B1.1 Logic
Lecture 15

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14. Prenex normal form

A formula is in **prenex normal form (PNF)** if it is of the form

$$Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_k x_{i_k} \, \psi,$$

where each Q_i is a quantifier (i.e. either $\forall$ or $\exists$), and where ψ is a formula containing no quantifiers.

14.1 PNF-Theorem
Every $\phi \in \mathrm{Form}(\mathcal{L})$ *is logically equivalent to an* $\mathcal{L}$*-formula in PNF.*

Proof: Induction on ϕ (working in the language with $\forall, \exists, \neg, \wedge$, recalling that $\{\neg, \wedge\}$ is adequate for propositional logic):

- ϕ atomic: ϕ is already in PNF.

- $\phi = \neg\chi$ with χ in PNF:
  say $\phi = \neg Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_k x_{i_k}\, \psi$.

  Then $\phi \models\!\!\!=\!\!\!\mid\ Q_1^- x_{i_1} \cdots Q_k^- x_{i_k}\, \neg\psi$,
  where $Q^- = \exists$ if $Q = \forall$,
  and $Q^- = \forall$ if $Q = \exists$.

- $\phi = (\chi \wedge \rho)$ with χ, ρ in PNF:

  Note that $\forall x_i \alpha \models\!\!\!=\!\!\!\mid\ \forall x_j \alpha[x_j/x_i]$
  if x_j does not occur in α.

  Swapping variables in this way, we may assume that the variables quantified over in χ do not occur in ρ, and vice versa.

  But then, e.g.
  $(\forall x_1 \alpha \wedge \exists x_2 \beta) \models\!\!\!=\!\!\!\mid\ \forall x_1 \exists x_2 (\alpha \wedge \beta)$. □