B1.1 Logic
Lecture 16

Martin Bays

Oxford University, MT 23

16 An algebraic application (non-examinable)

16.1 ACF

Let $\mathcal{L}_{\text{ring}} := \{+, -, \cdot, \bar{0}, \bar{1}\}$. Let ACF be the $\mathcal{L}_{\text{ring}}$-theory whose models are precisely the algebraically closed fields:

$$\text{ACF} := \text{ [Field axioms] } \cup \{\forall z_0, \ldots, z_n \left(\neg z_n \doteq \bar{0} \to \exists x \sum_{i=0}^{n} z_i x^i \doteq \bar{0} \right) : n \geq 1\}.$$

Let

$$\text{ACF}_0 := \text{ACF} \cup \{\neg \bar{n} \doteq \bar{0} : n \in \mathbb{N}\},$$

where for $n \geq 1$, $\bar{n} := \bar{1} + \ldots + \bar{1}$ (n times). So the models of ACF_0 are precisely the algebraically closed fields of characteristic 0. In particular, $\langle \mathbb{C}; +, -, \cdot, 0, 1 \rangle \vDash \text{ACF}_0$. We aim to show that ACF_0 is maximally consistent, i.e. axiomatises $\text{Th}(\langle \mathbb{C}; +, -, \cdot, 0, 1 \rangle)$.

We can prove this analogously to the case of $\langle \mathbb{Q}; < \rangle$, but working with uncountable sets.

From now on, we assume the axiom of choice. We will explain this and the related notion of the **cardinality** ("size") $|A|$ of a set A in the Set Theory course; for now it suffices to know that $|A| = |B|$ if and only if there exists a bijection $A \to B$, and cardinalities are linearly ordered.

Fact 16.1. *Any characteristic 0 algebraically closed field* $\langle K; +, -, \cdot, 0, 1 \rangle \vDash \text{ACF}_0$ *with the same cardinality as* $\mathbb{C}$ *is isomorphic to* $\langle \mathbb{C}; +, -, \cdot, 0, 1 \rangle$.

Sketch proof. A subset $A \subseteq K$ is **algebraically independent** if there are no non-trivial polynomial relations between its elements, i.e. $f(a_1, \ldots, a_n) \neq 0$ for any $f \in \mathbb{Z}[X_1, \ldots, X_n] \setminus \{0\}$ and $\{a_1, \ldots, a_n\} \subseteq A$.

Then just as for linear independence in vector spaces, an algebraically closed field has a well-defined dimension ("transcendence degree") which is the cardinality of any maximal algebraically independent subset, this dimension determines an algebraically closed field of a given characteristic up to isomorphism, and the dimension of an uncountable ACF is equal to its cardinality. □

Fact 16.2. *Let* $\mathcal{L}$ *be a possibly uncountable first-order language, i.e. with sets of constant, function, and relation symbols of arbitrary cardinality. Let* $|\mathcal{L}|$ *be the cardinality of the language, i.e. that of the alphabet.*

Let $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$*, and suppose any finite subset of* Σ *has a model. Then* Σ *has a model of cardinality (i.e. with domain of cardinality)* $\leq |\mathcal{L}|$.

Sketch proof. Our proof for countable $\mathcal{L}$ mostly goes through directly.

The only place we used the countability assumption was in extending a consistent set Σ to a maximal consistent witnessing set. We can use Zorn's lemma here in the uncountable case – the union of a chain of consistent witnessing sets containing Σ is still consistent and witnessing, so there exists a maximal such with respect to inclusion, which (as in the proof in the countable case) is maximal consistent witnessing. □

Corollary 16.3. ACF_0 *is maximal consistent, hence axiomatises* $\mathrm{Th}(\mathbb{C})$.

Proof. Let $\mathcal{A} \vDash \mathrm{ACF}_0$. Note that $\mathcal{A}$ is infinite, since it has characteristic 0.

Let $C = \{c_a : a \in \mathbb{C}\}$ be a set of constant symbols of cardinality $|\mathbb{C}|$, and let $\mathcal{L}' := \mathcal{L}_{\mathrm{ring}} \cup C$. Let $\Sigma := \mathrm{Th}^{\mathcal{L}_{\mathrm{ring}}}(\mathcal{A}) \cup \{\neg c_a \doteq c_b : a, b \in \mathbb{C},\ a \neq b\} \subseteq \mathrm{Sent}(\mathcal{L}')$. Then since $\mathcal{A}$ is infinite, any finite subset of Σ has as model $\mathcal{A}$ with the finitely many c_a which appear interpreted as distinct elements. So by Fact 16.2, Σ has a model $\mathcal{B}$ of cardinality $\leq |\mathcal{L}'| = |\mathbb{C}|$. Considering the interpretations of the c_a, we actually have $|\mathcal{B}| = |\mathbb{C}|$. Let $\mathcal{B}'$ be the $\mathcal{L}_{\mathrm{ring}}$ structure obtained from $\mathcal{B}$ by ignoring the c_a. Then by Fact 16.1, $\mathcal{B}' \cong \mathbb{C}$. So $\mathcal{A} \equiv \mathcal{B}' \equiv \mathbb{C}$.

So we conclude that any two models of ACF_0 are elementary equivalent, so ACF_0 is maximal consistent. □

Theorem 16.4 (Ax-Grothendieck)**.** *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial map, i.e.* $F(a_1, \ldots, a_n) = (F_1(a_1, \ldots, a_n), \ldots, F_n(a_1, \ldots, a_n))$*, where* $F_i \in \mathbb{C}[\overline{X}]$.

If F *is injective, then* F *is surjective.*

Proof. Fact: The algebraic closure of the finite field $\mathbb{F}_p$ is the union of a chain of finite subfields, $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_k \mathbb{F}_{p^{k!}}$.

Claim 16.5. *Let* p *be prime. Any injective polynomial map* $F : (\mathbb{F}_p^{\mathrm{alg}})^n \to (\mathbb{F}_p^{\mathrm{alg}})^n$ *is surjective.*

Proof. Let k_0 be such that the coefficients of F are in $\mathbb{F}_{p^{k_0!}}$.

Let $k \geq k_0$. Then $F({\mathbb{F}_{p^{k!}}}^n) \subseteq {\mathbb{F}_{p^{k!}}}^n$, and so by injectivity, finiteness of ${\mathbb{F}_{p^{k!}}}^n$, and the pigeonhole principle, $F({\mathbb{F}_{p^{k!}}}^n) = {\mathbb{F}_{p^{k!}}}^n$.

Hence $F((\mathbb{F}_p^{\mathrm{alg}})^n) = (\mathbb{F}_p^{\mathrm{alg}})^n$. □

Let $n, d \in \mathbb{N}$. Let $\sigma_{n,d}$ be an $\mathcal{L}_{\mathrm{ring}}$-sentence expressing that any injective polynomial map $F : K^n \to K^n$ consisting of polynomials of degree $\leq d$ is surjective:

$$\sigma_{n,d} := \forall z_{1,0}, \ldots, z_{n,d}\ (\forall \overline{x}, \overline{y}\ ((\bigwedge_i \sum_j z_{i,j} {x_i}^j \doteq \sum_j z_{i,j} {y_i}^j) \to \bigwedge_i x_i \doteq y_i)$$
$$\to \forall \overline{y}\ \exists \overline{x}\ \bigwedge_i \sum_j z_{i,j} {x_i}^j \doteq y_i).$$

Suppose $\mathbb{C} \nvDash \sigma_{n,d}$. Then by maximal consistency of ACF_0, $\mathrm{ACF}_0 \vDash \neg\sigma_{n,d}$. Then by compactness, for some $m \in \mathbb{N}$,

$$\mathrm{ACF} \cup \{\neg \bar{i} \doteq \bar{0} : 0 < i < m\} \vDash \neg\sigma_{n,d}.$$

So if $p > m$ is prime, $\mathbb{F}_p^{\mathrm{alg}} \vDash \neg\sigma_{n,d}$. But this contradicts the Claim. □