Infinite Groups

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In the last lecture

We used an induction on the Hirsch length and

Proposition

Let G be a finitely generated nilpotent group and let $\varphi \in Aut(G)$. Then the polycyclic group $P = G \rtimes_{\varphi} \mathbb{Z}$ is

1. *either virtually nilpotent;*
2. *or has exponential growth.*

to prove

Theorem (Wolf's Theorem)

A polycyclic group is either virtually nilpotent or has exponential growth.

Milnor's theorem

Theorem (J. Milnor)

A finitely generated solvable group is either polycyclic or has exponential growth.

Lemma

If a finitely generated group G has sub-exponential growth then for all $\beta_1, \dots, \beta_m, g \in G$, the set of conjugates

$$\{g^k \beta_i g^{-k} \mid k \in \mathbb{Z}, i = 1, \dots, m\}$$

generates a finitely generated subgroup $N \leqslant G$.

Proof.

Exercise on Ex. Sheet 4. □

Proof of Milnor's theorem

Proof. By induction on the derived length d of G.
$d = 1$: G is finitely generated abelian, the statement is immediate.
Assume that the alternative is true for finitely generated solvable groups of derived length $\leqslant d$. Consider G of derived length $d+1$. $H = G/G^{(d)}$ is finitely generated solvable of derived length d. By the inductive assumption, either H has exponential growth or H is polycyclic. If H has exponential growth then G has exponential growth too.

Assume therefore that H is polycyclic. Milnor's Theorem will follow from:

Lemma

Consider a short exact sequence

$$1 \to A \to G \xrightarrow{\pi} H \to 1\,, \quad \textit{with } A \textit{ abelian and } G \textit{ finitely generated.} \tag{1}$$

If H is polycyclic then G is either polycyclic or has exponential growth.

Proof of the final lemma

Proof Assume G has sub-exponential growth. We will prove that G is polycyclic. The group G is polycyclic iff A is finitely generated.
Since H is polycyclic, it is finitely presented. This allows us to begin by proving that A is normally generated by a finite set.
An older set of ideas: We had proven that finite presentability is independent of the generating set.

Proposition

Assume $G = \langle S \mid R \rangle$ finite presentation, and $G = \langle X \mid T \rangle$ is such that X is finite. Then $\exists$ finite subset $T_0 \subset T$ such that $G = \langle X \mid T_0 \rangle$.

This can be reformulated as follows: If G is finitely presented, X is finite and

$$1 \to N \to F(X) \to G \to 1$$

is a short exact sequence, then N is normally generated by finitely many elements $n_1, \dots, n_k$.

Generalization of 'independence of finite presentability from the generating set'

This can be generalized to an arbitrary short exact sequence:

Lemma

Consider a short exact sequence

$$1 \to N \to K \stackrel{\pi}{\to} G \to 1\,, \quad \text{with } K \text{ finitely generated.} \tag{2}$$

If G is finitely presented, then N is normally generated by finitely many elements $n_1, \ldots, n_k \in N$.

Proof Let S be a finite generating set of K; then $\overline{S} = \pi(S)$ is a finite generating set of G. Since G is finitely presented, there exist finitely many words $r_1, \ldots, r_k$ in S such that

$$\langle \overline{S} \mid r_1(\overline{S}), \ldots, r_k(\overline{S}) \rangle$$

is a presentation of G.

Proof of generalization of 'independence of finite presentability from the generating set'

Define $n_j = r_j(S)$, an element of N by the assumption. We prove that the finite set $\{n_1, \ldots, n_k\}$ normally generates N.
Let n be an arbitrary element in N and $w(S)$ a word in S such that $n = w(S)$ in K. Then $w(\overline{S}) = \pi(n) = 1$, whence in $F(S)$ the word $w(S)$ is a product of finitely many conjugates of $r_1, \ldots, r_k$. When projecting such a relation *via* $F(S) \to K$ we obtain that n is a product of finitely many conjugates of $n_1, \ldots, n_k$. □

Back to our final Lemma: we have a short exact sequence

$$1 \to A \to G \xrightarrow{\pi} H \to 1\,, \quad \text{with } A \text{ abelian, } G \text{ fin. gen., } H \text{ polycyclic.} \qquad (3)$$

We assume G has sub-exponential growth, and deduce that G is polycyclic by proving that A is finitely generated.
H is polycyclic, hence finitely presented. Hence there exist finitely many elements $a_1, \ldots, a_k$ in A such that every element in A is a product of G-conjugates of $a_1, \ldots, a_k$.

H is polycyclic $\Rightarrow$ it has the bounded generation property: there exist finitely many elements $h_1, \dots, h_q$ in H such that every element $h \in H$ can be written as

$$h = h_1^{m_1} h_2^{m_2} \cdots h_q^{m_q}\,, \text{ with } m_1, m_2, \dots, m_q \in \mathbb{Z}\,.$$

Choose $g_i \in G$ such that $\pi(g_i) = h_i$ for every $i \in \{1, 2, \dots, q\}$. Then every element $g \in G$ can be written as

$$g = g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q} a\,, \text{ with } m_1, m_2, \dots, m_q \in \mathbb{Z} \text{ and } a \in A\,. \tag{4}$$

We have that $A = \langle\langle a_1, \dots, a_k \rangle\rangle$, and all the conjugates of a_j are of the form

$$g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q} a_j \left(g_1^{m_1} g_2^{m_2} \cdots g_q^{m_q} \right)^{-1}\,. \tag{5}$$

The subgroup A_q generated by the conjugates $g_q^m a_j g_q^{-m}$ with $m \in \mathbb{Z}$ and $j \in \{1, \dots, k\}$ is finitely generated. Let S_q be its finite generating set.

Proof of final Lemma, continued

The conjugates $g_{q-1}^n g_q^m a_j g_q^{-m} g_{q-1}^{-n}$ with $m, n \in \mathbb{Z}$ and $j \in \{1, \ldots, k\}$ are in the subgroup A_{q-1} of A generated by $g_{q-1}^n s g_{q-1}^{-n}$ with $n \in \mathbb{Z}$ and $s \in S_q$. The subgroup A_{q-1} is finitely generated. Continuing inductively, we conclude that the group A is finitely generated. Hence G is polycyclic. □

This also concludes the proof of Milnor's Theorem. □

By combining the theorems of Milnor and Wolf we obtain:

Theorem

Every finitely generated solvable group either is virtually nilpotent or it has exponential growth.

This was strengthened by J. Rosenblatt as follows:

Theorem (J. Rosenblatt)

Every finitely generated solvable group either is virtually nilpotent or it contains a free non-abelian subsemigroup.

Milnor's conjecture is true for linear groups.

Theorem (The Alternative Theorem of Jacques Tits)

Let F be a field of zero characteristic and let G be a fin. gen. subgroup of $GL(n, F)$. Then either G is virtually nilpotent or it has exponential growth.

In fact, what J. Tits proved is that G as above is either virtually solvable or it contains a free non-abelian subgroup.

This combined with Milnor-Wolf yields the result.

Milnor formulated a second conjecture: is a group with polynomial growth virtually nilpotent?

Theorem (Gromov's Polynomial Growth Theorem)

Every finitely generated group of growth at most polynomial is virtually nilpotent.

This is a typical example of an algebraic property that may be recognized *via* a, seemingly, weak geometric information.

Gromov's proof uses the Alternative Theorem.

Later, Y. Shalom and T. Tao proved the following effective version of Gromov's Theorem:

Theorem (Shalom–Tao Effective Polynomial Growth Theorem)

There exists a constant C such that for any finitely generated group G and $d > 0$, if for some $R \geqslant \exp\left(\exp\left(Cd^C\right)\right)$, the ball of radius R in G has at most R^d elements, then G has a finite index nilpotent subgroup of class less than C^d.

The following questions related to growth remain open.

Question

What is the set Growth(groups) of the equivalence classes of growth functions of finitely generated groups?

Question

Does there exist a finitely presented group of intermediate growth (that is, subexponential and superpolynomial) ?

Question

What are the equivalence classes of growth functions for finitely presented groups?

Clearly, $Growth(f.p.groups) \subset Growth(groups)$.
This inclusion is proper since R. Grigorchuk proved that there exist uncountably many nonequivalent growth functions of finitely generated groups, while there are only countably many finitely presented groups, up to isomorphism.

Theorem (Grigorchuk's Subexponential Growth theorem)

Let $f(n)$ be an arbitrary sub-exponential function larger than $2^{\sqrt{n}}$. Then there exists a finitely generated group G with subexponential growth function $\mathfrak{G}(n)$ such that:

$$f(n) \leqslant \mathfrak{G}(n)$$

for infinitely many $n \in \mathbb{N}$.

Question (R. Grigorchuk)

Is it true that if the growth of a finitely generated group is below $e^{\sqrt{n}}$ then it is polynomial?