1. Fix a basis $\mathcal{C}$ for X of size $\kappa = w(X)$ and suppose $\mathcal{B}$ is any basis for X.

   Every open set is the union of at most κ many elements of $\mathcal{B}$: For U open and $x \in U$ choose $B \in \mathcal{B}$ and $C_x \in \mathcal{C}$ such that $x \in C_x \subseteq B \subseteq U$ (noting that the union of the C_x is U). For each $C \in \mathcal{C}' = \{C_x : x \in C\} \subseteq \mathcal{C}$ choose some $B_C \in \mathcal{B}$ such that $C \subseteq B_C \subseteq U$ (at least one such exists by the choice of the C_x). Then $\{B_D : D \in \mathcal{C}'\}$ has size $\leq \kappa$ and union U.

   Apply to elements of $\mathcal{C}$: For each $C \in \mathcal{C}$, choose a $\mathcal{B}_C \subseteq \mathcal{B}$ of size $\leq \kappa$ with $C = \bigcup \mathcal{B}_C$.

   $\bigcup_{C\in\mathcal{C}} \mathcal{B}_C$ is a basis of size κ: It is a union of κ many sets of size κ and hence has size κ. To see that it is a basis, suppose $x \in U$ open $\subseteq X$. First find $C \in \mathcal{C}$ with $x \in C \subseteq U$ and then $B \in \mathcal{B}_C$ with $x \in B \subseteq C \subseteq U$.

   An alternative proof (by Arthur Pander): Fix a basis $\mathcal{C}$ of size κ and $\mathcal{B}$ any basis.

   Fix a $B_0 \in \mathcal{B}$.

   For each $(C, C') \in \mathcal{C}^2$ such that there exists $B \in \mathcal{B}$ with $C \subseteq B \subseteq C'$ choose a $B_{C,C'} \in \mathcal{B}$ such that $C \subseteq B_{C,C'} \subseteq C'$. Otherwise set $B_{C,C'} = B_0$.

   Set $\mathcal{B}' = \left\{B_{C,C'} : (C, C') \in \mathcal{C}^2\right\} \subseteq \mathcal{B}$ which is of size at most $\left|\mathcal{C}^2\right| = |C|$.

   It is a basis, since if $x \in U$ open, then find $C' \in \mathcal{C}$ with $x \in C' \subseteq U$, then $B \in \mathcal{B}$ with $x \in B \subseteq C'$ and finally $C \in \mathcal{C}$ with $x \in C \subseteq B$ to see that $x \in B_{C,C'} \subseteq U$.

2. Write $A = \bigcup_n C_n$ and $B = \bigcup_n D_n$ where $(C_n), (D_n)$ are without loss of generality (take unions of initial segments) increasing sequences of closed sets. Note that $C_n \cap \overline{B} \subseteq A \cap \overline{B} = \emptyset$ and similarly $D_n \cap \overline{A} = \emptyset$. Thus for each n, we can find open U_n, V_n such that $C_n \subseteq U_n \subseteq \overline{U_n} \subseteq X \setminus \overline{B}$ and $D_n \subseteq V_n \subseteq \overline{V_n} \subseteq X \setminus \overline{A}$. Again, by taking finite unions of initial segments we may assume that the (U_n) and (V_n) are increasing. Then let $U = \bigcup_n (U_n \setminus \overline{V_n})$ and $V = \bigcup_n (V_n \setminus \overline{U_n})$. These are unions of open sets, so open. Since $A = \bigcup_n C_n \subseteq \bigcup_n U_n$ and $A \cap \overline{V_n} = \emptyset$ for each n, $A \subseteq U$ and similarly $B \subseteq V$. Finally if $x \in U \cap V$ then for some n, m we have $x \in U_n \setminus \overline{V_n}$ and $x \in V_m \setminus \overline{U_m}$. Wlog $n \leq m$ so that $x \in U_n \subseteq U_m \subseteq \overline{U_m}$ a contradiction.

   Alternative solution using Urysohn's Lemma: Again, write $A = \bigcup_n C_n$ and $B = \bigcup_n D_n$ for (C_n), (D_n) increasing sequences of closed sets and as above note that $\overline{A} \cap D_n = \emptyset = \overline{B} \cap C_n$. Thus by Urysohn's Lemma, we can find continuous functions $f_n, g_n : X \to [0, 2^{-n}]$ such that $f_n(C_n) \subseteq \{2^{-n}\}$, $f_n\left(\overline{B}\right) \subseteq \{0\}$, $g_n(D_n) \subseteq \{2^{-n}\}$, $g_n\left(\overline{A}\right) \subseteq \{0\}$. By the M-test, $h = \sum_n f_n - \sum_n g_n$ is a continuous real-valued function on X.

If $x \in A$ then some $f_n(x) > 0$ and all $g_n(x) = 0$ so that $h(x) > 0$. If $x \in B$ then all $f_n(x) = 0$ and some $g_n(x) > 0$ so that $h(x) < 0$. Thus $h^{-1}((-\infty, 0))$ and $h^{-1}((0, \infty))$ are the required open sets.

Remarks: The first proof is very similar to Lindelöf + regular implies normal. A little bit of countability added to regularity gives normality.

3. Note that since C is closed, closed subsets of C are closed in X. We will thus not specify whether we mean C-closed or X-closed as these are equivalent. Similarly, F_σ-subsets of C are also F_σ-subsets of X and again, we do not need to specify which topology we refer to.

   $A = f^{-1}([0, r)) \subseteq f^{-1}([0, r])$ and the right hand side is closed and disjoint from $B = f^{-1}((r, 1])$. Thus $\overline{A} \cap B = \emptyset$ and similarly $\overline{B} \cap A = \emptyset$ showing that they are separated. Also $[0, r) = \bigcup_n [0, r - 2^{-n}]$ is an F_σ and hence $A = f^{-1}([0, r)) = f^{-1}(\bigcup_n [0, r - 2^{-n}]) = \bigcup_n f^{-1}([0, r - 2^{-n}])$ is an F_σ and similarly for B.

4. Well order $\mathbb{Q} \cap (0, 1)$ as $\{q_0, q_1, q_2, \ldots\}$. q_0 and q_1 are special cases (some of the Us are empty) which work by modifying the proof below in the obvious way.

   Assume U_{q_i} have been defined satsifying the condition for $i < 2 \leq n$. Let $i_0, i_1 < n$ be such that q_{i_0} is the maximal element of $\{q_i \colon i < n, q_i < q_n\}$ and q_{i_1} is the minimal element of $\{q_i \colon i < n, q_n < q_i\}$. Write $A = f^{-1}([0, q_n))$ and $B = f^{-1}((q_n, 1])$. Then $A' = \overline{U_{q_{i_0}}} \cup A$ and $B' = B \cup X \setminus U_{q_{i_1}}$ are F_σ sets. We claim that these are separated: first note that finite unions and closures commute. Next

   - $\overline{A} \cap B = \emptyset = A \cap \overline{B}$ by the previous part;
   - $\overline{A} \subseteq f^{-1}([0, q_n]) \subseteq f^{-1}([0, q_{i_1})) \subseteq U_{q_{i_1}}$ is disjoint from $X \setminus U_{q_{i_1}} = \overline{X \setminus U_{q_{i_1}}}$;
   - similarly $\overline{B} \subseteq f^{-1}([q_n, 1]) \subseteq f^{-1}((q_{i_0}, 1])$ is disjoint from $\overline{U_{i_0}}$;
   - $\overline{U_{q_{i_0}}} \subseteq U_{q_{i_1}}$ is disjoint from $X \setminus U_{q_{i_1}} = \overline{X \setminus U_{q_{i_1}}}$ (by inductive hypothesis).

   Hence A' and B' are separated and we can thus (by the first part) find an open U_{q_n} with $A' \subseteq U_{q_n} \subseteq \overline{U_{q_n}} \subseteq X \setminus B'$ as required.

   As in Urysohn's Lemma, the function defined by $F(x) = \inf\{q \in \mathbb{Q} \cap (0, 1) \colon x \in U_q\} = \inf\left\{q \in \mathbb{Q} \cap (0, 1) \colon x \in \overline{U_q}\right\}$ is continuous.

   Finally, for all $q \in \mathbb{Q} \cap (0, 1)$ and $x \in C$ we have $f(x) < q \implies x \in U_q \implies F(x) \leq q$ and $f(x) > q \implies x \notin U_q \implies F(x) \geq q$ and hence that F extends f.

5. If C, D are disjoint closed then $1_D \colon C \cup D \to [0, 1]$ is a continuous $[0, 1]$-valued function on the closed set $C \cup D$. By Tietze's Theorem this extends to some continuous $[0, 1]$-valued function F, which is a Urysohn function for C, D.

6. Note that all closed and bounded intervals (in $\mathbb{R}$) are obviously homeomorphic.

   A, B are C-closed, so X-closed. Thus by Urysohn's Lemma there is a continuous function $g\colon X \to [-1/3, 1/3]$ such that $g(A) \subseteq \{-1/3\}$ and $g(B) \subseteq \{1/3\}$. Now for $x \in A$ we have $f(x)-g(x) \geq -1-(-1/3) = -2/3$, for $x \in B$ we have $f(x) - g(x) \leq 1 - 1/3 = 2/3$ and if $x \in X \setminus (A \cup B)$ then $-1/3 < f(x) < 1/3$ so that $-2/3 = -1/3 - 1/3 < f(x) - g(x) < 1/3 - (-1/3) = 2/3$.

   Rescaling the above argument we obtain that if $a \in \mathbb{R}^+$ and $f\colon C \to [-a, a]$ is continuous then there is $g\colon X \to [-a/3, a/3]$ such that $(f - g)(X) \subseteq [-2a/3, 2a/3]$.

   Starting with $f_0 = f$, obtain g_0 as described. Inductively let $f_{n+1} = f_n - g_n = f - \sum_{k\leq n} g_k\colon C \to [-(2/3)^n, (2/3)^n]$ and let $g_{n+1}\colon X \to [-(2/3)^n/3, (2/3)^n/3]$ be continuous with $(f_{n+1} - g_{n+1})(X) \subseteq [-(2/3)^{n+1}, (2/3)^{n+1}]$ (possible by the note).

   Then let $F = \sum_n g_n\colon X \to [-1, 1]$. By the M-test, this is well defined and continuous and $f - F = 0$ on C.

7. A continuous function $[0, 1]$-valued on X is determined by its values on a dense subset: If Y is Hausdorff then $\Delta_Y = \{(y, y)\colon y \in Y\}$ is closed in Y^2 (if $y \neq y'$ then the disjoint open sets $U \ni y$, $V \ni y'$ witness this via $(y, y') \in U \times V \subseteq Y^2 \setminus \Delta_Y$) and so $\{x \in X\colon f(x) = g(x)\} = (f\Delta g)^{-1}(\Delta_Y)$ is closed. Thus $f|_D = g|_D$ for a subset D of X, then $f|_{\overline{D}} = g|_{\overline{D}}$ as required.

   If X has a countable dense subset D, then $f \mapsto f|_D$ is an injection from $\mathcal{C}(X, [0, 1])$ into $\mathcal{C}(D, [0, 1])$ so that there are at most $\left|[0, 1]^D\right| = (2^{\mathbb{N}})^{\mathbb{N}} = 2^{\mathbb{N}\times\mathbb{N}} = 2^{\mathbb{N}}$ many continuous $[0, 1]$-valued functions on X.

   On the other hand, by Tietze's Theorem there are at least as many continuous $[0, 1]$-valued functions on X as there are on any closed subset of X (because each continuous $[0, 1]$-valued function on a closed subset extends to some continuous function on X). If C is a discrete subspace of size $2^{\mathbb{N}}$ of X then every function on C is continuous so there are at least $\left|[0, 1]^C\right| = (2^{\mathbb{N}})^{(2^{\mathbb{N}})} = 2^{\mathbb{N}\times 2^{\mathbb{N}}} = 2^{2^{\mathbb{N}}}$ many continuous $[0, 1]$-valued functions on X.

   Since $2^{2^{\mathbb{N}}} > 2^{\mathbb{N}}$ (Cantor's Theorem), a normal space with countable dense subset cannot have a continuum-sized closed discrete subspace.

8. The antidiagonal of the Sorgenfrey Plane is a closed discrete subspace of size continuum. But $\mathbb{Q} \times \mathbb{Q}$ is a countable dense subset of the Sorgenfrey Plane. (Draw a picture for both.) Thus the Sorgenfrey Plane cannot be normal.

   Normality of the Sorgenfrey is easiest shown using halving operators: we define $H(x, [x, x + \epsilon)) = [x, x + \epsilon/2)$ (or in fact $[x, x + \epsilon)$) and extend in the obvious way: if $x \in U$ open, then choose $\epsilon > 0$ such that $[x, x+\epsilon) \subseteq U$ and define $H(x, U) = H(x, [x, x + \epsilon))$. It is easy to see that this works.

It cannot be metrizable as then the Sorgenfrey Plane would be metrizable as well so the Sorgenfrey Plane would be normal.

Since the Sorgenfrey Line is normal, it cannot be second countable (by Urysohn's Metrization Theorem).

Remarks: The Sorgenfrey Line is first countable ($\{[a, a+2^{-n})\colon n \in \mathbb{N}\}$ is a countable neighbourhood basis at a).

Note that the Sorgenfrey Line is Lindelöf: just like showing compactness of $[0,1]$ in the usual topology, we can show that $[0,1]$ is Lindelöf in the Sorgenfrey topology: if $\mathcal{U}$ is an open cover of $[0,1]$ then let $\alpha = \sup\{x \in [0,1]\colon [0,x] \text{ is covered by a countable subcover}\}$ and for $n \in \mathbb{N}$ choose countable subcovers $\mathcal{V}_n$ covering $[0, \alpha - 2^{-n}]$ (possible by Approximation Property) as well as $U \in \mathcal{U}$ with $\alpha \in [\alpha, \alpha+\epsilon) \in U$ for some $\epsilon > 0$. Then $\bigcup_n \mathcal{V}_n \cup \{U\}$ is a countable open cover of $[0, \alpha + \epsilon/2]$ showing that $\alpha = 1$ and that $[0,1]$ has a countable subcover. Clearly $[n, n+1]$ is homeomorphic to $[0,1]$ (in the Lindelöf topology) and a countable union of Lindelöf spaces is Lindelöf giving that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1]$ is Lindelöf.

Hence (regularity + Lindelöf implies paracompact) the Sorgenfrey Line is paracompact.

You can also show that the Sorgenfrey Line is perfectly normal, but this is a little harder.