Summary sheet for elliptic PDEs

Luc Nguyen

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Let L denote the uniformly elliptic operator

$$Lu := -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu$$

where (a_{ij}) is symmetric and b_i and c are bounded. Let B denote the associated bilinear form.

Energy estimates

$$\frac{\lambda}{2}\|\nabla u\|^2_{L^2(\Omega)} \leq B[u,u] + C\|u\|^2_{L^2(\Omega)}.$$

Proof: Use Cauchy-Schwarz' inequality.

Existence 1.

If $b \equiv 0$ and $c \geq 0$ a.e. in Ω, then the boundary value problem for L is uniquely solvable for right hand side in $H^{-1}(\Omega)$ and zero boundary data. (The case of general Dirichlet boundary data follows.)

Proof 1: Show that $B[\cdot,\cdot]$ defines an equivalent inner product on $H^1_0(\Omega)$ and use the Riesz representation theorem.

Proof 2: Use the direct method of the calculus of variations with the following steps:

- Construct a variational functional whose critical points are exactly the solution of the desired boundary value problem.
- Show that every minimizing sequence (u_m) is bounded in H^1.
- Pass to a subsequence to obtain the weak convergence of (u_m) to u. Then prove the weak lower semi-continuity property

  $$\liminf_{m\to\infty} B[u_m,u_m] \geq B[u,u].$$

  Deduce that u minimizes the constructed function and deduces the existence of the solution to the BVP.
- Proof uniqueness by energy estimates.

Existence 2.

Fredholm alternative: (i) The Dirichlet boundary value problem is uniquely solvable if and only if $L|_{H_0^1(\Omega)}$ is injective. (ii) The kernels of $L|_{H_0^1(\Omega)}$ and $L^*|_{H_0^1(\Omega)}$ are finite dimensional and have the same dimension. (iii) The Dirichlet boundary value problem is solvable if and only if the right hand side satisfies the right 'orthogonality condition' with the kernel of $L^*|_{H_0^1(\Omega)}$.

Proof: Suffice to consider the case of zero boundary datum so that we only need to work in $H_0^1(\Omega)$. Let $L_0 = -\partial_i(a_{ij}\partial_j)$. Then $Lu = T$ if and only if

$$(I - K)u := u - L_0^{-1}(-b_i\partial_i u - cu) = L_0^{-1}T.$$

Then check that K is compact and use the functional analytic Fredholm alternative theorem.

Existence 3.

Via spectrum of compact operators.

Regularity 1.

Local version: Suppose $a \in C^1(\Omega)$ and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ satisfies $Lu = f$ in Ω then $u \in H^2_{loc}(\Omega)$, and for any open $\omega \Subset \Omega$ we have

$$\|u\|_{H^2(\omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Global version: Suppose that $a \in C^1(\bar{\Omega})$, $\partial\Omega$ is C^2, $f \in L^2(\Omega)$, $u_0 \in H^2(\Omega)$. If $u \in H_0^1(\Omega)$ satisfies $Lu = f$ in Ω and $u = u_0$ on $\partial\Omega$, then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u_0\|_{H^2(\Omega)}).$$

Proof of local version for constant coefficients:

- Reduce to Laplace operator by rotating/rescaling axes.
- Reduce to whole space using cut-off function.
- Reduce to the case of smooth function with compact support via convolution.
- Prove estimate for $u \in C_c^\infty$ using integration by parts.

Regularity 2.

If $u \in H^1(\Omega)$ satisfies $Lu = f$ in Ω and $f \in L^q(\Omega)$ with $q > \frac{n}{2}$, then u is locally Hölder continuous, and for any open $\omega \Subset \Omega$,

$$\|u\|_{C^{0,\alpha}(\omega)} \leq C(\|f\|_{L^q(\Omega)} + \|u\|_{H^1(\Omega)}).$$