Infinite Groups

Cornelia Druțu

University of Oxford

Part C course MT 2023, Oxford

Comparison between solvable and nilpotent: growth

Final topic of this course: distinguishing f.g nilpotent groups in the larger class of f.g. solvable groups *via* their growth.

This is the celebrated Milnor-Wolf Theorem.

Byproducts of the proof: new features that allow to distinguish between solvable and polycyclic, polycyclic and nilpotent.

Let $G = \langle S \rangle$, where S finite, $S^{-1} = S$, $1 \notin S$.
Let dist_S be the word metric associated to S.
The growth function of G with respect to S is

$$\mathfrak{G}_{G,S}(R) := \mathrm{card}\, \bar{B}(1, R).$$

When there is no risk of confusion, we write simply $\mathfrak{G}_S(R)$.
Question: How much does $\mathfrak{G}_{G,S}$ depend on S?

Growth functions

Definition

Given $f, g : X \to \mathbb{R}$ with $X \subset \mathbb{R}$, we define an asymptotic inequality $f \preceq g \Leftrightarrow \exists a, b > 0,\ c \geqslant 0$ and $x_0 \in \mathbb{R}$ such that $\forall x \in X, x \geqslant x_0$, $bx + c \in X$ and $f(x) \leqslant ag(bx + c)$.

$f \asymp g \Leftrightarrow f \preceq g$ and $g \preceq f$; we say that f and g are asymptotically equal.

Lemma

Assume that (G, dist_S) and (H, dist_X) are bi-Lipschitz equivalent, i.e. $\exists L > 0$ and a bijection $f : G \to H$ such that

$$\frac{1}{L}\mathrm{dist}_S(g, g') \leqslant \mathrm{dist}_X(f(g), f(g')) \leqslant L\mathrm{dist}_S(g, g')\,, \forall g, g' \in G\,. \qquad (1)$$

Then $\mathfrak{G}_{G,S} \asymp \mathfrak{G}_{H,X}$.
In particular true when $(H, \mathrm{dist}_X) = (G, \mathrm{dist}_{S'})$, $G = \langle S' \rangle$.

Growth functions

Corollary

If S, S' are two finite generating sets of G then $\mathfrak{G}_S \asymp \mathfrak{G}_{S'}$. Thus, one can speak of growth function $\mathfrak{G}_G$ of a group G, well defined up to $\asymp$.

Examples

1. If $G = \mathbb{Z}^k$ then $\mathfrak{G}_S \asymp x^k$ for every finite generating set $S = S^{-1}$.
2. If $G = F_k$, the free group of finite rank $k \geqslant 2$, and X is the set of k letters/symbols then

$$\mathfrak{G}_{X \sqcup X^{-1}}(n) = 1 + (q^n - 1)\frac{q+1}{q-1}, \quad q = 2k - 1.$$

Growth functions: properties

Proposition

1. *If G is infinite, $\mathfrak{G}_G|_{\mathbb{N}}$ is strictly increasing.*
2. *If $H \leqslant G$ then $\mathfrak{G}_H \preceq \mathfrak{G}_G$.*
3. *If $H \leqslant G$ finite index then $\mathfrak{G}_H \asymp \mathfrak{G}_G$.*
4. *If $N \lhd G$ then $\mathfrak{G}_{G/N} \preceq \mathfrak{G}_G$.*
5. *If $N \lhd G$, N finite, then $\mathfrak{G}_{G/N} \asymp \mathfrak{G}_G$.*
6. *For each finitely generated group G, $\mathfrak{G}_G(r) \preceq 2^r$.*
7. *The growth function is sub-multiplicative:*

$$\mathfrak{G}_{G,S}(r+t) \leqslant \mathfrak{G}_{G,S}(r)\mathfrak{G}_{G,S}(t).$$

$\mathfrak{G}_{G,S}$ sub-multiplicative $\Rightarrow \ln \mathfrak{G}_{G,S}(n)$ sub-additive.

By Fekete's Lemma, there exists a (finite) limit

$$\lim_{n\to\infty} \frac{\ln \mathfrak{G}_{G,S}(n)}{n}.$$

Hence, we also have a finite limit

$$\gamma_{G,S} = \lim_{n\to\infty} \mathfrak{G}_{G,S}(n)^{\frac{1}{n}},$$

called growth constant. The property (1) implies that $\mathfrak{G}_{G,S}(n) \geqslant n$; whence, $\gamma_{G,S} \geqslant 1$.

Definition

If $\gamma_{G,S} > 1$ then G is said to be of exponential growth. If $\gamma_{G,S} = 1$ then G is said to be of sub-exponential growth.

Note that if there exists a finite generating set S such that $\gamma_{G,S} > 1$ then $\gamma_{G,S'} > 1$ for every other finite generating set S'. Likewise for the equality to 1.

Two examples of order of growth

Example

For every $n \geqslant 2$, the group $SL(n, \mathbb{Z})$ has exponential growth.

Definition

Let G be a finitely generated nilpotent group of class k. Let m_i denote the free rank of the abelian group $C^i G / C^{i+1} G$. The homogeneous dimension of G is

$$d(G) = \sum_{i=1}^{k} i m_i .$$

Theorem (Bass–Guivarc'h Theorem)

The growth function of G satisfies

$$\mathfrak{G}_G(n) \asymp n^d . \qquad (2)$$