1. Fix $x \in X$ and find open $U \ni x$ that meets only finitely many $A \in \mathcal{A}$. As U is open $U \cap \overline{A} \neq \emptyset \iff U \cap A \neq \emptyset$ giving the result.

2. Fix an open cover $\mathcal{U}$. For each $x \in X$ find U_x and open V_x with $x \in V_x \subseteq \overline{V_x} \subseteq U_x$. Then $\{V_x : x \in X\}$ is an open cover and hence has a locally finite refinement $\mathcal{A}$ covering X. For each $A \in \mathcal{A}$ choose $U_A \in \mathcal{U}$ such that $\overline{A} \subseteq U_A$ and for each $U \in \mathcal{U}$ let

$$F_U = \overline{\bigcup \{A \in \mathcal{A} \colon U_A = U\}} = \bigcup \left\{\overline{A} \colon A \in \mathcal{A}, U_A = U\right\}.$$

by local finiteness of $\mathcal{A}$. Note that for open $V \subseteq X$, $V \cap \overline{A} \neq \emptyset \iff V \cap A \neq \emptyset$. Hence, if $V \ni x$ witnesses local finiteness of $\mathcal{A}$ at x then it witnesses local finiteness of $\left\{\overline{A} \colon A \in \mathcal{A}\right\}$ at x and thus meets only finitely many F_U. (This last step seems difficult for students, so here are two (?) more detailed explanations: each $A \in \mathcal{A}$ is only used in one F_U. So if V only meets $A_1, \ldots, A_n$ and no other $A \in \mathcal{A}$, then V only meets $F_{U_{A_1}}, \ldots, F_{U_{A_n}}$: if V meets F_U then V meets some $A \in \mathcal{A}$ with $U_A = U$.)

   It remains to argue that $\{F_U \colon U \in \mathcal{U}\}$ is an open cover of X refining $\mathcal{U}$. Clearly $F_U \subseteq U$ (since $U_A = U$ implies $\overline{A} \subseteq U$). For covering, fix $y \in X$. Then $y \in A$ for some $A \in \mathcal{A}$ and $A \subseteq F_{U_A}$.

3. Suppose $\mathcal{U} = \bigcup_n \mathcal{U}_n$ is an open cover of X such that each $\mathcal{U}_n$ is locally finite. Write $U_n = \bigcup_{k<n} \bigcup \mathcal{U}_k$ (the bit of X covered by the families $\mathcal{U}_1, \ldots, \mathcal{U}_{n-1}$) and let

$$\mathcal{A} = \bigcup_n \{U \setminus U_n \colon U \in \mathcal{U}_n\}.$$

(so from the elements of $\mathcal{U}_n$, throw away what has been covered previously). We claim that $\mathcal{A}$ is the required locally finite refinement of $\mathcal{U}$ covering X. Clearly $\mathcal{A}$ refines $\mathcal{U}_n$ and hence $\mathcal{U}$.

   Next, it covers X since for each $x \in X$ we can find the minimal n such that $x \in \bigcup \mathcal{U}_n$ and $U \in \mathcal{U}_n$ such that $x \in U$. Then $x \in U \setminus U_n$ (by minimality of n). If $V_1, \ldots, V_n$ witness that $\mathcal{U}_1, \ldots, \mathcal{U}_n$ are locally finite at x then $W = V_1 \cap \ldots \cap V_n \cap U_{n+1}$ will only meet finitely many elements of $\mathcal{A}$: if $k > n$ then $U \setminus U_k, U \in \mathcal{U}_k$, does not meet W (because of the U_{n+1}) and each V_i ensures that W meets only finitely many $U \setminus U_k, U \in \mathcal{U}_k$ for $k \leq n$.

4. We have done everything except for (i) implies (ii) which is trivial and (iv) implies (i). For this we use the notation of the hint.

   Observe that each $D \in \mathcal{D}$ is contained in some $W \in \mathcal{W}$ and hence can only meet those V_C where W meets C. As $\mathcal{W}$ witnesses local finiteness of $\mathcal{C}$ there are only finitely many such. So for each $D \in \mathcal{D}$ there is a finite $\mathcal{C}_D \subseteq \mathcal{C}$ such that $D \cap V_C = \emptyset$ unless $C \in \mathcal{C}_D$.

   Now let $x \in X$ and let A witness local finiteness of $\mathcal{D}$ at x. Then $\mathcal{D}_A = \{D \in \mathcal{D} \colon A \cap D \neq \emptyset\}$ is finite and $A \subseteq \mathcal{D}_A$ as $\mathcal{D}$ is a cover. Thus A can only meet V_C for $C \in \bigcup_{D \in \mathcal{D}_A} \mathcal{C}_D$ and this is a finite union of finite sets. Hence $\{V_C : C \in \mathcal{C}\}$ is locally finite.

Observe that for each $C \in \mathcal{C}$, $C \subseteq V_C$ and choose $U_C \in \mathcal{U}$ such that $C \subseteq U_C$ so that $C \subseteq V_C \cap U_C$. Then $\{V_C \cap U : C \in \mathcal{C}\}$ is a locally finite refinement of $\mathcal{U}$ covering X, as required.

5. Examining the proof of (b), we see that if all elements of $\mathcal{A}$ are open then $\bigcup \left\{A \in \mathcal{A} : \overline{A} \subseteq U\right\}$ is open as required.

6. Let $\mathcal{U}$ be an open cover, and use a previous part to obtain a locally finite open collection $\mathcal{V} = \{V_U : U \in \mathcal{U}\}$ such that $\forall U \in \mathcal{U}\ \overline{V_U} \subseteq U$. Let D be a countable dense set and for $d \in D$ choose $V_d \in \mathcal{V}$ such that $d \in V_d$. Now note that by local finiteness of $\mathcal{V}$,
$$\bigcup_{d \in D} U_{V_d} \supseteq \bigcup_{d \in D} \overline{V_d} = \overline{\bigcup_{d \in D} V_d} \supseteq \overline{D} = X$$
so that $\{U_{V_d} : d \in D\}$ is the required countable subcover.

Alternative solution (by Christopher Turner) without using regularity: We show that in a separable space every locally finite family of (non-empty) open sets is countable: suppose $\mathcal{V}$ is a locally finite family of non-empty open sets and that D is a countable dense subset. Because $\mathcal{V}$ is locally finite, for $d \in D$ we have open $W \ni d$ that meets only finitely many elements of $\mathcal{V}$. Hence each $d \in D$ is only contained in finitely many elements of $\mathcal{V}$. Write $\mathcal{V}_d = \{V \in \mathcal{V} : d \in V\}$.

Because D is dense and each $V \in \mathcal{V}$ is non-empty open, it contains some $d \in D$. Thus $\mathcal{V} = \bigcup_{d \in D} \mathcal{V}_d$ which is a countable union of finite sets so countable.

Now, if $\mathcal{U}$ is an open cover then choose a locally finite open refinement $\mathcal{V}$ still covering X. By the above $\mathcal{V}$ is countable and if we choose for each $V \in \mathcal{V}$ a $U_V \in \mathcal{U}$ such that $V \subseteq U_V$ then $\{U_V : V \in \mathcal{V}\}$ is a countable subset of $\mathcal{U}$ covering X: if $x \in X$ then $x \in V$ for some $V \in \mathcal{V}$ ($\mathcal{V}$ covers) and so $x \in V \subseteq U_V$.

7. Suppose that $I = \bigcap \mathcal{U} \neq \emptyset$.As $X \setminus I \in \mathcal{U}$ leads to $I = \emptyset$ we must have $I \in \mathcal{U}$. If $x, y \in I$ are distinct then one of $\{x\}, X \setminus \{x\} \in \mathcal{U}$. If $\{x\} \in \mathcal{U}$ then $y \notin I$. If $X \setminus \{x\} \in \mathcal{U}$ then $x \notin I$. So I is a singleton $\{x\}$ and hence $\mathcal{U} = \mathcal{P}_x$.

   For the converse, simply observe that $\bigcap \mathcal{P}_x = \{x\}$.

   If $\mathcal{U}$ is free then for each x there must be $A \ni x$ with $X \setminus A \in \mathcal{U}$. But then $X \setminus \{x\} \supseteq X \setminus A$, so $X \setminus \{x\} \in \mathcal{U}$. Finally, taking finite intersections of $X \setminus \{x\}$ we obtain $X \setminus F \in \mathcal{U}$ for any finite set F.

8. Assume that $\mathcal{U}$ is an ultrafilter with countable basis $\{B_n : n \in \omega\}$. By taking finite intersections of the first n elements (for each n) we may assume that B_n is decreasing. If B_n is eventually constant equal to B

(say), then $\bigcap \mathcal{U} = B$, contradiction. So by passing to a subsequence B_n is wlog strictly decreasing. We then note that $\emptyset = \bigcap \mathcal{U} = \bigcap_n B_n$. Setting $A = \bigcup_k B_{2k} \setminus B_{2k+1}$ and $B = X \setminus A = \bigcup_k B_{2k+1} \setminus B_{2k+2}$ we have that $B_j \nsubseteq A$ and $B_j \nsubseteq B$ for every j, so (by the fact that the B_n form a filter basis for $\mathcal{U}$), neither A nor $X \setminus A$ belong to $\mathcal{U}$ a contradiction.

9. For each n, consider the family $\{[k/2^n, (k+1)/2^n] \colon k = 0, \dots, 2^n - 1\}$. This covers $[0, 1]$ and hence at least one element must belong to $\mathcal{U}$. Picking one such interval for each n gives a countable family of closed intervals whose intersection is non-empty (by compactness of $[0, 1]$) and at most a singleton (as the length go to 0). Hence this intersection is a singleton $\{x\} \in \mathcal{U}$, giving that $\mathcal{U} = \mathcal{P}_x$.