1. Let $f\colon X \to [0,1]$ be continuous mapping A to 0 and B to 1. Extend f to βf and note $A \subseteq \beta f^{-1}(0), B \subseteq \beta f^{-1}(1)$ and these two are closed and disjoint, hence contain $\overline{A}$ and $\overline{B}$ respectively, so the latter are disjoint.

   Note (for later) that if $A \cup B = X$ then $\overline{A}^{\beta X} \cup \overline{B}^{\beta X} = \beta X$, so that $\overline{A}^{\beta X}$ and $\overline{B}^{\beta X}$ are both clopen in βX.

2. Let D be a countable dense subset of X and choose some surjection $f\colon \mathbb{N} \to D$. As $\mathbb{N}$ has the discrete topology, f is continuous and can thus be extended to a continuous βf. But then $\beta f(\beta\mathbb{N})$ is compact (so closed) and contains $f(\mathbb{N}) = D$, so contains $\overline{D} = X$. Hence βf is onto.

3. We will first show that $\mathcal{N}_p \cap \mathbb{N} = \{N \cap \mathbb{N}\colon N \in \mathcal{N}_p\}$ is an ultrafilter on $\mathbb{N}$: note that by density of $\mathbb{N}$ in $\beta\mathbb{N}$, we have that $\mathcal{N}_p \cap \mathbb{N}$ is a filter. If $A \notin \mathcal{N}_p \cap \mathbb{N}$ then by the previous part $\overline{A}, \overline{X \setminus A}$ partition $\beta\mathbb{N}$ into disjoint clopen sets and by assumption and $\overline{A} \cap \mathbb{N} = \overline{A}^{\mathbb{N}} = A$ we must have $\overline{X \setminus A} \in \mathcal{N}_p$ so that $X \setminus A \in \mathcal{N}_p \cap \mathbb{N}$. Because each $\beta\mathbb{N} \setminus \{n\} \in \mathcal{N}_p, n \in \mathbb{N}$, $\mathcal{N}_p \cap \mathbb{N}$ is a free ultrafilter on $\mathbb{N}$. By a question from the last sheet $\mathcal{N}_p \cap \mathbb{N}$, and hence $\mathcal{N}_p$, cannot have a countable filter basis.

   Metric spaces are first countable, showing that $\beta\omega$ is not metrizable. Similarly, $\omega \cup \{p\}$ can not be second countable (as otherwise the character of p would be at most countable). It is clearly countable and as only p is a non-trivial limit point of any closed set it is easily seen to be normal.

4. Writing $X \times Y = X \times \{y_0\} \cup \bigcup_{x \in X} \{x\} \times Y$ for some $y_0 \in Y$ and observing that $X \sim X \times \{y_0\}$ meets each $\{x\} \times Y \sim Y$ gives that if X, Y are connected, then so is $X \times Y$. By induction, a finite product of connected spaces is connected.

   If $X_i, i \in I$ is a collection of connected (non-empty) spaces, fix $x = (x_i) \in \prod_i X_i$. For each F finite $\subseteq I$ the set
   $$D_F = \prod_{i \in F} X_i \times \prod_{i \in I \setminus F} \{x_i\}$$
   is homeomorphic to $\prod_{i \in F} X_i$ and hence connected and clearly contains x. Thus
   $$D = \bigcup_{F \text{ finite } \subseteq I} D_F$$
   is connected.

   If F finite $\subseteq I$ and $\emptyset \neq U_i$ open $\subseteq X_i$ for $i \in F$ then picking $y_i \in U_i$ for $i \in F$ and $y_i = x_i$ for $i \in I \setminus F$ gives a point
   $$y \in D \cap {\pi_i}^{-1}(U_i).$$
   Thus D is dense in $\prod_i X_i$ and hence $\prod_i X_i = \overline{D}$ is connected.

5. Suppose X is compact Hausdorff and $C_n, n \in \omega$ is a decreasing sequence of closed connected sets. Let $C = \bigcap_n C_n$ and note that C is a closed subset of X. If C can be partitioned into non-empty C-clopen $A, B \subseteq C$ then A, B are in fact X-closed, and X is compact Hausdorff, so there are disjoint X-open $U \supseteq A, V \supseteq B$. Now $\bigcap_n C_n \subseteq U \cup V$ which is open and hence (by the Lemma about compactness) there is some $N \in \omega$ with $C_N \subseteq U \cup V$. Then $C_N \cap U$ and $C_N \cap V$ are disjoint C_N-open and hence C_N-clopen sets partitioning C_N, so one of them, say $V \cap C_N$ must be empty. But then $B \subseteq C \cap V \subseteq C_N \cap V = \emptyset$, a contradiction.

   For the example, let

$$X = \{(0,0), (1,0)\} \cup \{0,1\} \times [0,1] \cup \bigcup_{n \in \omega} [0,1] \times \left\{2^{-n}\right\} \subseteq \mathbb{R}^2$$

   and $C_n = X \cap [0,1] \times [0, 2^{-n}]$

6. Clearly if f is injective it is mono. If f is mono, let $x, y \in X$ with $f(x) = f(y)$, let $Z = \{\star\}$ and $g, h : Z \to X$ given by $g(\star) = x$ and $h(\star) = y$. Then $fg = fh$ and so $g = h$ which gives $x = y$ as required.

   Again, surjectivity clearly implies epi. For the converse, suppose there is $y \in Y \setminus f(X)$. As X is compact, so is $f(X)$ and hence there is clopen $C \ni y$ with $C \cap f(X) = \emptyset$. Let $g = \chi_C$ be the indicator function on C and set $h = 0$. Then $gf = 0 = hf$ but $g \neq h$.

7. Suppose $f : X \to Y$ is injective. Then f is a homeomorphism onto its image (since X is compact Hausdorff) and so wlog $X \subseteq Y$ and $f : X \to Y; x \mapsto x$ by identifying X with its image under f. Note that $\{C \cap X \colon C \text{ clopen } \subseteq Y\}$ is a basis for X. If D is X-clopen then it is a closed subset of a compact space and thus compact. Hence it is a finite union of basic X-clopen sets and thus there are Y-clopen $C_1, \ldots, C_n$ with

$$D = (C_1 \cup \ldots \cup C_n) \cap X = f^{-1}(C_1 \cup \ldots \cup C_n).$$

   Hence $D = \mathcal{B}_f(C_1 \vee \cdots \vee C_n)$. Since $D \in \mathcal{B}_X$ was arbitrary, $\mathcal{B}_f : \mathcal{B}_Y \to \mathcal{B}_X$ is surjective.

   Now assume $\phi : B \to A$ is surjetive and let $\mathcal{U}, \mathcal{V} \in \mathcal{S}(A)$ be distinct ultrafilters. Note that $\mathcal{S}(\phi)(\mathcal{U}) = \phi^{-1}(\mathcal{U})$. Let $a \in \mathcal{U} \setminus \mathcal{V}$ (wlog) and take $b \in B$ with $\phi(b) = a$ so $b \in \mathcal{S}(\phi)\mathcal{U} \setminus \mathcal{S}(\phi)\mathcal{V}$, so that $\mathcal{S}(\phi)$ is indeed injective.

   Assume $f : X \to Y$ is surjective and let $C, D \in \mathcal{B}_Y$ be distinct clopen subsets of Y. Find (wlog) $y \in C \setminus D$ and $x \in X$ with $f(x) = y$. Then $x \in f^{-1}(C) \setminus f^{-1}(D)$ so $\mathcal{B}_f(D) \neq \mathcal{B}_f(C)$ and $\mathcal{B}_f$ is indeed injective.

   Assume $\phi : B \to A$ is injective and let $\mathcal{U} \in \mathcal{S}(B)$ be an ultrafilter on B. Then $\{\phi(b) \colon b \in \mathcal{U}\}$ is closed under $\wedge$ and does not contain $\mathbf{0}$, since ϕ is injective so can only map $\mathbf{0}$ to $\mathbf{0}$, and thus can be extended to an ultrafilter $\mathcal{V}$ on A and by construction $\mathcal{S}(\phi)(\mathcal{V}) = \phi^{-1}(\mathcal{V}) = \mathcal{U}$. Hence $\mathcal{S}(\phi)$ is surjective.

8. Again, injective implies mono and surjective implies epi are clear.

   Assume that $\phi : B \to A$ is mono. We show that $\mathcal{S}(\phi) : \mathcal{S}(A) \to \mathcal{S}(B)$ is epi, hence surjective and thus ϕ must be injective.

   Thus let Z be a Stone space and $g, h : \mathcal{S}(B) \to Z$ be continuous with $\mathcal{S}(\phi)\, g = \mathcal{S}(\phi)\, h$. By Stone Duality, $Z = \mathcal{S}(C)$ and $g = \mathcal{S}(\psi_1), h = \mathcal{S}(\psi_2)$ for a Boolean Algebra C and homomorphisms $\psi_1, \psi_2 : C \to B$ and $\psi_1\phi = \psi_2\phi$. Since ϕ is mono this mean $\psi_1 = \psi_2$ and thus $g = \mathcal{S}(\psi_1) = \mathcal{S}(\psi_2) = h$ as required.

   [C] by hand Let $b, b' \in B$ with $f(b) = f(b')$ and consider the Boolean Algebra $C = \mathcal{P}(\{0,1\})$ (ordered by $\subseteq$). Let $g, h : C \to B$ be given by $g(\{1\}) = b$ (so that $g(\{0\}) = \neg b$, $g(\mathbf{0}) = \mathbf{0}$, $g(\mathbf{1}) = \mathbf{1}$) and $h(\{1\}) = b'$. Then $fh = fg$ since $f(h(\{0\})) = f(\neg b') = \neg f(b') = \neg f(b) = f(\neg b) = f(\{0\})$ and thus $g = h$ which gives $b = g(\{1\}) = h(\{1\}) = b'$.

   Now assume $\phi : B \to A$ is epi. We show that $\mathcal{S}(\phi) : \mathcal{S}(A) \to \mathcal{S}(B)$ is mono, hence injective and thus ϕ must be surjective.

   Thus let Z be a Stone space and $g, h : Z \to \mathcal{S}(A)$ be continuous with $g\mathcal{S}(\phi) = h\mathcal{S}(\phi)$. By Stone Duality, $Z = \mathcal{S}(C)$ and $g = \mathcal{S}(\psi_1), h = \mathcal{S}(\psi_2)$ for a Boolean Algebra C and homomorphisms $\psi_1, \psi_2 : A \to C$ and $\phi\psi_1 = \phi\psi_2$. Since ϕ is epi this mean $\psi_1 = \psi_2$ and thus $g = \mathcal{S}(\psi_1) = \mathcal{S}(\psi_2) = h$ as required.

   [C] by hand This is very tricky indeed.

9. **[A]** Clearly $\leq$ is a partial order on $A \times B$ with least element $(\mathbf{0}, \mathbf{0})$ and greatest element $(\mathbf{1}, \mathbf{1})$.

   We then note that $(a, b) \wedge (a', b') = (a \wedge a', b \wedge b')$, $(a, b) \wedge (a', b') = (a \vee a', b \vee b')$ and $\neg(a, b) = (\neg a, \neg b)$ work as expected.

   From these definitions it is immediate that π_A and π_B are Boolean Algebra homomorphisms.

   [B] An explicit isomorphism is provided by $\mathcal{B}_{X \oplus Y} \to \mathcal{B}_X \times \mathcal{B}_Y : C \mapsto (C \cap X, C \cap Y)$ with inverse $(A, B) \mapsto A \cup B$.