1}{x}\big)$ in particular has the boundedness property and hence is a distribution.

For $r > 0$ and $\phi \in \mathscr{D}(\mathbb{R})$ we calculate:

$$\begin{aligned}
\left\langle d_r \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle &= \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \frac{1}{r} d_{\frac{1}{r}} \phi \right\rangle \\
&= \lim_{a \searrow 0} \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\phi(r^{-1}x)}{rx} \, \mathrm{d}x \\
&= \lim_{a \searrow 0} \left(\int_{-\infty}^{-a/r} + \int_{a/r}^{\infty} \right) \frac{\phi(y)}{ry} \, \mathrm{d}y \\
&= r^{-1} \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle .
\end{aligned}$$

This shows that the distribution $\mathrm{pv}\big(\frac{1}{x}\big)$ is homogeneous of degree -1, as required.

Finally we note that $\log|\cdot| \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ so that we may consider it as a distribution. For $\phi \in \mathscr{D}(\mathbb{R})$ we calculate:

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x} \log|x|, \phi \right\rangle &= \langle \log|x|, -\phi' \rangle \\
&= -\int_{-\infty}^{\infty} \log|x| \phi'(x) \, \mathrm{d}x \\
&= -\lim_{\varepsilon \searrow 0} \left(\int_{-1/\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{1/\varepsilon} \right) \log|x| \phi'(x) \, \mathrm{d}x \\
&\overset{parts}{=} \lim_{\varepsilon \searrow 0} \left(\int_{-1/\varepsilon}^{-\varepsilon} + \int_{\varepsilon}^{1/\varepsilon} \right) \frac{\phi(x) - \phi(0)}{x} \, \mathrm{d}x \\
&= \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle ,
\end{aligned}$$

as required.

(b) It follows from (a) that the order is at most 1. To see that it is 1 we assume for a contradiction that it is 0. Then we find a constant $c \geq 0$ such that

$$\left| \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi \right\rangle \right| \leq c \max|\phi|$$

holds for all $\phi \in \mathscr{D}(\mathbb{R})$ supported in $[-1, 1]$. For the standard mollifier $\big(\rho_\varepsilon\big)_{\varepsilon > 0}$ on $\mathbb{R}$ put for each $j \in \mathbb{N} \setminus \{1, 2\}$,

$$\phi_j = \rho_{1/j} * \mathbf{1}_{(1/j, 1/2)}.$$

Then $\phi_j \in \mathscr{D}(\mathbb{R})$ has support $[0, \frac{1}{2} + \frac{1}{j}]$ and $\max|\phi_j| = 1$, so $\big|\big\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi_j \big\rangle\big| \leq c$ holds for all large $j \in \mathbb{N}$. But ϕ_j is also nonnegative and equals 1 on $[\frac{2}{j}, \frac{1}{2} - \frac{1}{j}]$, so we can estimate

$$\left| \left\langle \mathrm{pv}\big(\frac{1}{x}\big), \phi_j \right\rangle \right| \geq \int_{\frac{2}{j}}^{\frac{1}{2} - \frac{1}{j}} \frac{\mathrm{d}x}{x} = \log\big(\frac{1}{4} j - \frac{1}{2}\big)$$

and this is clearly impossible for large $j \in \mathbb{N}$. The order is therefore 1.

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(c) For $\phi \in \mathscr{D}(\mathbb{R})$ we calculate,

$$\begin{aligned}\left\langle x\mathrm{pv}\left(\frac{1}{x}\right), \phi\right\rangle &= \left\langle \mathrm{pv}\left(\frac{1}{x}\right), x\phi\right\rangle \\ &= \lim_{a\searrow 0}\left(\int_{-\infty}^{-a} + \int_{a}^{\infty}\right)\frac{x\phi(x)}{x}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x,\end{aligned}$$

hence $x\mathrm{pv}\left(\frac{1}{x}\right) = 1$, as required. Because the equation (1) is linear we get the general solution by use of question 5: $u = \mathrm{pv}\left(\frac{1}{x}\right) + c\delta_0$, where $c \in \mathbb{C}$ is arbitrary.

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Section C

7. (A fundamental solution for the differential operator $\mathrm{d}^{m+1}/\mathrm{d}x^{m+1}$)

   Denote $x_+ = \max\{0, x\}$ for $x \in \mathbb{R}$ and fix $m \in \mathbb{N}_0$. Show that if

$$E_m(x) = \frac{x_+^m}{m!},$$

   where for $m = 0$ we interpret this as the Heaviside function, then

$$\frac{\mathrm{d}^{m+1}}{\mathrm{d}x^{m+1}} E_m = \delta_0 \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

   Solution: It is clear that E_m is a piecewise continuous function and therefore that it in particular represents a regular distribution. Note that $E_0' = \delta_0$ and that by Question 1 (since E_1 is piecewise C^1), $E_1' = E_0$ so $E_1'' = \delta_0$. Now assume that $E_m^{(m+1)} = \delta_0$ for some $m \in \mathbb{N}_0$. Note that $E_{m+1} = \frac{x_+}{m+1} E_m = \frac{x}{m+1} E_m$, so that by Leibniz' rule

$$\begin{aligned} E_{m+1}^{(m+2)} &= \frac{\mathrm{d}^{m+2}}{\mathrm{d}x^{m+2}} \left(\frac{x}{m+1} E_m \right) \\ &= \frac{1}{m+1} \sum_{j=0}^{m+2} \binom{m+2}{j} x^{(j)} E_m^{(m+2-j)} \\ &= \frac{1}{m+1} \big(x E_m^{(m+2)} + (m+2) E_m^{(m+1)} \big) \\ &= \frac{1}{m+1} \big(x\delta_0' + (m+2)\delta_0 \big). \end{aligned}$$

   Because $x\delta_0' = -\delta_0$ it follows that $E_{m+1}^{(m+2)} = \delta_0$, and so we may conclude by induction on m.

8. (Continuation of Question 4 about homogeneous distributions) Let $\beta \in \mathbb{C}$ and assume that $u \in \mathscr{D}'(\mathbb{R}^n)$ satisfies

$$\sum_{j=1}^{n} x_j \partial_j u = \beta u$$

   in the sense of distributions on $\mathbb{R}^n$. Prove that u is homogeneous of degree β.

   Solution: Euler's relation can be rewritten as $x \cdot \nabla u = \beta u$, that is,

$$\big\langle u, (n+\beta)\psi + x \cdot \nabla \psi \big\rangle = 0$$

   holds for all $\psi \in \mathscr{D}(\mathbb{R}^n)$. In order to show that $d_r u = r^\beta u$ holds for all $r > 0$ we show that, for each $\phi \in \mathscr{D}(\mathbb{R}^n)$, the function $h(r) = \big\langle u, r^{-\beta}\phi_r \big\rangle$, $r > 0$, is constant. Note that

$$\partial_r \Big(r^{-\beta}\phi_r(x) \Big) = -r^{-\beta-1} \Big((\beta+n)\phi_r(x) + x \cdot \nabla \phi_r(x) \Big),$$

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and therefore applying Euler's relation above with $\psi = \phi_r$ we are done provided we can show that h is differentiable and that its derivative can be calculated by differentiation behind the u-sign. Fix $r > 0$ and consider for $s > 0$ distinct from r the difference-quotient

$$\frac{h(s) - h(r)}{s - r} = \left\langle u, \frac{s^{-\beta}\phi_s - r^{-\beta}\phi_r}{s - r} \right\rangle = \langle u, \Delta_s \rangle,$$

say. It is not difficult to see that $\Delta_s(x) \to \partial_r\bigl(r^{-\beta}\phi_r(x)\bigr)$ uniformly in $x \in \mathbb{R}^n$ as $s \to r$, but in order pass to the limit under the u-sign we must use that u is a distribution and show that the above convergence of Δ_s takes place in $\mathscr{D}(\mathbb{R}^n)$. We thus require control of the supports and uniform convergence of all partial derivatives. Take $R > 0$ such that $B_R(0)$ contains the support of ϕ and note that $B_{rR}(0)$ then contains the support of ϕ_r. Therefore $K = \overline{B_{(r+1)R}(0)}$ is a compact set containing the supports of all the Δ_s when $s \in (0, r) \cup (r, r+1)$. Next, for a multi-index $\alpha \in \mathbb{N}_0^n$ we have

$$\begin{aligned} \partial^\alpha \Delta_s(x) &= \frac{s^{-\beta-|\alpha|}\bigl(\partial^\alpha\phi\bigr)_s(x) - r^{-\beta-|\alpha|}\bigl(\partial^\alpha\phi\bigr)_r(x)}{s - r} \\ &\to \partial_r\bigl(r^{-\beta-|\alpha|}\bigl(\partial^\alpha\phi\bigr)_r(x)\bigr) = \partial^\alpha\partial_r\bigl(r^{-\beta}\phi_r(x)\bigr) \end{aligned}$$

uniformly in $x \in \mathbb{R}^n$ as $s \to r$. Consequently, the difference-quotients Δ_s converge in $\mathscr{D}(\mathbb{R}^n)$ as required and so we conclude the proof using that u is $\mathscr{D}$ continuous.

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