B1.1 Logic
Lecture 15

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15 Applications of the Completeness Theorem

Throughout, $\mathcal{L}$ denotes a countable first-order language.

15.1 Elementary equivalence

Definition 15.1.

- An **$\mathcal{L}$-theory** is a set of $\mathcal{L}$-sentences $\Sigma \subseteq \text{Sent}(\mathcal{L})$.
- Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Then the **(first-order) theory of** $\mathcal{A}$ is the $\mathcal{L}$-theory
$$\text{Th}(\mathcal{A}) = \text{Th}^{\mathcal{L}}(\mathcal{A}) := \{\sigma \in \text{Sent}(\mathcal{L}) \mid \mathcal{A} \models \sigma\},$$
the set of all $\mathcal{L}$-sentences true in $\mathcal{A}$.
- $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent**, written $\mathcal{A} \equiv \mathcal{B}$, if $\text{Th}(\mathcal{A}) = \text{Th}(\mathcal{B})$.

Exercise 15.2. An $\mathcal{L}$-theory $\Sigma \subseteq \text{Sent}(\mathcal{L})$ is maximal consistent if and only if Σ has a model and $\mathcal{A} \equiv \mathcal{B}$ for any two models $\mathcal{A}$ and $\mathcal{B}$ of Σ.

15.2 Axiomatisations

Definition 15.3. An **axiomatisation** of the theory $\text{Th}(\mathcal{A})$ of an $\mathcal{L}$-structure $\mathcal{A}$ is a maximal consistent subset of $\text{Th}(\mathcal{A})$; i.e. a set of sentences which hold of $\mathcal{A}$ and which suffice to deduce any sentence which holds of $\mathcal{A}$.

Recall Hilbert's programme from Lecture 1. Now we have established the Completeness Theorem, the programme would call for us to find "finitary" (i.e. computable) axiomatisations of the structures in mathematics.

In general this is *impossible*: Gödel's first incompleteness theorem shows that already the theory of arithmetic $\text{Th}(\langle \mathbb{N}; +, \cdot \rangle)$ has no computable axiomatisation. But for some interesting structures it is possible, as we will now begin to see.

15.3 A criterion for maximal consistency

Definition 15.4. Let $\mathcal{A} = \langle A; \ldots \rangle$ and $\mathcal{B} = \langle B; \ldots \rangle$ be $\mathcal{L}$-structures. An **isomorphism** of $\mathcal{A}$ with $\mathcal{B}$ is a bijection $\theta : A \to B$ such that

- $\theta(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for c a constant symbol;
- $\theta(f^{\mathcal{A}}(a_1, \ldots, a_k)) = f^{\mathcal{B}}(\theta(a_1), \ldots, \theta(a_k))$ for f a k-ary function symbol and $a_i \in A$;
- $(a_1, \ldots, a_k) \in P^{\mathcal{A}} \Leftrightarrow (\theta(a_1), \ldots, \theta(a_k)) \in P^{\mathcal{B}}$ for P a k-ary relation symbol and $a_i \in A$.

We write $\mathcal{A} \cong \mathcal{B}$ to mean that there exists such an isomorphism.

Exercise 15.5. $\mathcal{A} \cong \mathcal{B}$ implies $\mathcal{A} \equiv \mathcal{B}$.
The converse fails (e.g. due to Löwenheim-Skolem).

Theorem 15.6. *Suppose $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$ has a unique countable model up to isomorphism, i.e. Σ is consistent and if $\mathcal{A}, \mathcal{B} \vDash \Sigma$ are countable then $\mathcal{A} \cong \mathcal{B}$.*
Then Σ is maximal consistent.

Proof. Let $\mathcal{A}, \mathcal{B} \vDash \Sigma$. We conclude by showing $\mathcal{A} \equiv \mathcal{B}$.
By Weak Downward Löwenheim-Skolem (Theorem 13.10), there are countable $\mathcal{A}' \equiv \mathcal{A}$ and $\mathcal{B}' \equiv \mathcal{B}$. Then $\mathcal{A}', \mathcal{B}' \vDash \Sigma$, so $\mathcal{A}' \cong \mathcal{B}'$, and so $\mathcal{A}' \equiv \mathcal{B}'$ by Exercise 15.5. Hence $\mathcal{A} \equiv \mathcal{A}' \equiv \mathcal{B}' \equiv \mathcal{B}$. □

Remark 15.7. The converse fails. We will see an example in the next lecture.

Example 15.8. Let $\mathcal{L}_= := \emptyset$, the language with no non-logical symbols. For $n \geq 2$, set $\tau_n := \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i \doteq x_j$. Then the models of

$$\Sigma_\infty := \{\tau_n : n \geq 2\}$$

are precisely the infinite $\mathcal{L}_=$-structures (i.e. the infinite sets). By Theorem 15.6, Σ_∞ is maximal consistent.

15.4 Example: axiomatising $\mathrm{Th}(\langle \mathbb{Q}; < \rangle)$

Definition 15.9. Let $\mathcal{L}_< := \{<\}$ and let σ_{DLO} be the following $\mathcal{L}_<$-sentence, whose models are the dense linear orderings without endpoints:

$$\begin{aligned}
\sigma_{\mathrm{DLO}} := \forall x\, \forall y\, \forall z\, (\neg x < x \\
&\wedge (x < y \vee x = y \vee y < x) \\
&\wedge ((x < y \wedge y < z) \to x < z) \\
&\wedge (x < y \to \exists w\, (x < w \wedge w < y)) \\
&\wedge \exists w\, w < x \\
&\wedge \exists w\, x < w).
\end{aligned}$$

Note that $\langle \mathbb{Q}; < \rangle \vDash \sigma_{\mathrm{DLO}}$, and also $\langle \mathbb{R}; < \rangle \vDash \sigma_{\mathrm{DLO}}$.

Theorem 15.10 (Cantor)**.** *σ_{DLO} has a unique countable model up to isomorphism (so any countable model is isomorphic to $\langle \mathbb{Q}; < \rangle$).*

Proof. *("Back-and-forth argument")*
Let $\mathcal{M}, \mathcal{N} \vDash \sigma_{\text{DLO}}$ be countable. By the non-existence of endpoints, each is infinite.

A **partial isomorphism** $\theta : \mathcal{M} \dashrightarrow \mathcal{N}$ is a partially defined injective map such that for all $a, b \in \text{dom}(\theta)$,

$$\mathcal{M} \vDash a < b \quad \Leftrightarrow \quad \mathcal{N} \vDash \theta(a) < \theta(b).$$

Enumerate the domains of $\mathcal{M}$ and $\mathcal{N}$ as $(m_i)_{i \in \mathbb{N}}$ and $(n_i)_{i \in \mathbb{N}}$ respectively. We recursively construct a chain of partial isomorphisms $\theta_i : \mathcal{M} \dashrightarrow \mathcal{N}$ such that

$$\text{dom}(\theta_i) \text{ is finite, and for all } j < i, \text{ we have } m_j \in \text{dom}\,\theta_i \text{ and } n_j \in \text{im}\,\theta_i. \quad (*)$$

Let $\theta_0 := \emptyset$.

Given θ_i satisfying (*), we first extend θ_i by finding $n \in \mathcal{N}$ such that setting $\theta_i'(m_i) := n$ yields a partial isomorphism $\theta_i' : \mathcal{M} \dashrightarrow \mathcal{N}$ with $\text{dom}\,\theta_i' = \text{dom}\,\theta \cup \{m_i\}$.

Say $\text{dom}(\theta_i) = \{a_1, \ldots, a_s\}$ with $\mathcal{M} \vDash a_k < a_l$ for $1 \le k < l \le s$, and similarly $\text{im}(\theta_i) = \{b_1, \ldots, b_s\}$ with $\mathcal{N} \vDash b_k < b_l$ for $1 \le k < l \le s$. There are four cases:

(i) $m_i = a_k$ (some $k \in [1, s]$): set $n := b_k$.

(ii) $m_i < a_1$: let $n \in \mathcal{N}$ be such that $n < b_1$ (n exists, since $\mathcal{N}$ has no endpoint).

(iii) $m_i > a_s$: let $n \in \mathcal{N}$ be such that $n > b_s$ (n exists, since $\mathcal{N}$ has no endpoint).

(iv) $a_j < m_i < a_{j+1}$ (some $j \in [1, s-1]$): let $n \in \mathcal{N}$ be such that $a_i < n < a_{i+1}$ (n exists, since $\mathcal{N}$ is dense).

In all cases, θ_i' is a partial isomorphism.

Symmetrically, $(\theta_i')^{-1} : \mathcal{N} \dashrightarrow \mathcal{M}$ extends to $\theta_i'' : \mathcal{N} \dashrightarrow \mathcal{M}$ with $n_i \in \text{dom}\,\theta_i{}''$;
then $\theta_{i+1} := (\theta_i'')^{-1} : \mathcal{M} \dashrightarrow \mathcal{N}$ is a partial isomorphism satisfying (*).

Then $\theta := \bigcup_i \theta_i : \mathcal{M} \xrightarrow{\cong} \mathcal{N}$ is an isomorphism. □

Applying Theorem 15.6, we obtain:

Corollary 15.11. $\{\sigma_{\text{DLO}}\}$ *is maximal consistent. Hence* $\{\sigma_{\text{DLO}}\}$ *axiomatises* $\text{Th}(\langle \mathbb{Q}; < \rangle)$.

Corollary 15.12. *Completeness of a linear order is not a first-order property: there is no* $\mathcal{L}_<$*-theory* Σ *such that the models of* Σ *are precisely the complete linear orders.*

Proof. Suppose such a Σ exists. Then $\langle \mathbb{R}; < \rangle \vDash \Sigma$ since $\langle \mathbb{R}; < \rangle$ is a complete linear order. But $\langle \mathbb{R}; < \rangle \equiv \langle \mathbb{Q}; < \rangle$, since both satisfy the maximal complete theory $\{\sigma_{\text{DLO}}\}$, so then also $\langle \mathbb{Q}; < \rangle \vDash \Sigma$. But $\langle \mathbb{Q}; < \rangle$ is not a complete linear order, contradicting the desired property of Σ. □