§1. Projective geometry.

Lectures 1–3 will introduce projective geometry.

Reading: Hitchin notes, §1.

§1.1. Projective spaces

Let $\mathbb{F}$ be a field. We usually take $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, but you could also take $\mathbb{F} = \mathbb{Z}_p$, p prime, etc.

Define the projective space $P(V)$ associated to V to be the set of 1-dimensional vector subspaces of V.

For each $v \neq 0$ in V, write $[v] = \{\lambda v : \lambda \in \mathbb{F}\}$.

Then $P(V) = \{[v] : 0 \neq v \in V\}$, where $[v] = [w]$ iff $v = \lambda w$ for $0 \neq \lambda \in \mathbb{F}$.

Define the dimension of $P(V)$ to be $\dim V - 1$.

$\dim P(V) = 0 \Rightarrow P(V)$ is a point.

$\dim P(V) = 1 \Rightarrow P(V)$ is a projective line.

$\dim P(V) = 2 \Rightarrow P(V)$ is a projective plane.

Suppose that U is a vector subspace of V. (The notation for this is $U \leq V$.) Then $P(U) \subseteq P(V)$.

We say that $P(U)$ is a *projective linear subspace* of $P(V)$. A linear subspace $P(U)$ with $\dim P(U) = 1$ is called a *line* in $P(V)$. A linear subspace $P(U)$ with $\dim P(U) = \dim P(V) - 1$ is called a *hyperplane* in $P(V)$.

Write $\mathbb{RP}^n = P(\mathbb{R}^{n+1})$, $\mathbb{CP}^n = P(\mathbb{C}^{n+1})$, and $\mathbb{P}^n$ or $\mathbb{P}_n$ for n-dimensional projective space.

§1.2. Coordinates on projective spaces.

Let V be a finite dim'l vector space over $\mathbb{F}$, with $\dim V = n+1$. Choose a basis $v_0, v_1, \ldots, v_n$ for V. Then every $v \in V$ can be written uniquely as

$$v = \sum_{j=0}^{n} x_j v_j, \quad \text{for } x_j \in \mathbb{F}.$$

Definition Let $x_0, \ldots, x_n \in \mathbb{F}$ be not all zero. Write $[x_0, x_1, \ldots, x_n]$ for the point $p = [x_0 v_0 + \cdots + x_n v_n]$ in $P(V)$. Every point in $P(V)$ can be written this way. We call $[x_0, \ldots, x_n]$ *homogeneous coordinates* for p.

Note that $[x_0, x_1, \ldots, x_n]$ and $[y_0, y_1, \ldots, y_n]$ represent the same point in $P(V)$ iff the vectors $x_0 v_0 + \cdots + x_n v_n$ and $y_0 v_0 + \cdots + y_n v_n$ are *proportional*, that is, iff $y_j = \lambda x_j$, $j = 0, \ldots, n$, some $\lambda \in \mathbb{F} \setminus \{0\}$. So, the homogeneous coordinates of a point are not unique, but can always be rescaled by a nonzero constant.

Now let $P(V)$ be a projective space of dimension n, with homogeneous coordinates $[x_0, x_1, \dots, x_n]$. Let U_0 be the set of points $[x_0, x_1, \dots, x_n]$ in $P(V)$ with $x_0 \neq 0$. Put $y_j = \frac{x_j}{x_0}$, $j = 1, \dots, n$. Then $[x_0, x_1, \dots, x_n] = [1, y_1, \dots, y_n]$, rescaling by x_0. Represent the point $[1, y_1, \dots, y_n] \in U_0$ by $(y_1, \dots, y_n)$. We say that $(y_1, \dots, y_n)$ are <u>inhomogeneous coordinates</u> on U_0.

Inhomogeneous coordinates are coordinates of the usual sort (not like homogeneous coordinates). They give a 1-1 correspondence between U_0 and $\mathbb{F}^n$. They are defined only on a subset U_0 of $P(V)$.

Now $P(V) \setminus U_0$ is the set of points $[x_0, x_1, \dots, x_n]$ with $x_0 = 0$, which is $P(W)$, where W is the vector subspace of V spanned by $v_1, \dots, v_n$, so that $P(V)$ is a projective space of dimension $n-1$.

That is, $\mathbb{C}P^n = \mathbb{C}^n \amalg \mathbb{C}P^{n-1}$,

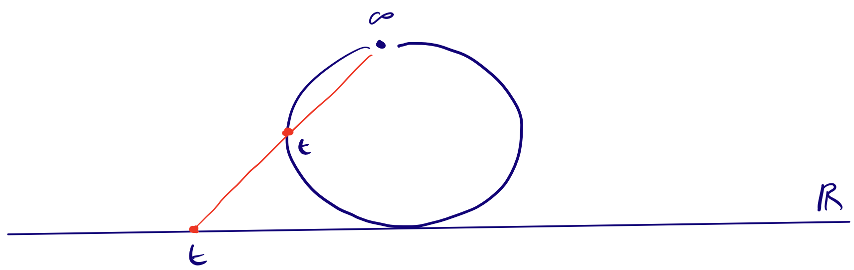
$\mathbb{R}P^n = \mathbb{R}^n \amalg \mathbb{R}P^{n-1}$,

$\mathbb{P}^n = \mathbb{F}^n \amalg \mathbb{P}^{n-1}$.

Think of $\mathbb{C}P^{n-1}$ as "points at infinity in $\mathbb{C}^n$".

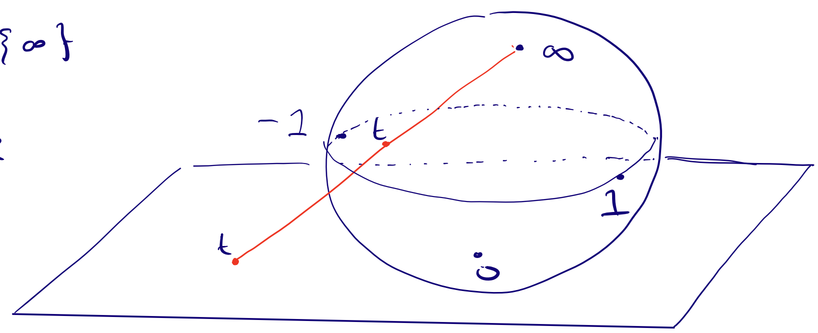
So, for example, $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$

is a circle



$\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$

is a 2-sphere



Call $\mathbb{R}^n$, $\mathbb{C}^n$ affine space, in comparison with $\mathbb{R}P^n, \mathbb{C}P^n$.

Things are often simpler in projective geometry than in affine geometry.

Example. In $\mathbb{R}^2$, two distinct lines meet in a unique point unless they are parallel.

In $\mathbb{R}P^2$, two distinct lines meet in a unique point.

Parallel lines in $\mathbb{R}^2$ meet in a "point at infinity" in $\mathbb{R}P^2$.

§1.3. Examples of proofs in projective geometry.

Proposition Let ℓ_1, ℓ_2, ℓ_3 be lines in $P(\mathbb{R}^4)$ that do not intersect pairwise. Then there are an infinite number of lines ℓ that intersect ℓ_1, ℓ_2 and ℓ_3.

Proof. Let $\ell_j = P(U_j)$, $j = 1, 2, 3$. Then U_1, U_2, U_3 are vector subspaces of $\mathbb{R}^4$, $\dim U_j = 2$.

Since $\ell_1 \cap \ell_2 = \emptyset$, $U_1 \cap U_2 = \{0\}$. Hence $\mathbb{R}^4 = U_1 \oplus U_2$, counting dimensions.

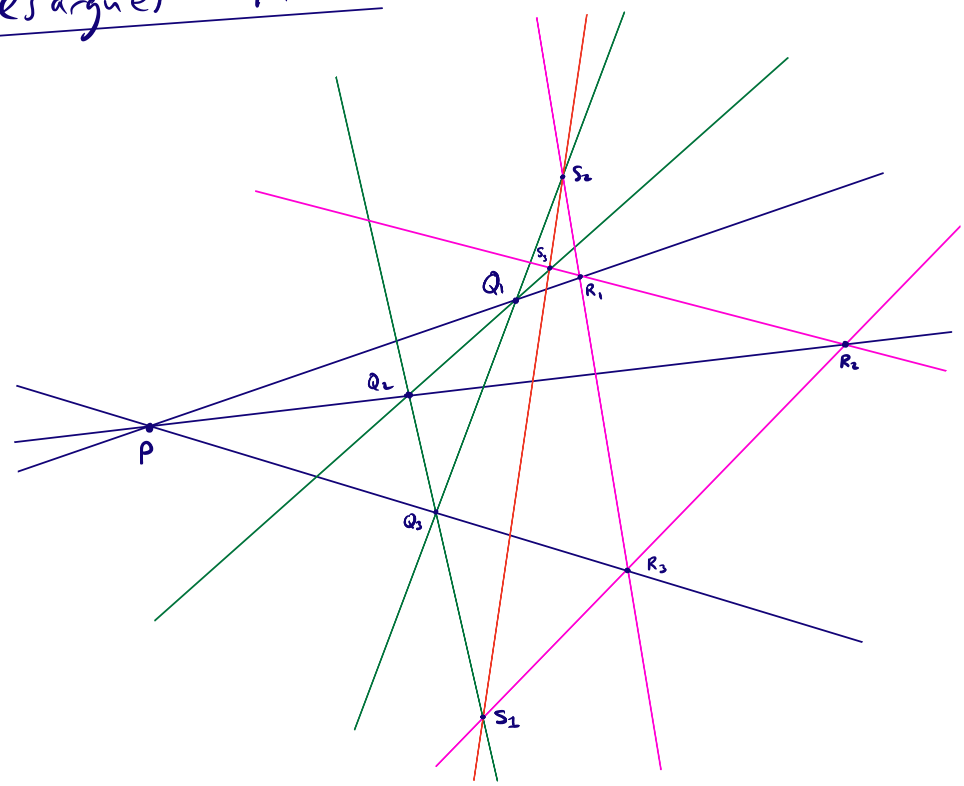
Let $v \in U_3$ be nonzero. Then $v = u_1 + u_2$ with $u_1 \in U_1$, $u_2 \in U_2$, as $U_3 \subset \mathbb{R}^4 = U_1 \oplus U_2$.

If $u_1 = 0$ then $[v] \in \ell_2 \cap \ell_3$, $\times$, and if $u_2 = 0$ then $[v] \in \ell_1 \cap \ell_3$, $\times$. Hence u_1, u_2 and v are all nonzero.

Let $U_4 = \langle u_1, u_2 \rangle$. Then $\dim U_4 = 2$, so $\ell_4 = P(U_4)$ is a line in $P(\mathbb{R}^4)$. Also, ℓ_4 contains $[u_1] \in \ell_1$, $[u_2] \in \ell_2$ and $[v] = [u_1 + u_2] \in \ell_3$. So ℓ_4 intersects ℓ_1, ℓ_2 and ℓ_3. By construction, there exists such a line ℓ through every point $[v]$ of ℓ_3, and different $[v]$ give different lines ℓ, so there are infinitely many such lines. Q.E.D.

We have proved a nontrivial geometric fact, using algebraic methods. This is typical of proofs in projective geometry. Here is a more difficult proof.

Desargues Theorem



Let $P, Q_1, Q_2, Q_3, R_1, R_2, R_3$ be distinct points in a projective plane $P(V)$, such that the lines Q_1R_1, Q_2R_2, Q_3R_3 are distinct and concurrent at P.
Let S_1 be the intersection of the lines Q_2Q_3, R_2R_3.
Let S_2 be the intersection of the lines Q_3Q_1, R_3R_1
Let S_3 be the intersection of the lines Q_1Q_2, R_1R_2.
Let S_3 be the intersection of

Then S_1, S_2, S_3 are collinear.

<u>Proof.</u> Choose representatives p, q_1, r_1 etc. in V for P, Q_1, R_1, etc.

As P, Q_1, R_1 lie on a line, p, q_1, r_1 are linearly dependent.

By rescaling q_1, r_1, we can ensure that

$$p = q_1 + r_1,$$ and similarly $$p = q_2 + r_2,$$ $$p = q_3 + r_3.$$

As $p = q_2 + r_2 = q_3 + r_3$, $q_2 - q_3 = r_3 - r_2 = s_1$, say.

Since Q_2, Q_3 are distinct, $q_2 - q_3$ is non zero. Hence, s_1 is the representative of a point on $Q_2 Q_3$. But $s_1 = r_3 - r_2$, so s_1 represents a point on $R_2 R_3$.

Thus s_1 represents S_1, the intersection of $Q_2 Q_3$ and $R_2 R_3$.

Similarly put $s_2 = q_3 - q_1 = r_1 - r_3$, and

$s_3 = q_1 - q_2 = r_2 - r_1$, and

s_2, s_3 represent S_2, S_3.

Now $s_1 + s_2 + s_3 = (q_2 - q_3) + (q_3 - q_1) + (q_1 - q_2) = 0$.

Hence S_1, S_2, S_3 are collinear, as s_1, s_2, s_3 are linearly dependent.

Q.E.D.

§ 1.4. Projective transformations

Let V, W be vector spaces of the same dimension over $\mathbb{F}$.
Let $T: V \to W$ be an invertible linear transformation.
If $v \neq 0$ lies in V, then $T([v]) = [Tv]$, where $[v]$ is a vector subspace of V, and $[Tv]$ a vector subspace of W.
As T is invertible, $Tv \neq 0$. Thus $[v] \in P(V)$ and $[Tv] \in P(W)$.
Hence, T induces an invertible map $\tau: P(V) \longrightarrow P(W)$ by $\tau([v]) = [Tv]$ for all $0 \neq v \in V$.
We call τ a <u>projective transformation</u>.
— In fact, we can define a transformation $\tau: P(V) \to P(W)$ if T is only <u>injective</u>, rather than invertible. But if T is not injective, then τ is not well-defined on the subset $P(\operatorname{Ker} T)$ of $P(V)$.
If $0 \neq \lambda \in \mathbb{F}$, then the linear transformations T and λT define the same projective transformation τ.
When $V = W$, the projective transformations $\tau: P(V) \to P(W)$ define a <u>group</u> under composition.

In: §1.4 Projective transformations.

Recall: V, W vector spaces of the same dimension over $\mathbb{F}$.
$T: V \to W$ invertible linear transformation.
Then $\tau: P(V) \to P(W)$, $\tau([v]) = [Tv]$
is a projective transformation.

Example. $\mathbb{C}P^1 = P(\mathbb{C}^2)$ can be written as
$\mathbb{C}P^1 = \{[1,z] : z \in \mathbb{C}\} \cup \{[0,1]\}$.

Identify $[1,z] \cong z$, $[0,1] \cong \infty$.

Then $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$, the Riemann sphere from complex analysis.

Let $\tau: \mathbb{C}P^1 \to \mathbb{C}P^1$ be a projective transformation induced by a linear transformation $T: \mathbb{C}^2 \to \mathbb{C}^2$.

Write action on $[z_0, z_1]$ as

N.B. unconventional to use $\begin{pmatrix} z_1 \\ z_0 \end{pmatrix}$ not $\begin{pmatrix} z_0 \\ z_1 \end{pmatrix}$.

$$T: \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_0 \end{pmatrix} = \begin{pmatrix} az_1 + bz_0 \\ cz_1 + dz_0 \end{pmatrix}.$$

So, $\tau: [z_0, z_1] \longmapsto [cz_1 + dz_0, az_1 + bz_0]$.

If $z \in \mathbb{C}$ and $cz + d \neq 0$, we have $\tau([1,z]) = \left[1, \frac{az+b}{cz+d}\right]$.

Therefore, in inhomogeneous coordinates, we have

$$\tau(z) = \begin{cases} \dfrac{az+b}{cz+d}, & z \in \mathbb{C},\ cz+d \neq 0, \\ \infty, & z \in \mathbb{C},\ cz+d = 0, \\ a/c, & z = \infty. \end{cases}$$

This is a <u>Möbius transformation</u>, from complex analysis.
Hence, projective transformations of $\mathbb{C}P^1$ are the same as Möbius transformations.

§1.5. Points in general position.

Let V be a vector space of dimension $n+1$, and $P(V)$ the associated projective space of dimension n.
Let $p_0, \dots, p_{n+1}$ be points in $P(V)$, and let $v_0, \dots, v_{n+1}$ be representative vectors for $p_0, \dots, p_{n+1}$, respectively.

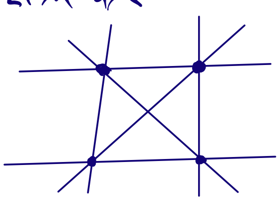
<u>Definition.</u> We say that $p_0, p_1, \dots, p_{n+1}$ are in <u>general position</u> if

(i) no $n+1$ of $v_0, v_1, \dots, v_{n+1}$ are linearly dependent, or equivalently, (ii) no $n+1$ of $p_0, p_1, \dots, p_{n+1}$ lie in a hyperplane.

(i) algebraic condition; (ii) geometric condition.

[<u>Recall</u>: A <u>hyperplane</u> is a projective linear subspace of $P(V)$, of dimension $\dim P(V) - 1$.]

<u>Examples</u>
- $n=1$, $P(V)$ projective line. 3 points are in general position if they are <u>distinct</u>.
- $n=2$, $P(V)$ projective plane. 4 points are in general position if no 3 of them are collinear.



Here is the main result about points in general position.

<u>Theorem 1.1.</u> Let V, W be vector spaces of dimension $n+1$, and let $p_0, \dots, p_{n+1}$ and $q_0, \dots, q_{n+1}$ be in general position in $P(V)$ and $P(W)$, respectively. Then there exists a unique projective transformation $\tau : P(V) \to P(W)$ such that $\tau(p_i) = q_i$, $i = 0, \dots, n+1$.

<u>Proof.</u> Choose $v_0, \dots, v_{n+1}$ in V representing $p_0, \dots, p_{n+1}$. Then $v_0, \dots, v_n$ are linearly independent, as $p_0, \dots, p_{n+1}$ are in general position. So $v_0, \dots, v_n$ form a <u>basis</u> for V. Thus $v_{n+1} = \lambda_0 v_1 + \dots + \lambda_n v_n$ for some $\lambda_0, \dots, \lambda_n \in \mathbb{F}$.

Suppose any $\lambda_j = 0$. Then $v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n, v_{n+1}$ are linearly independent, a contradiction. Thus $\lambda_j \neq 0$, all j. Replace v_j by $\lambda_j v_j$ for $j = 0, \dots, n$. Then we have representative vectors $v_0, \dots, v_{n+1}$ for $p_0, \dots, p_{n+1}$ with

$$v_{n+1} = v_0 + \dots + v_n.$$

Similarly, choose representative vectors $w_0, \dots, w_{n+1}$ in W for $q_0, \dots, q_{n+1}$ with $w_{n+1} = w_0 + \dots + w_n$.

Since $v_0, \dots, v_n$ and $w_0, \dots, w_n$ are bases for V, W there exists a unique invertible linear transformation $T : V \to W$ with $T(v_j) = w_j$, $j = 0, \dots, n$.

But then $T(v_{n+1}) = T(v_0 + \dots + v_n) = w_0 + \dots + w_n = w_{n+1}$.

So $T(v_j) = w_j$, $j = 0, \dots, n+1$.

Let $\tau: P(V) \to P(W)$ be the induced projective transformation. Then $\tau(p_j) = q_j$, $j = 0, \dots, n+1$.

Suppose $\sigma: P(V) \to P(W)$ is projective with $\sigma(p_j) = q_j$, all j. Let σ be induced by $S: V \to W$.

Then $Sv_j = \lambda_j w_j$, some constants $\lambda_j \in \mathbb{F} \setminus \{0\}$.

So $\lambda_{n+1}(w_0 + \dots + w_n) = \lambda_{n+1} w_{n+1} = S(v_{n+1}) = S(v_0 + \dots + v_n)$

$= \lambda_0 w_0 + \dots + \lambda_n w_n$.

As $w_0, \dots, w_n$ is a basis, $\lambda_j = \lambda_{n+1}$, $j = 0, \dots, n$.

Thus $S = \lambda_{n+1} T$, $\sigma = \tau$, and τ is unique. Q.E.D.

Remark. A set of points in general position in a projective space, is very similar to a basis of a vector space.

Example If p_0, p_1, p_2 and q_0, q_1, q_2 are triples of unique points in $\mathbb{C}P^1$, there is a unique Möbius transformation/projective transformation $\tau: \mathbb{C}P^1 \to \mathbb{C}P^1$ with $\tau(p_i) = q_i$, $i = 0, 1, 2$.

Find τ with $\tau[1,0] = [1,i]$, $\tau[0,1] = [1,-i]$, $\tau[1,1] = [1,1]$.

Let τ be induced by $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$\begin{pmatrix} a \\ c \end{pmatrix} = \lambda_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\begin{pmatrix} b \\ d \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \Rightarrow \lambda_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_0 \begin{pmatrix} 1 \\ i \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$.

Choose $\lambda_0 = 1$. Solve to get $\lambda_1 = i$, $\lambda_2 = 1+i$,

$$T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

Remark Our favourite set of points in general position in $\mathbb{C}P^n$ is $[1,0,\dots,0], [0,1,0,\dots,0], \dots, [0,\dots,0,1], [1,1,\dots,1]$.

§2. Plane curves

§2.1. Affine and projective curves

Definition. An <u>algebraic curve in $\mathbb{C}^2$</u> is $C = \{(x,y) \in \mathbb{C}^2 : p(x,y) = 0\}$,

for $p(x,y)$ some nonzero polynomial,

i.e. $p(x,y) = \sum_{\substack{j,k \geq 0 \\ j+k \leq d}} a_{jk} x^j y^k$, $a_{jk} \in \mathbb{C}$, $a_{jk} \neq 0$, some $j+k = d$, $d =$ <u>degree</u> of p.

Example $x^2+y^2 = 1 \iff p(x,y) = x^2+y^2-1$.

These are sometimes called <u>affine curves</u> (affine = to do with $\mathbb{C}^n$). Can also work in $\mathbb{F}^2$, $\mathbb{F}$ field.

Definition Let $P(x,y,z)$ be a complex polynomial. Call P <u>homogeneous of degree d</u> if

$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x,y,z)$ for all $\lambda, x, y, z \in \mathbb{C}$.

This holds iff $P(x,y,z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k$ with

$a_{ijk} = 0$ unless $i+j+k = d$.

An <u>algebraic curve in $\mathbb{CP}^2$</u> is

$C' = \{[x,y,z] \in \mathbb{CP}^2 : P(x,y,z) = 0\}$, for some $P(x,y,z)$ nonzero and homogeneous of degree $d > 0$.

Also in $\mathbb{RP}^2$, $\mathbb{FP}^2$, $\mathbb{F}$ field.

Note that for $0 \neq \lambda \in \mathbb{C}$, $[x,y,z] = [\lambda x, \lambda y, \lambda z]$.

So $P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x,y,z)$, and

$P(\lambda x, \lambda y, \lambda z) = 0 \iff P(x,y,z) = 0$.

Thus the condition $P(x,y,z) = 0$ is independent of the choice of representative for $[x,y,z]$, if P homogeneous.

Here is how to relate algebraic curves in $\mathbb{C}^2$ and $\mathbb{CP}^2$.

Embed $\mathbb{C}^2$ in $\mathbb{CP}^2$ by $(x,y) \mapsto [x,y,1]$,

so $\mathbb{C}^2 = \{[x,y,z] \in \mathbb{CP}^2 : z \neq 0\}$.

If C' is a curve in $\mathbb{CP}^2$ defined by $P(x,y,z)$ then $C = C' \cap \mathbb{C}^2$ is a curve in $\mathbb{C}^2$ defined by $p(x,y) = P(x,y,1)$.

If C is a curve in $\mathbb{C}^2$ defined by $p(x,y)$, of degree d, then $P(x,y,z) = z^d p\left(\frac{x}{z}, \frac{y}{z}\right)$ is a homogeneous polynomial of degree d, and if C' is the curve in $\mathbb{CP}^2$ associated to P, then $C = C' \cap \mathbb{C}^2$.

So $C' = C \cup \underbrace{\{\text{points at infinity}\}}_{= \{[x,y,0] \in \mathbb{CP}^2 : P(x,y,0) = 0\}}$.

To make P: replace $a_{jk} x^j y^k$ in p by $a_{jk} x^j y^k z^{d-j-k}$.

As a topological space, $\mathbb{CP}^2$ is compact, and algebraic curves are closed, so algebraic curves in $\mathbb{CP}^2$ are compact.

Algebraic curves in $\mathbb{C}^2$ are always non compact, so adding points at infinity to make projective curves compactifies them.

<u>Example</u>: $C = \{(x,y) \in \mathbb{C}^2 : x^2+y^2=1\}$ is isomorphic to $\mathbb{C} \setminus 0 = \mathbb{CP}^1 \setminus (2 \text{ points})$.

The corresponding projective curve is

$$C' = \{[x,y,z] \in \mathbb{CP}^2 : x^2+y^2-z^2=0\} \cong \mathbb{CP}^1 \cong S^2$$

$$= C \cup \{\underbrace{[1,i,0], [1,-i,0]}\}$$

solutions of $x^2+y^2=0$; points at infinity.

§2.2. Facts from algebra and algebraic geometry.

Recall facts about commutative rings, e.g. in Herstein, "Topics in Algebra".

- A <u>unit</u> in a commutative ring R is $a \in R$ s.t. $\exists\, b \in R$ with $ab=1$.
- An <u>irreducible element</u> $a \in R$ is such that if $a = bc$ in R then b or c is a unit.
- Call R a <u>unique factorization domain</u> (<u>UFD</u>) if R has no zero divisors, and every element of R has a factorization into irreducible elements, unique up to order and multiplication by units.

<u>Theorem</u> Let $\mathbb{F}$ be a field. Then the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ is a U.F.D. Units are constants $\mathbb{F} \setminus 0$. So, can always factorize polynomials into irreducible (prime) factors.

– like factorizing integers into prime factors.

A field $\mathbb{F}$ is called <u>algebraically closed</u> if every polynomial $p(x)$ over $\mathbb{F}$ factorizes into linear factors. $\mathbb{C}$ is algebraically closed. $\mathbb{R}$ is not algebraically closed (x^2+1 doesn't factorize). $\mathbb{Z}_p$ is not alg. closed.

"zero theorem" in German.

<u>Theorem</u> (Hilbert's Nullstellensatz).

Let $\mathbb{F}$ be an algebraically closed field, and $P, Q \in \mathbb{F}(x_1, \dots, x_n)$. Then

$$\{(x_1,\dots,x_n) \in \mathbb{F}^n : P(x_1,\dots,x_n) = 0\} = \{(x_1,\dots,x_n) \in \mathbb{F}^n : Q(x_1,\dots,x_n) = 0\}$$

$\iff$ P divides Q^m and Q divides P^n, some $m, n > 0$

$\iff$ P and Q have the same irreducible factors, possibly with different multiplicities.

— Proof beyond the scope of the course.

<u>Consequence of Hilbert's Nullstellensatz:</u>

If C' is a curve in $\mathbb{CP}^2$ defined by a polynomial $P(x,y,z)$ with no repeated factors, then C' determines P up to rescalings $P \mapsto \lambda P$, $\lambda \in \mathbb{C} \setminus 0$.

Not true in $\mathbb{RP}^2$, e.g. $x^2+y^2+z^2=0$, $x^2+2y^2+3z^2=0$ give the same curve $C' = \emptyset$.

§2.3 Irreducible and nonsingular curves

Definition Let C be an algebraic curve in $\mathbb{CP}^2$. Then C is defined by a homogeneous polynomial $P(x,y,z)$. Taking P to have no repeated factors, P is determined by C uniquely up to rescaling, $P \mapsto \lambda P$, $\lambda \in \mathbb{C} \setminus 0$ (Nullstellensatz).

The degree of C is the degree of P.

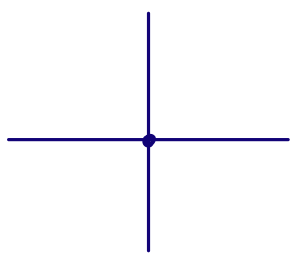
We call C irreducible if P is irreducible, i.e. the only factors of P are constants and multiples of P.

N.B. Notion of irreducible not well-behaved over non algebraically closed fields, as C does not determine P up to scale.

An irreducible curve D is called a component of C if its defining polynomial Q is a factor of P. Then $D \subseteq C$.

Since P has a factorization $P = Q_1 \cdots Q_n$ into irreducible factors, C has a decomposition $C = D_1 \cup \cdots \cup D_n$ into irreducible components, unique up to order.

Example: $C: xy = 0$ is the union of two components $x=0$ and $y=0$.



Definition Let C be a curve in $\mathbb{CP}^2$ defined by $P(x,y,z)$ without repeated factors.

We say that $[a,b,c]$ is a <u>singular point</u> of C if $P(a,b,c)=0$ and $\frac{\partial P}{\partial x}(a,b,c)=\frac{\partial P}{\partial y}(a,b,c)=\frac{\partial P}{\partial z}(a,b,c)=0$.

Remark. If P is homogeneous of degree d then differentiating $P(\lambda x,\lambda y,\lambda z)=\lambda^d P(x,y,z)$ w.r.t. λ at $\lambda=1$ gives <u>Euler's relation</u>:

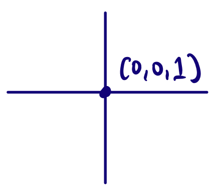
$$x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z}=d\,P.$$

Learn this.

Hence $\frac{\partial P}{\partial x}(a,b,c)=\frac{\partial P}{\partial y}(a,b,c)=\frac{\partial P}{\partial z}(a,b,c)=0$ imply $P(a,b,c)=0$.

Example. Let $P(x,y,z)=xy$.

Then $\frac{\partial P}{\partial x}(a,b,c)=b$, $\frac{\partial P}{\partial y}(a,b,c)=a$, $\frac{\partial P}{\partial z}(a,b,c)=0$, so the only singular point is $[0,0,1]$, the intersection point of the two components $x=0$, $y=0$.



A singular point is where the curve is not smooth, i.e. does not look like a small piece of $\mathbb{C}$.

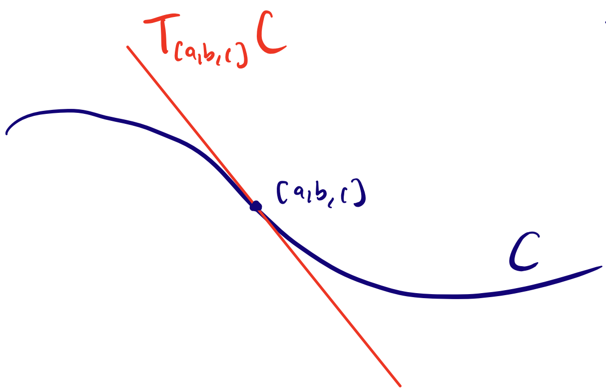
A point of C which is not singular is called <u>nonsingular</u>.

A curve C is <u>nonsingular</u> if it has no singular points.

Definition Let C be a curve in $\mathbb{CP}^2$ defined by $P(x,y,z)$ without repeated factors, and suppose $[a,b,c]$ is a nonsingular point of C. Then the <u>tangent line</u> to C at $[a,b,c]$ is

$$T_{[a,b,c]}C = \left\{ [x,y,z] \in \mathbb{CP}^2 : \ x\frac{\partial P}{\partial x}(a,b,c) + y\frac{\partial P}{\partial y}(a,b,c) + z\frac{\partial P}{\partial z}(a,b,c) = 0 \right\}.$$

This is a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$, since $\frac{\partial P}{\partial x}(a,b,c), \dots, \frac{\partial P}{\partial z}(a,b,c)$ are not all zero, as $[a,b,c]$ is nonsingular.



The tangent line passes through $[a,b,c]$ (by Euler's relation) and "touches" C there.

At a singular point there may be more than one tangent line — see Sheet 2, q. 6.

e.g. at $(0,0,1)$ in $C: xy = 0$, two tangent lines $x = 0$ and $y = 0$.

§2.4. Conics

A <u>conic</u> C in $\mathbb{CP}^2$ is a curve of degree 2, that is, C is defined by a homogeneous quadratic polynomial $P(x,y,z)$. We can write P in matrix form as

$$P(x,y,z) = (x \ y \ z)\, A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A is a 3×3 symmetric matrix over $\mathbb{C}$.

We are interested in classifying conics up to projective transformations.

Definition. A <u>symmetric bilinear form</u> on a vector space V is a map $B: V \times V \to \mathbb{C}$ with $B(v,w) = B(w,v)$ and $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1,w) + \lambda_2 B(v_2,w)$, all $v, v_1, v_2, w \in V$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

For $\mathbb{F}$ not of characteristi 2.

Then $P(v) = B(v,v)$ is a homogeneous quadratic polynomial on V. Conversely, any homogeneous quadratic polynomial comes from a unique symmetric bilinear form.

$$B(v,w) = \frac{1}{2}\left(P(v+w) - P(v) - P(w) \right).$$

Theorem 2.1. Let V be a complex vector space of dimension n, and B a symmetric bilinear form on V. Then there exists a basis $v_1, \dots, v_n$ for V such that if $v = \sum_{i=1}^{n} z_i v_i$ then $B(v,v) = \sum_{i=1}^{m} z_i^2$, some $0 \le m \le n$.

$$A = \begin{array}{cc} & \begin{array}{cc} m & n-m \end{array} \\ \begin{array}{c} m \\ n-m \end{array} & \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right) \end{array}$$

$m =$ rank of matrix.

Last time we stated:

<u>Theorem 2.1.</u> Let V be a complex vector space of dimension n, and B a symmetric bilinear form on V. Then there exists a basis $v_1, \dots, v_n$ for V such that if $v = \sum_{i=1}^{n} z_i v_i$ then $B(v,v) = \sum_{i=1}^{m} z_i^2$, some $0 \le m \le n$.

<u>Proof.</u> We will construct $(v_1, \dots, v_k)$ by induction on $k = 1, \dots, m$, and then $v_{m+1}, \dots, v_n$ in one step.

<u>Inductive hypothesis.</u> Suppose we have chosen linearly independent $v_1, \dots, v_k$ such that if $v = z_1 v_1 + \dots + z_k v_k$ then $B(v,v) = z_1^2 + \dots + z_k^2$.

<u>First step</u> If $B = 0$ then any basis $v_1, \dots, v_n$ will do, with $m = 0$.

If $B \ne 0$ then choose v' with $B(v',v') \ne 0$ and set $v_1 = \frac{1}{B(v',v')^{1/2}} v'$.

N.B. doesn't work over $\mathbb{R}$: need arbitrary square roots.

<u>Inductive step.</u> Suppose $v_1, \dots, v_k$ given, $k < n$.

Define $W = \{ w \in V : B(v_i, w) = 0, \ i = 1, \dots, k \}$.

As $B(v_i, v_j) = \delta_{ij}$, have $V = \langle v_1, \dots, v_k \rangle \oplus W$,

so $\dim W = n - k > 0$.

If $B|_{W \times W} = 0$, let $v_{k+1}, \dots, v_n$ be any basis for W, and finish with $m = k$.

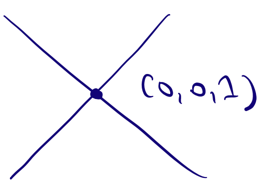
Otherwise choose $w' \in W$ with $B(w', w') \neq 0$, and set $v_{k+1} = \frac{1}{B(w',w')^{1/2}} w'$. Then proof holds by induction. $\square$

<u>Corollary.</u> Let C be a conic in $\mathbb{CP}^2$. Then after a projective transformation, C is equivalent to one of the following:

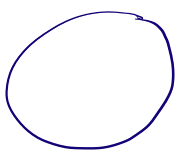
(a) $x^2 = 0$: the (double) line $x = 0$.

Really this is a degree 1 curve $\mathbb{CP}^1$, not a true conic. Considered as a conic, every point of C is singular. So this does not count as a true conic.

(b) $x^2 + y^2 = 0$: this is a pair of lines $x + iy = 0$, $x - iy = 0$, meeting at $(0, 0, 1)$, which is a singular point.

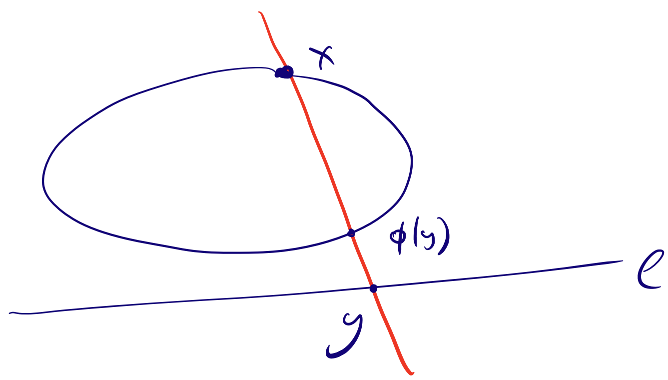


(c) $x^2 + y^2 + z^2 = 0$: a <u>nonsingular</u> conic



— So all nonsingular conics are equivalent under projective transformations.

Theorem 2.2. Let C be a nonempty, nonsingular conic in a projective plane $P(V)$ over any field $\mathbb{F}$. Choose a point $x \in C$ and a line ℓ in $P(V)$ not containing x. Then there is a unique bijection $\phi : \ell \to C$ such that x, y and $\phi(y)$ are collinear for all $y \in \ell$.



Sketch proof. (See Hitchin notes § 2.3, Sheet 0, q.3.)
Let C be defined by homogeneous quadratic $Q : V \to \mathbb{F}$. Let $y \in \ell$, and $x = [u]$, $y = [v]$. Then $Q(\alpha u + \beta v) = a\alpha^2 + b\alpha\beta + c\beta^2$, for $u, v \in \mathbb{F}$, some $a, b, c \in \mathbb{F}$. As $x = [u] \in C$, $a = 0$.
As C is nonsingular it does not contain a line, so b, c not both zero. Thus $cu - bv \neq 0$, and

$$Q(cu - bv) = 0 - b^2c + b^2c = 0,$$

so $\phi(y) = [cu - bv] \in C$, and $\phi(y)$ is collinear with $x = [u]$ and $y = [v]$.
Can show this satisfies the conditions of the theorem. $\square$

§ 3. Intersections and inflections.

Today: overview of next 2 lectures, state main results.

§ 3.1. Bézout's Theorem

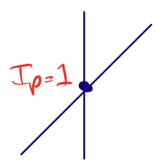
Theorem 3.1 (Bézout's Theorem, weak form)

Let C, D be curves in $\mathbb{CP}^2$ of degrees m, n, with no common component. Then C and D intersect in at most mn points.

Proof: next 2 lectures.

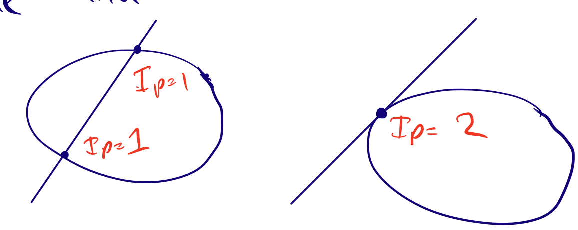
Examples

2 lines



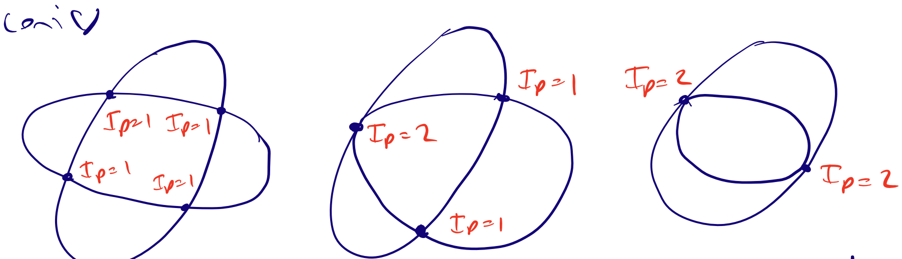
$m = n = 1$.
1 point of intersection.

line and conic



1 or 2 points of intersection.

2 conics



2, 3 or 4 points of intersection.

Think of cases with $< mn$ points as several intersection points "run together".

We deal with this with a notion of

<u>intersection multiplicity</u> $I_p(C,D)$: ← not very easy to define

the "number of times C and D intersect at p",

a positive integer if $p \in C \cap D$.

Then:

<u>Theorem 3.2 (Bézout's Theorem, strong form)</u>

Let C and D be curves in $\mathbb{C}P^2$ of degrees

m and n, with no common component.

Then $$\sum_{p \in C \cap D} I_p(C,D) = mn.$$

<u>Proof:</u> next 2 lectures.

So Bézout's theorem gives a precise count of the intersections of two curves.

§3.2. Consequences of Bézout's Theorem

Lemma 3.3. Every curve C in $\mathbb{CP}^2$ is nonempty, and has infinitely many points.

Proof. Take $p \in \mathbb{CP}^2 \setminus C$. Then every line ℓ through p intersects C, by Bézout (strong form). There are infinitely many such lines, and intersections with distinct lines are distinct, as $p \notin C$. $\square$

Lemma 3.4. Every two curves C, D in $\mathbb{CP}^2$ intersect.

Proof. If C, D have a common component E, then $E \neq \emptyset$ by Lemma 3.3. Otherwise $C \cap D \neq \emptyset$ by Bézout. $\square$

Lemma 3.5. An irreducible curve C has finitely many singular points.

Proof. Let C be defined by $P(x,y,z)$, irreducible of degree n. If $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z} \equiv 0$ then $P =$ constant, contradiction. Suppose WLOG that $\frac{\partial P}{\partial x} \not\equiv 0$. Let D be the curve $\frac{\partial P}{\partial x} = 0$. Then C, D have no common components, as C irreducible degree n, $\deg D = n-1 < n$. So $C \cap D$ is finite by Bézout. But $\{\text{singular points of } C\} \subseteq C \cap D$. $\square$

Lemma 3.6. A nonsingular curve is irreducible.

Proof. Suppose C is not irreducible. Then $D = D_1 \cup \cdots \cup D_k$, $k > 1$. But $D_1 \cap D_2 \neq \emptyset$ by Lemma 3.4, and can show $D_1 \cap D_2$ are singular points of C. $\square$

§3.3. Resultants.

Resultants are the main tool in the proof of Bézout's Theorem, and the definition of intersection multiplicity.

Definition Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ and $q(x) = b_0 + b_1(x) + \cdots + b_n x^n$ be polynomials, $a_m, b_n \neq 0$. Then the resultant $R_{p,q}$ is the determinant of the $(m+n)\times(m+n)$ matrix

$$R_{pq} = \begin{array}{|ccccccccc|l}
 & & & & & & & n-1 & & \\
a_0 & a_1 & \cdots & \cdots & a_m & 0 & 0 & \cdots & 0 & \\
0 & a_0 & a_1 & \cdots & a_{m-1} & a_m & 0 & \cdots & 0 & n \text{ rows} \\
\vdots & & & & \vdots & \vdots & & & \vdots & \\
0 & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & \cdots & a_m & \\
b_0 & b_1 & \cdots & & b_n & 0 & \cdots & \cdots & 0 & \\
0 & b_0 & b_1 & \cdots & b_{n-1} & b_n & 0 & \cdots & 0 & m \text{ rows} \\
\vdots & & & & \vdots & \vdots & & & \vdots & \\
0 & \cdots & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & b_n & \\
 & m-1 & & & & & & & &
\end{array}$$

Then R_{pq} is degree n in the a_i and degree m in the b_i.

Example Let $p(x) = x - \lambda$, $q(x) = x - \mu$.

Then $m = n = 1$, $R_{p,q} = \begin{vmatrix} -\lambda & 1 \\ -\mu & 1 \end{vmatrix} = \mu - \lambda$.

Proposition 3.7. If $p(x) = \alpha(x-\lambda_1)\cdots(x-\lambda_m)$,
$q(x) = \beta(x-\mu_1)\cdots(x-\mu_n)$

then $R_{p,q} = \alpha^n \beta^m \prod_{\substack{1 \le i \le m, \\ 1 \le j \le n}} (\mu_j - \lambda_i)$.

Proof. Kirwan p. 68-9. $\square$

Corollary 3.8. $R_{p,q} = 0$ iff p, q have a common (non-constant) factor.

Proof. $R_{p,q} = 0 \iff \mu_j = \lambda_i$ some $i,j \iff (x-\lambda_i)$ divides p, q. $\square$

Corollary 3.9. Let p, q, r be polynomials.

Then $R_{p,(qr)} = R_{p,q} \cdot R_{p,r}$.

Proof. Exercise. $\square$

§ 3.4. Resultants in several variables.

Let $P(x,y,z)$, $Q(x,y,z)$ be polynomials in x,y,z, degrees m, n.

Can write $P(x,y,z) = a_0(y,z) + a_1(y,z)x + \dots + a_m(y,z)x^m$,

$Q(x,y,z) = b_0(y,z) + b_1(y,z)x + \dots + b_n(y,z)x^n$.

Then can define $R_{P,Q}$ as before, but this time $R_{P,Q}$ is a polynomial in y,z.

Call $R_{P,Q}$ <u>the resultant of P,Q with respect to x.</u>

<u>Lemma 3.10</u> If P,Q homogeneous of degree m,n in x,y,z, then $R_{P,Q}$ is homogeneous of degree mn in y,z.

<u>Proof.</u> Write $P(x,y,z) = \alpha(x-\lambda_1)\cdots(x-\lambda_m)$,
$Q(x,y,z) = \beta(x-\mu_1)\cdots(x-\mu_n)$,

$\alpha,\beta \in \mathbb{C}$, λ_i, μ_j functions of y,z, homogeneous of degree 1.

Then $R_{P,Q} = \alpha^m\beta^n \prod_{i,j}(\mu_j-\lambda_i)$, product of mn factors, homogeneous of degree 1 $\Rightarrow$ degree mn. $\square$

<u>Lemma 3.11.</u> Let $P(x,y,z)$, $Q(x,y,z)$ be polynomials. Then $R_{P,Q} \equiv 0$ as polynomial in y,z iff P,Q have a common factor as polynomials in x,y,z.

<u>Proof.</u> 'If' follows from Corollary 3.8 for each y,z.
'Only if' as in Kirwan, p. 68: essentially it is Cor. 3.8 over the field of rational functions in y,z, rather than over $\mathbb{C}$. $\square$

§3.5. Proof of Bézout's Theorem, weak form.

Recall: Theorem 3.1. Let C, D be curves in $\mathbb{CP}^2$ of degrees m, n, with no common component. Then C, D intersect in at most mn points.

N.B. Proof of Bézout's Theorem not examinable.

Proof. Let C, D be curves in $\mathbb{CP}^2$ of degrees m, n, defined by $P(x,y,z)$, $Q(x,y,z)$ of degrees m, n. Suppose $p_1, \ldots, p_{mn+1}$ are distinct points in $C \cap D$. We will show that P, Q have a common factor, so C, D have a common component.

Choose a point $q \in \mathbb{CP}^2$, $q \notin C$, $q \notin D$ and q does not lie on any line through two of $p_1, \ldots, p_{mn+1}$. After applying a projective transformation, can assume $q = [1,0,0]$. Let $p_j = [a_j, b_j, c_j]$, b_j, c_j not both zero as $p_j \neq [1,0,0]$.

Consider $R(y,z) = R_{P,Q}$

Now: $x - a_j$ divides $P(x, b_j, c_j)$ and $Q(x, b_j, c_j)$ as $p_j \in C \cap D$.

$\Rightarrow R(b_j, c_j) = 0$ by Cor. 3.8.

$\Rightarrow (c_j y - b_j z) \mid R(y,z)$, as R homogeneous.

As $[1,0,0]$ does not lie on line through $[a_i, b_i, c_i]$, $[a_j, b_j, c_j]$, $c_i y - b_i z$, $c_j y - b_j z$ are coprime, not proportional, for $i \neq j$.

Hence $(c_1 y - b_1 z) \cdots (c_{mn+1} y - b_{mn+1} z) \mid R(y,z)$.

But R is homogeneous of degree mn, by Lemma 3.10, and l.h.s. has degree $mn+1$, so $R=0$.

Hence $P(x,y,z)$, $Q(x,y,z)$ have a common factor by Lemma 3.11, say $P = X \cdot Y$, $Q = X \cdot Z$.

Then any component of the curve $X=0$ is a component of C and D, so C, D have a common component.

Conversely, if C, D have no common component, there are at most mn distinct points in $C \cap D$. $\square$

§ 3.6. Intersection multiplicity.

<u>Definition</u> Let C, D be curves in $\mathbb{CP}^2$, of degrees m, n, with no common component, defined by $P(x,y,z)$, $Q(x,y,z)$. Then $C \cap D$ is finite by Bézout (weak form).

Let $q \in \mathbb{CP}^2$ with $q \notin C$, $q \notin D$, and q does not lie on a line joining any two points of $C \cap D$.

Apply a projective transformation to make $q = [1,0,0]$.

Let $R(y,z) = R_{P,Q}$ as in § 3.5.

If $p \in C \cap D$, $p = [a,b,c]$, then as in § 3.5, $cy - bz$ divides $R(y,z)$.

Define the <u>intersection multiplicity</u> $I_p(C,D)$ of C, D at $p=[a,b,c]$ to be the largest positive integer $k>0$ such that $(cy-bz)^k \mid R(y,z)$.

If $p \notin C \cap D$, define $I_p(C,D)=0$.

The following two propositions are proved in Kirwan, §3.1.

<u>Proposition 3.12.</u> The definition of $I_p(C,D)$ is independent of the choice of q and projective transformation.

<u>Proposition 3.13.</u> Let $p \in C \cap D$. Then $I_p(C,D)=1$ iff p is a nonsingular point of C and D, and the tangent lines to C and D at p are distinct. Learn this.

§3.7. Proof of Bézout's Theorem, strong form

<u>Recall:</u> <u>Theorem 3.2.</u> Let C, D be curves in $\mathbb{CP}^2$ with no common component, of degrees m, n. Then

$$\sum_{p \in C \cap D} I_p(C,D) = mn.$$

<u>Proof.</u> By weak Bézout, $|C \cap D| \leq mn$.
Write $C \cap D = \{p_1, \dots, p_k\}$, $k \leq mn$.
As in §3.5, choose q, $q \notin C$, $q \notin D$, q not collinear

with p_i, p_j, $i \neq j = 1, \dots, k$.

Apply projective transformation to take q to $[1,0,0]$.

Then let $R(y,z) = R_{P,Q}$.

Write $p_i = [a_i, b_i, c_i]$, $i = 1, \dots, k$.

<u>Claim:</u> $$R(y,z) = \alpha \prod_{i=1}^{k} (c_i y - b_i z)^{I_{p_i}(C,D)}, \qquad (1)$$

$0 \neq \alpha \in \mathbb{C}$.

<u>Proof.</u> By definition of $I_{p_i}(C,D)$, $(c_i y - b_i z)^{I_{p_i}(C,D)}$ is the largest power of $c_i y - b_i z$ dividing $R(y,z)$.

Also $(c_i y - b_i z)$, $(c_j y - b_j z)$ coprime for $i \neq j$ as $[1,0,0]$, $[a_i, b_i, c_i]$, $[a_j, b_j, c_j]$ not collinear.

So the r.h.s. of (1) divides the l.h.s.

Suppose $R(y,z)$ has some other factor $cy - bz$, $b/c \neq b_i/c_i$ any $i = 1, \dots, k$.

Then $R_{P(x,b,c),\, Q(x,b,c)} = 0$, so $P(x,b,c)$, $Q(x,b,c)$ have a common factor $x-a$, $[a,b,c] \neq p_i$, $i = 1, \dots, k$ and $[a,b,c] \in C \cap D$, $\times$.

This proves (1) for some $\alpha \in \mathbb{C}$. But $\alpha \neq 0$ by Lemma 3.11. Hence the degree of l.h.s. of (1) is mn, and the degree of the r.h.s. is $\sum_{i=1}^{k} I_{p_i}(C,D)$, which proves the theorem. $\square$

Extra material.

As I am getting through the lecture faster than I expected, I'm going to add some more material on intersection multiplicity. The rest of the lectures don't depend on it.

§3.7½. Axiomatic definition of intersection multiplicity

We can extend the definition of intersection multiplicity $I_p(C,D)$ so that it is defined for any two curves C, D in $\mathbb{CP}^2$ (common components are allowed) and for any $p \in \mathbb{CP}^2$ (don't require $p \in C \cap D$), as follows:

* If $p \notin C \cap D$ then $I_p(C,D) = 0$.
* If p lies in a common component E of C and D then $I_p(C,D) = \infty$.
* If $p \in C \cap D$ does not lie in a common component of C and D then $I_p(C,D) = I_p(C', D')$, where C' is the union of components of C containing p, and D' is the union of components of D containing p.

Then $p \in C' \cap D'$ and C', D' have no common component, and $I_p(C', D')$ is defined as in §3.6. This is well defined as with the old definition,
$$I_p(C_1 \cup C_2, D) = I_p(C_1, D) \text{ if } p \notin C_2.$$

With this extended notion of intersection multiplicity, we can give an axiomatic characterization of $I_p(C,D)$ (Kirwan Th. 3.18, Hitchin Th. 14).

<u>Theorem</u>. There is a unique intersection multiplicity $I_p(C,D) \in \mathbb{N} \cup \{\infty\}$, defined for all curves C, D in $\mathbb{CP}^2$ and all $p \in \mathbb{CP}^2$, satisfying:

(i) $I_p(C,D) = I_p(D,C)$.

(ii) $I_p(C,D) = \infty$ if p lies on a common component of C and D, and otherwise $I_p(C,D) \in \mathbb{N}$.

(iii) $I_p(C,D) = 0$ if $p \notin C \cap D$.

(iv) $I_p(L_1, L_2) = 1$ if L_1, L_2 are distinct lines and $p = L_1 \cap L_2$.

(v) If C_1, C_2 have no common components then
$$I_p(C_1 \cup C_2, D) = I_p(C_1, D) + I_p(C_2, D).$$

(vi) If C, D are defined by polynomials P, Q and E is defined by $PR + Q$ then
$$I_p(C,D) = I_p(C,E).$$

<u>Sketch proof</u>. (a) One shows that (extended) intersection multiplicities defined using resultants satisfy (i)–(vi).
(b) One shows that (i)–(vi) determine $I_p(C,D)$ uniquely.
The important part of the proof is a kind of downward induction: by replacements like $(P,Q) \mapsto (P, PR+Q)$ in (vi).

we can eventually reduce P, Q to products of linear factors, and then uniqueness follows from (i) – (v). $\square$

3.7 $\frac{2}{3}$. Intersection multiplicity as a dimension.

This material not examinable.

Let C, D be curves in $\mathbb{CP}^2$ defined by $P(x,y,z)$, $Q(x,y,z)$. To define $I_{(0,0,1)}(C,D)$, consider the power series ring $\mathbb{C}[[x,y]]$ of formal power series $f(x,y) = \sum_{m,n \geq 0} a_{m,n} x^m y^n$, with no convergence criteria.

Then $I_{(0,0,1)}(C,D) = \dim_{\mathbb{C}} \left(\frac{\mathbb{C}[[x,y]]}{(P(x,y,1), Q(x,y,1))} \right)$,

where $(P(x,y,1), Q(x,y,1))$ is the ideal in $\mathbb{C}[[x,y]]$ generated by $P(x,y,1)$, $Q(x,y,1)$.

This gives the correct answer in $\mathbb{N} \cup \{\infty\}$.

<u>Example</u>. If $P(x,y,z) = yz^{k-1} + x^k$, $Q(x,y,z) = y$, then

$$\frac{\mathbb{C}[[x,y]]}{(P(x,y,1), Q(x,y,1))} = \frac{\mathbb{C}[[x,y]]}{(y+x^k, y)} = \frac{\mathbb{C}[[x,y]]}{(y, x^k)}$$

$= \langle 1, x, \dots, x^{k-1} \rangle_{\mathbb{C}}$, so $I_p(C,D) = k$.

§3.8. The Hessian and points of inflection.

Definition. Let $P(x,y,z)$ be a homogeneous polynomial of degree d. The Hessian $H_P(x,y,z)$ of P is the polynomial defined by

$$H_P(x,y,z) = \det\begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix},$$

where $P_x = \frac{\partial P}{\partial x}$, $P_{xy} = \frac{\partial^2 P}{\partial x \partial y}$, etc. Then $H_P(x,y,z)$ is a homogeneous polynomial of degree $3(d-2)$.

Definition Let C be a curve in $\mathbb{CP}^2$ defined by $P(x,y,z)$. A nonsingular point $p=[a,b,c]$ in C is called a point of inflection (or flex) if $H_P(a,b,c)=0$.

From Sheet 3 q. 3 we have:

Lemma 3.14. If P is homogeneous of degree $d>1$ then

$$z^2 H_P(x,y,z) = (d-1)^2 \det\begin{pmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & dP/(d-1) \end{pmatrix}.$$

We also show that if $P(x,y,1) = y - g(x)$ then $[a,b,1]$ is a point of inflection of C iff $b=g(a)$ and $g''(a)=0$. So, this definition matches the usual definition of a point of inflection of the graph of a function $g(x)$.

Proposition 3.15. A nonsingular point p in a curve C in $\mathbb{CP}^2$ is a point of inflection iff $I_p(C,L) \geq 3$, where L is the tangent line to C at p.

Proof. After a projective transformation can take $p = [0,0,1]$ and $L = \{x = 0\}$. Then
$P(0,0,1) = P_y(0,0,1) = P_z(0,0,1) = 0$, and $P_x(0,0,1) \neq 0$
as p is nonsingular. So by Lemma 3.14

$$H_p(0,0,1) = (d-1)^2 \det \begin{pmatrix} P_{xx}(0,0,1) & P_{xy}(0,0,1) & P_x(0,0,1) \\ P_{yx}(0,0,1) & P_{yy}(0,0,1) & 0 \\ P_x(0,0,1) & 0 & 0 \end{pmatrix}$$

$$= -P_x(0,0,1)^2 P_{yy}(0,0,1).$$

Thus p is a point of inflection iff $P_{yy}(0,0,1) = 0$.
But $I_p(C,L)$ is the largest power of $(y - 0z)$ dividing $R_{P(x,y,z),x} = P(0,y,z)$
As $P(0,0,1) = P_y(0,0,1) = 0$, y^2 divide $P(0,y,z)$,
and y^3 divides $P(0,y,z)$ iff $P_{yy}(0,0,1) = 0$.
Hence $I_p(C,L) \geq 2$, and $I_p(C,L) \geq 3$ iff p is a point of inflection. $\square$

<u>Proposition 3.16.</u> Let C be a nonsingular curve in $\mathbb{CP}^2$ of degree d.

(a) If $d=1$ then every point of C is a point of inflection.

(b) If $d \geq 2$ then C has at most $3d(d-2)$ points of inflection.

(c) If $d \geq 3$ then C has at least one point of inflection.

<u>Proof.</u> (a) Trivial as $H_P = \det\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$

(b) Points of inflection are intersections of curve C, with degree d, and $H_P = 0$, with degree $3(d-2)$. Can show C is not a component of $H_P = 0$ (Kirwan Lem. 3.3.2).

So (b) follows by weak Bézout, and (c) by Lemma 3.4. $\square$

§3.9. The normal form of a cubic.

<u>Theorem 3.17.</u> Let C be a nonsingular cubic curve in $\mathbb{CP}^2$. Then C is equivalent under a projective transformation to the curve $y^2 z = x(x-z)(x-\lambda z)$ for some $\lambda \in \mathbb{C} \setminus \{0,1\}$.

<u>Proof.</u> By Prop. 3.16(c), C has at least one point of inflection. Apply a projective transformation to make $[0,1,0]$ a point of inflection, with tangent line $z=0$.

Let C be defined by $P(x,y,z)$. Then

$P(0,1,0) = P_x(0,1,0) = 0$, $P_z(0,1,0) \neq 0$, $H_P(0,1,0) = 0$.

From Lemma 3.14 with y, z reversed, get

$$y^2 H_P(x,y,z) = 4 \begin{vmatrix} P_{xx} & P_x & P_{xz} \\ P_x & \frac{3}{2}P & P_z \\ P_{zx} & P_z & P_{zz} \end{vmatrix}$$

So $$0 = H_P(0,1,0) = 4 \begin{vmatrix} P_{xx}(0,1,0) & 0 & P_{xz}(0,1,0) \\ 0 & 0 & P_z(0,1,0) \\ P_{zx}(0,1,0) & P_z(0,1,0) & P_{zz}(0,1,0) \end{vmatrix}$$

$= -4 \, P_z(0,1,0)^2 P_{xx}(0,1,0)$, so $P_{xx}(0,1,0) = 0$ as $P_z(0,1,0) \neq 0$.

Hence: P has no y^3 or xy^2 or x^2y terms as

$P(0,1,0) = P_x(0,1,0) = P_{xx}(0,1,0)$.

Thus $P(x,y,z) = yz(\alpha x + \beta y + \gamma z) + Q(x,z)$,

$\beta = P_z(0,1,0) \neq 0$.

Apply projective transformation $(x,y,z) \mapsto (x, y + \frac{\alpha x + \gamma z}{2\beta}, z)$.

Takes P to $\beta^2 yz + \tilde{Q}(x,z) =: \tilde{P}(x,y,z)$.

C irreducible, so $z \nmid \tilde{P}$, so coefficient of x^3 in $\tilde{Q} \neq 0$.

Can rescale x, y to get

$\tilde{\tilde{P}} = y^2 z - (x-az)(x-bz)(x-cz)$.

If $a = b$, say, then $(a,0,1)$ is a singular point, $\times$.

So a, b, c are distinct.

Finally apply projective transformation

$$(x, y, z) \longmapsto \left(\frac{x - az}{b - a}, (b-a)^{-3/2} y, z \right)$$

to get $\bar{\bar{\bar{p}}}(x, y, z) = y^2 z - x(x-z)(x-\lambda z),$

$$\lambda = \frac{c-a}{b-a} \qquad \lambda \neq 0, 1, \quad \lambda \in \mathbb{C}.$$

$\square$

The equation $y^2 z = x(x-z)(x-\lambda z)$ is called a <u>normal form</u>, i.e. a standard way to write the cubic.

Parameter count : cubics in x, y, z = $\mathbb{C}^{10}$

acted on by $GL(3, \mathbb{C})$, dim 9.

$\Rightarrow$ Expected dimension of cubics / projective transformation

is $10 - 9 = 1$, parametrized by $\underset{\text{dim } 1.}{\lambda \in \mathbb{C} \setminus \{0, 1\}}$

§4. Algebraic curves and Riemann surfaces.

§4.1. Topological surfaces and Riemann surfaces

— Recap material from B3.2: Geometry of Surfaces.

Several kinds of surfaces:

- topological surfaces: topological spaces locally like $\mathbb{R}^2$.
- smooth surfaces
- Riemann surfaces

$\left.\right\}$ topological surfaces with an extra structure, given by an 'atlas of charts'.

For smooth surface S, have notion of smooth map $f: S \to \mathbb{R}$. Real 2-dimensional.

For Riemann surface S, have notion of holomorphic map $f: S \to \mathbb{C}$. Complex 1-dimensional.

We'll deal only with topological and Riemann surfaces.

Definition A topological surface is a Hausdorff topological space S such that every point $x \in S$ has an open neighbourhood U homeomorphic to an open set in $\mathbb{R}^2$.

Definition. Let S be a topological surface. A chart on S is a triple (φ, U, V) where $U \subseteq S$ is open, $V \subseteq \mathbb{C}$ is open, and $\varphi: U \to V$ is a homeomorphism.

Let $(\varphi_\alpha, U_\alpha, V_\alpha)$ and $(\varphi_\beta, U_\beta, V_\beta)$ be charts on S.

Then we have the <u>transition function</u>

$$\begin{tikzcd} & \varphi_\alpha(U_\alpha \cap U_\beta) \subset V_\alpha \subset \mathbb{C} \arrow[d, "\varphi_\beta \circ \varphi_\alpha^{-1}"] \\ S \supset U_\alpha \cap U_\beta \arrow[ur, "\varphi_\alpha"] \arrow[dr, "\varphi_\beta"'] & \\ & \varphi_\beta(U_\alpha \cap U_\beta) \subset V_\beta \subset \mathbb{C}. \end{tikzcd}$$

φ_α^{-1}

$\leftarrow$ transition function

Call $(\varphi_\alpha, U_\alpha, V_\alpha)$ and $(\varphi_\beta, U_\beta, V_\beta)$ <u>compatible</u> if $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a <u>holomorphic</u> map of open subsets of $\mathbb{C}$.

A <u>holomorphic atlas</u> $\mathcal{A}$ on S is a collection of compatible charts

$$\mathcal{A} = \{ (\varphi_\alpha, U_\alpha, V_\alpha) : \alpha \in A \}$$

with
$$S = \bigcup_{\alpha \in A} U_\alpha .$$

A <u>Riemann surface</u> is a topological surface S with a holomorphic atlas $\mathcal{A}$.

Let $(S, \mathcal{A})$ and $(S', \mathcal{A}')$ be Riemann surfaces.

A continuous map $f : S \to S'$ is called <u>holomorphic</u> if the composition

$$\begin{tikzcd} \mathbb{C} \supset V_\alpha \supset \varphi_\alpha(f^{-1}(U'_\beta)) \arrow[r, "\varphi_\alpha"] \arrow[d, "\text{composition}"] & U_\alpha \cap f^{-1}(U'_\beta) \arrow[d, "f"] \\ \mathbb{C} \supset V'_\beta & U_\beta \arrow[l, "\varphi'_\beta"'] \end{tikzcd}$$

is a holomorphic map of open subsets of $\mathbb{C}$ for all $(\varphi_\alpha, U_\alpha, V_\alpha) \in \mathcal{A}$ and $(\varphi'_\beta, U'_\beta, V'_\beta) \in \mathcal{A}'$.

Example Define charts (φ_0, U_0, V_0) and (φ_1, U_1, V_1) on $\mathbb{CP}^1$ by $U_0 = \{[z_0, z_1] \in \mathbb{CP}^1 : z_0 \neq 0\}$, $V_0 = \mathbb{C}$, $\varphi_0 : [z_0, z_1] \mapsto \frac{z_1}{z_0}$,

$U_1 = \{[z_0, z_1] \in \mathbb{CP}^1 : z_1 \neq 0\}$, $V_1 = \mathbb{C}$, $\varphi_1 : [z_0, z_1] \mapsto \frac{z_0}{z_1}$.

Transition function is $\varphi_1 \circ \varphi_0^{-1} : \mathbb{C}\setminus 0 \to \mathbb{C} \setminus 0$, $z \longmapsto z^{-1}$, which is holomorphic.

So $\mathcal{A} = \{(\varphi_0, U_0, V_0), (\varphi_1, U_1, V_1)\}$ is a holomorphic atlas, and $\mathbb{CP}^1$ is a Riemann surface.

Proposition 4.1. Let C be a nonsingular projective curve in $\mathbb{CP}^2$. Then C has a holomorphic atlas, and so is a Riemann surface.

Proof. Let C be defined by $P(x,y,z) = 0$. Let $p = [a,b,c] \in C$. Consider the 6 cases:

(a) $a \neq 0$, $P_y(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto z/x$.

(b) $a \neq 0$, $P_z(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto y/x$.

(c) $b \neq 0$, $P_x(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto z/y$.

(d) $b \neq 0$, $P_z(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto x/y$.

(e) $c \neq 0$, $P_x(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto y/z$.

(f) $c \neq 0$, $P_y(a,b,c) \neq 0$, $\varphi : [x,y,z] \longmapsto x/z$.

Since $(a,b,c) \neq (0,0,0)$, $aP_x(a,b,c) + bP_y(a,b,c) + cP_z(a,b,c) = 0$, and p nonsing pt of $C \Rightarrow P_x(a,b,c)$, $P_y(a,b,c)$, $P_z(a,b,c)$ not all zero, we see that one of (a)–(f) always holds.

In each case, φ is <u>well-defined</u> and a <u>homeomorphism</u> near p.

(To show (local) homeomorphism, use $P_y(a,b,1) \neq 0$ in (a), (f), etc., and the Implicit Function Theorem.)

So, can choose U_p open neighborhood of p in C, s.t. $\varphi_p = \varphi|_{U_p}$ is a homeomorphism $\varphi_p : U_p \to \varphi(U_p) = V_p \underset{\text{open}}{\subset} \mathbb{C}$.

Thus, for each $p \in C$, have defined a <u>chart</u> (φ_p, U_p, V_p).

Let $\mathcal{A} = \{(\varphi_p, U_p, V_p) : p \in C\}$.

<u>Claim:</u> $\mathcal{A}$ is a holomorphic atlas.

<u>Proof.</u> Must show transition functions are holomorphic.

Let $p, q \in C$. Then look at $\varphi_{pq} = \varphi_p \circ \varphi_q^{-1}$.

Now φ_p, φ_q are z/x, y/x, z/y, x/y, y/z or x/z.

Every case is like one of:

(i) $\varphi_p = \varphi_q = z/x$. $\varphi_{pq}(w) = w$, holomorphic.

(ii) $\varphi_p = z/x$, $\varphi_q = x/z$. $\varphi_{pq}(w) = w^{-1}$, holomorphic.

(iii) $\varphi_p = x/z$, $\varphi_q = y/z$.

Then $P(\varphi_p(r), \varphi_q(r), 1) = 0$, so $P(\varphi_{pq}(w), w, 1) = 0$.

Thus, the transition function $\varphi_{pq}(w)$ satisfies a polynomial equation with w. Deduce $\varphi_{pq}(w)$ is holomorphic.

(Holomorphic Implicit Function Theorem.)

So $\mathcal{A}$ is a holomorphic atlas, and $(C, \mathcal{A})$ is a Riemann surface. $\square$

§4.2. Meromorphic functions

Recall from §4.1: have defined <u>Riemann surfaces</u> and <u>holomorphic maps</u> between Riemann surfaces. $\mathbb{CP}^1$ is a Riemann surface. Every nonsingular algebraic curve C in $\mathbb{CP}^2$ is a Riemann surface.

<u>Definition.</u> Identify $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$ by $[1,z] \leftrightarrow z \in \mathbb{C}$ and $[0,1] \cong \infty$. Let C be a Riemann surface. A <u>meromorphic function</u> on C is a holomorphic function $f: C \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ such that f does not take the constant value ∞ on any connected component of C. We allow f to take the value ∞, but only at isolated points in C.

If $f: C \longrightarrow \mathbb{C}$ is holomorphic, then f is meromorphic as a map to $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

So, think of meromorphic functions as holomorphic functions into $\mathbb{C}$ which are also allowed to take the value ∞.

<u>Example 4.2.</u> Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$. Then C is a Riemann surface, as in Prop 4.1.

Suppose $[1,0,0] \notin C$. Define $\pi: C \longrightarrow \mathbb{CP}^1$ by

$\pi: [x,y,z] \longmapsto [y,z]$.

We claim that π is a <u>meromorphic function</u>. This follows from the proof of Prop. 4.1: the charts (φ, U, V) on C

have $\phi([x,y,z]) = z/x,\ y/x,\ z/y,\ x/y,\ y/z$ or x/z.

The charts (φ', U', V') on $\mathbb{CP}^1$ have $\phi'([y,z]) = \frac{y}{z}$ or $\frac{z}{y}$.

Then $\varphi' \circ \varphi^{-1}$ is holomorphic as in Prop. 4.1. $\square$

Note that we can apply a projective transformation of $\mathbb{CP}^2$ to C before applying π. Thus, this construction yields many meromorphic functions $\pi : C \longrightarrow \mathbb{CP}^1$ (8 parameter family).

<u>Definition.</u> Let $f : C \longrightarrow D$ be a (non constant) holomorphic map of Riemann surfaces. Let $p \in C$. Choose a chart (φ, U, V) on C with $p = \varphi(0)$, $0 \in U$, and a chart (φ', U', V') on D with $f(p) \in U'$. Then $F(w) = \varphi' \circ f \circ \varphi^{-1}(w)$ is a holomorphic function $\mathbb{C} \longrightarrow \mathbb{C}$ defined near $w = 0$.

Also F is not constant, provided f not constant in C near p ($\Leftarrow$ f not constant, if C connected).

We call p a <u>ramification point</u>, and $f(p)$ a <u>branch point</u>, if $F'(0) = 0$.

Since F is a non constant holomorphic function, there is a unique integer $m \geqslant 2$ such that $\frac{d^kF}{dw^k}(0) = 0$ for $k = 1, 2, \dots, m-1$ and $\frac{d^mF}{dw^m}(0) \neq 0$.

We call m the <u>ramification index</u> $\nu_f(p)$ of p.
Can show these definitions are independent of the choice of (φ, U, V), (φ', U', V').
Changing coordinates on C, D, we can take $F(w) = w^m$ near $w=0$, m ramification index.
So, a holomorphic map $f: C \longrightarrow D$ with ramification index m at p looks locally like $w \longmapsto w^m$.

If p is not a ramification point, the ramification index $\nu_f(p)$ is 1.
If p has ramification index m, then for q close to $f(p)$, $f^{-1}(q)$ contains exactly m points close to p.

<u>Proposition 4.3.</u> Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$ with $[1,0,0] \notin C$, and $\pi: C \longrightarrow \mathbb{CP}^1$, $\pi: [x,y,z] \mapsto [y,z]$ be the meromorphic function of Example 4.2.
Then $p \in C$ is a ramification point of π iff the tangent line T_pC passes through $[1,0,0]$, and the ramification index $\nu_\pi(p)$ is $I_p(C, T_pC)$.
Hence by Prop. 3.15, the ramification index is >2 only if p is a <u>point of inflection</u> of C. $\square$

<u>Proof.</u> Hitchin, Prop. 19, p. 44.

§4.3. The genus of a Riemann surface.

Summarize material from B 3.2.

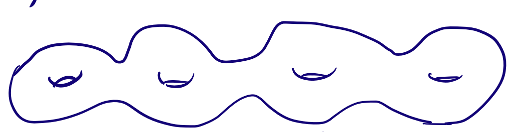
Let S be a topological surface. There is a notion of <u>orientable</u> for S, which we won't define. Roughly, to be oriented means to have a consistent notion of 'anticlockwise' everywhere on S. A Möbius strip is not orientable.

<u>Fact</u>: Every Riemann surface is orientable, since multiplication by $i \in \mathbb{C}$ on tangent spaces defines a notion of anticlockwise.

From the classification of surfaces, we have:

<u>Theorem 4.4.</u> Let S be a compact, connected, orientable, topological (or smooth) surface. Then S is isomorphic to a sphere with g holes for $g = 0, 1, 2, \dots$, called the <u>genus</u> of S.

e.g.



$g = 4$.

$g = 0$: S is a sphere S^2,

$g = 1$: S is a torus T^2.

<u>Proposition 4.5.</u> Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$. Then C is a compact, connected Riemann surface.

<u>Proof.</u> C is a Riemann surface by Prop 4.1. It is closed in $\mathbb{CP}^2$ which is compact, so it is compact. For connectedness, see Hitchin p. 45–6. $\square$

Thus we have:

<u>Corollary 4.6.</u> Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$. Then C is homeomorphic to a sphere with g holes for some $g = 0, 1, 2, \ldots$, called the <u>genus</u> of C.

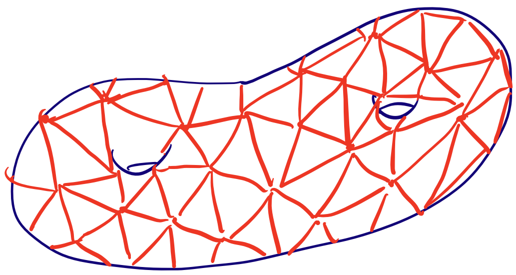
<u>Recall:</u> <u>Cor. 4.6.</u> A nonsingular algebraic curve C in $\mathbb{CP}^2$ is homeomorphic to a sphere with g holes, $g =$ <u>genus</u> of C.

Our goal today is to compute g in terms of the degree of C.

To compute the genus, we use an invariant called the <u>Euler characteristic</u>. There are many ways to define the Euler characteristic $\chi(S)$ of a surface S – see B3.2. We'll do it using a <u>triangulation</u>, a division of S into triangles. (Kirwan, Def. 4.9, App. C.2.)

Let S be a compact surface. Divide S into triangles. Let

$V = \#$ vertices,
$E = \#$ edges,
$F = \#$ faces.



Then $\chi(S) = V - E + F$.

It is independent of the choice of triangulation. For S a sphere with g holes, $\chi(S) = 2 - 2g$.

<u>§4.4. The degree-genus formula</u>

<u>Theorem 4.7.</u> Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$ of degree d. Then the genus of C is $g = \frac{1}{2}(d-1)(d-2)$. This is called the <u>degree-genus formula</u>.

Proof. If $d=1$ then $C \cong \mathbb{CP}^1$ and $g(C)=0$.
So suppose $d \geq 2$. Then C has finitely many points of inflection by Prop. 3.16(b).
Apply a projective transformation to C such that $[1,0,0] \notin C$, and also $[1,0,0]$ does not lie on the tangent line to any point of inflection.

Let $\pi: C \to \mathbb{CP}^1$, $\pi: (x,y,z) \mapsto (y,z)$, be as in §4.2.
Let C be defined by $P(x,y,z)$, of degree d.
By Prop 4.3, the point $p=[a,b,c]$ in C is a ramification point iff T_pC passes through $[1,0,0]$, i.e. if $P_x(a,b,c)=0$.

Also, if T_pC passes through $[1,0,0]$ then p is not a point of inflection, so the ramification index of π at p, which is $I_p(C,T_pC)$, is 2 by Prop. 4.3.
Let D be the degree $d-1$ curve $P_x(x,y,z)=0$.
Then $C \cap D$ is the set of ramification points of π.
If $p=[a,b,c] \in C \cap D$ then as p is not an inflection point of C and $[1,0,0] \in T_pC$, we have $P_{xx}(a,b,c) \neq 0$.
Thus, p is a nonsingular point of D, and T_pD does not pass through $[1,0,0]$, so $T_pC \neq T_pD$.
Hence $I_p(C,D)=1$ by Prop. 3.13.

By Bézout's Theorem, $\sum_{p \in C \cap D} I_p(C,D) = \sum_{p \in C \cap D} 1 = d(d-1)$,

so $|C \cap D| = d(d-1)$.

Write $C \cap D = \{p_1, \dots, p_{d(d-1)}\}$. Changing π a little bit, can assume that $\pi(p_1), \dots, \pi(p_{d(d-1)})$ are distinct in $\mathbb{CP}^1$. Write $q_i = \pi(p_i)$, $i = 1, \dots, d(d-1)$.

If $[b,c] \in \mathbb{CP}^1$, then $\pi^{-1}([b,c]) =$

$$C \cap \{ \underbrace{[x,y,z] \in \mathbb{CP}^2 : cy - bz = 0}_{L_{[b,c]}} \}.$$

This is the intersection of C, degree d, and a curve $L_{[b,c]}$ of degree 1 in $\mathbb{CP}^2$, so by Bézout $\sum_{p \in \pi^{-1}([b,c])} I_p(C, L_{[b,c]}) = d$.

But $I_p(C, L_{[b,c]})$ is the ramification index of π at p by Prop 4.3, which is 2 if $p = p_i$, some i, and 1 otherwise. Since $\pi(p_i) = q_i$, this implies that $|\pi^{-1}([b,c])| = d-1$ if $[b,c] = q_i$, some i, and $|\pi^{-1}([b,c])| = d$ otherwise.

Choose a triangulation of $\mathbb{CP}^1$ with vertices $q_1, \dots, q_{d(d-1)}$, and E edges, and F faces.

(Possible as $d \geqslant 2$, so $d(d-1) \geqslant 2$.)

Then $\chi(\mathbb{CP}^1) = 2 = d(d-1) - E + F$.

This triangulation lifts along π to a triangulation of C, with $\tilde{V}$ vertices, $\tilde{E}$ edges, and $\tilde{F}$ faces.

As $|\pi^{-1}(q_i)| = d-1$, $\tilde{V} = (d-1) \cdot d(d-1)$.

For $q \neq q_i$, $|\pi^{-1}(q)| = d$, so each edge or face in $\mathbb{CP}^1$ lifts to d edges or faces in C.

Hence $\tilde{E} = dE$, $\tilde{F} = dF$.

Therefore
$$\begin{aligned}\chi(C) &= \tilde{V} - \tilde{E} + \tilde{F} \\ &= (d-1)\, d(d-1) - dE + dF \\ &= d(d(d-1) - E + F) - d(d-1) \\ &= d \cdot 2 - d(d-1) \\ &= 3d - d^2 \\ &= 2 - 2\left(\tfrac{1}{2}(d-1)(d-2)\right) \\ &= 2 - 2g(C).\end{aligned}$$

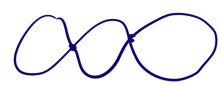
Thus the genus $g(C)$ is $\frac{1}{2}(d-1)(d-2)$. $\square$

Remarks (a) Consider the table

d	1	2	3	4	5	6	7	8	$\cdots$
g	0	0	1	3	6	10	15	21	$\cdots$

So no nonsingular algebraic curve in $\mathbb{CP}^2$ can have genus $2, 4, 5, 7, 8, 9, 11, \ldots$

Most Riemann surfaces are not isomorphic to nonsingular curves in $\mathbb{CP}^2$, though they can be immersed in $\mathbb{CP}^2$ with self-crossings,



(b) Can generalize the proof to show that if $f : C \longrightarrow \mathbb{CP}^1$ is meromorphic, C compact Riemann surface, then $\chi(C) = 2d - \sum_{p \in C} (v_f(p) - 1)$, the Riemann – Hurwicz formula from B3.2.

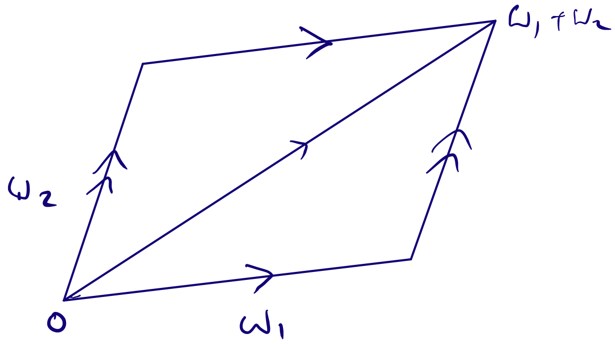
Here d is the degree of f $\Longleftrightarrow$ $|f^{-1}(q)| = d$ for generic $q \in \mathbb{CP}^1$, i.e. any q which is not a branch point.

§4.5. The torus and the cubic

The torus T^2 is $S^1 \times S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = \mathbb{R}^2/\mathbb{Z}^2$.

As a Riemann surface, we can consider $\mathbb{C}/\Lambda$, where $\Lambda \cong \mathbb{Z}^2$ is a <u>lattice</u> in $\mathbb{C}$, i.e.

$\Lambda = \{ m\omega_1 + n\omega_2 : m, n \in \mathbb{Z} \}$, where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$.



Think of $\mathbb{C}/\Lambda$ as a parallelogram in $\mathbb{C}$ with sides ω_1, ω_2, with opposite sides identified.

On $\mathbb{C}$, or $\mathbb{C}/\Lambda$, we can define a meromorphic function called the Weierstrass $\wp$-function, as in B3.2:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq w \in \Lambda} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right). \qquad (*)$$

This converges uniformly on compact subsets of $\mathbb{C} \setminus \Lambda$ to a holomorphic function. At each $w \in \Lambda$ it has a double pole, so $\wp : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$ is meromorphic, with $\wp(w) = \infty$ for $w \in \Lambda$. Also $\wp(z+w) = \wp(z)$ for all $z \in \mathbb{C}$, $w \in \Lambda$, so $\wp$ descends to a meromorphic function $\wp : \mathbb{C}/\Lambda \longrightarrow \mathbb{CP}^1$ on the compact Riemann surface $\mathbb{C}/\Lambda$.

Proposition 4.8. The Weierstrass $\wp$-function $\wp : \mathbb{C} \longrightarrow \mathbb{C} \cup \{\infty\}$ satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3,$$

for some $g_2, g_3 \in \mathbb{C}$ depending on Λ.

Proof. By complex analysis, $\wp$ has a Laurent series at 0. From $(*)$, we see that

$$\wp(z) = z^{-2} + a_2 z^2 + a_4 z^4 + \cdots,$$

since $\wp$ is even and the constant terms all cancel.

So $\wp'(z) = -2z^{-3} + 2a_2 z + 4a_4 z^3 + \cdots$

$$(\wp'(z))^2 = 4z^{-6} - 8a_2 z^{-2} + O(1)$$

$$\wp(z)^3 = z^{-6} + 3a_2 z^{-2} + O(1).$$

Set $g_2 = 20a_2$. Then

$$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) = [4z^{-6} - 8a_2 z^{-2} + O(1)] - 4[z^{-6} + 3a_2 z^{-2} + O(1)] + 20a_2[z^{-2} + O(1)] = O(1).$$

So $\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z)$ has no pole at 0, and thus is <u>holomorphic</u>, not just meromorphic. It is also doubly periodic, i.e. descends to a holomorphic function on $\mathbb{C}/\Lambda$. By the maximum principle (see Sheet 3, q.6) it is constant, i.e.

$$\wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) = -g_3, \text{ some } g_3 \in \mathbb{C}. \qquad \square$$

Now define $\Phi : \mathbb{C}/\Lambda \to \mathbb{CP}^2$ by

$$\Phi(\omega+\Lambda) = \begin{cases} [\wp(\omega), \wp'(\omega), 1], & \omega \notin \Lambda, \\ [0,1,0], & \omega \in \Lambda. \end{cases}$$

Since $\wp'(\omega)^2 = 4\wp(\omega)^3 - g_2\wp(\omega) - g_3$,

if $\Phi(\omega+\Lambda) = [x,y,z]$ then

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3.$$

That is, Φ maps $\mathbb{C}/\Lambda$ to the cubic curve

$Q(x,y,z) = 0$, where $Q(x,y,z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3$.

Can prove:

__Theorem 4.9.__ In the situation above, the algebraic curve C defined by $Q=0$ is nonsingular, and $\Phi : \mathbb{C}/\Lambda \to C$ is an isomorphism of Riemann surfaces.

__Proof.__ See Hitchin notes §4.4 or Kirwan. $\square$

Note that C has genus $\frac{1}{2}(3-1)(3-2) = 1$ by the degree-genus formula, so C is indeed a torus.

Also, by Theorem 3.17, every nonsingular cubic in $\mathbb{CP}^2$ is equivalent under a projective transformation to

$y^2 z = x(x-z)(x-\lambda z)$, $\quad \lambda \in \mathbb{C} \setminus \{0,1\}$, Compare

to similar form $y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3$.

Every nonsingular cubic C is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ in this way.

§5. The Riemann–Roch Theorem

§5.1. Divisors and meromorphic functions

<u>Definition.</u> Let C be a compact, connected Riemann surface.
A <u>divisor</u> on C is a formal sum $D = \sum_{p \in C} n_p p$,
$n_p \in \mathbb{Z}$, where $n_p = 0$ for all but finitely many $p \in C$.
Divisors form an abelian group $\mathrm{Div}(C)$ under addition.
The <u>degree</u> of a divisor D is $\deg D = \sum_{p \in C} n_p$.

Now let $f : C \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ be a meromorphic function, with $f \not\equiv 0$, $f \not\equiv \infty$.
If $p \in C$ with $f(p) = 0$ then choosing a holomorphic coordinate z on C near p with p at $z = a$, then near p we have $f(z) = (z-a)^m \cdot g(z)$ for some $m > 0$ and holomorphic g with $g(a) \neq 0$. We call m the <u>multiplicity</u> of the zero of f at p.
Similarly, if $f(p) = \infty$ then $f(z) = (z-a)^{-n} h(z)$ for $n > 0$ the <u>multiplicity</u> of the pole of f at p, and h holomorphic with $h(a) \neq 0$.

Let f have zeroes $p_1, \dots, p_k$ with multiplicities $m_1, \dots, m_k$, and poles $q_1, \dots, q_l$ with multiplicities $n_1, \dots, n_l$.
Define the <u>divisor</u> of f to be $(f) = \sum_{i=1}^{k} m_i p_i - \sum_{j=1}^{l} n_j q_j$.

Note that $(fg) = (f) + (g)$, $(f/g) = (f) - (g)$.
Like log.

<u>Recall:</u> C a compact, connected Riemann surface.

A <u>divisor</u> on C is $D = \sum_{p \in C} n_p p$, $n_p \in \mathbb{Z}$, $n_p = 0$ for all but finitely many $p \in C$.

The <u>degree</u> is $\deg D = \sum_{p \in C} n_p$.

If $f : C \to \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ is meromorphic, the <u>divisor</u> of f is $(f) = \sum_{i=1}^{k} m_i p_i - \sum_{j=1}^{l} n_j q_j$,

$p_1, \ldots, p_k$ zeroes of f, multiplicities $m_1, \ldots, m_k$,
$q_1, \ldots, q_l$ poles of f, multiplicities $n_1, \ldots, n_l$.

Note that $\deg(f) = 0$, since $\sum_{i=1}^{k} m_i = \sum_{j=1}^{l} n_j$ is the <u>degree</u> of $f : C \to \mathbb{CP}^1$, i.e. the number of points in $f^{-1}(c)$ counted with multiplicity for any $c \in \mathbb{CP}^1$, including $c = 0$, $c = \infty$ (Kirwan Prop. 6.29).

<u>Definition</u> A divisor D on C is called a <u>principal divisor</u> if $D = (f)$ for some meromorphic $f : C \to \mathbb{CP}^1$.

Divisors D, D' are called <u>linearly equivalent</u> if $D - D'$ is a principal divisor, written $D \sim D'$.

This is an <u>equivalence relation</u>: if $D \sim D'$ then $D - D' = (f)$, so $D' - D = (f^{-1})$, and $D' \sim D$. If $D \sim D'$, $D' \sim D''$ then $D' - D = (f)$, $D'' - D' = (g)$, so $D'' - D = (f) + (g) = (fg)$, and $D \sim D''$. As $D - D = 0 = (1)$, $D \sim D$. If $D \sim D'$ then $\deg D = \deg D'$, as $\deg D' - D = \deg (f) = 0$.

A divisor $D = \sum_{p \in C} n_p p$ is called <u>effective</u> if $n_p \geq 0$ for all $p \in C$. Write $D \geq D'$ if $D - D'$ is effective.

Let D be a divisor on C. Write $\mathcal{L}(D)$ for the set of meromorphic functions $f : C \to \mathbb{CP}^1$ such that $(f) + D \geq 0$, together with $f \equiv 0$.

If $D = \sum_{i=1}^{k} m_i p_i - \sum_{j=1}^{l} n_j q_j$, p_i, q_j distinct, $m_i, n_j > 0$, then $f \in \mathcal{L}(D)$ if f is meromorphic with poles of order $\leq m_i$ at p_i for $i = 1, \dots, k$ and no other poles, and f has zeroes of order $\geq n_i$ at q_j, $j = 1, \dots, l$.

Clearly $\mathcal{L}(D)$ is a vector space.

<u>Proposition 5.1</u> (i) $\mathcal{L}(D)$ is a finite-dimensional vector space.

(ii) If $\deg D < 0$ then $\mathcal{L}(D) = 0$.

(iii) If $D \sim D'$ then $\dim \mathcal{L}(D) = \dim \mathcal{L}(D')$.

(iv) The projective space $P(\mathcal{L}(D))$ is in 1-1 correspondence with the effective divisors linearly equivalent to D. $\square$

<u>Proof.</u> Hitchin p. 56.

Write $\ell(D) = \dim \mathcal{L}(D)$, well defined by (i).

Note that $\ell(D)$ can vary in complicated ways as we vary D — no simple formula for $\ell(D)$.

§5.2. Meromorphic differentials and canonical divisors

Let C be a compact, connected Riemann surface and $f: C \longrightarrow \mathbb{CP}^1$ a meromorphic function. What would it mean to differentiate f?

We could choose a local coordinate z on C and take $\frac{df}{dz}$. But the answer would depend on the choice of z (chain rule). So the derivative df of f is not a function, but a new kind of geometric object.

Definition. Let f, g be meromorphic functions on a compact, connected Riemann surface C, with g not constant. Then we call $f\,dg$ a meromorphic differential on C. If $\tilde{f}, \tilde{g}$ are also meromorphic on C, we say that $f\,dg = \tilde{f}\,d\tilde{g}$ iff for all charts (φ, U, V) on C we have

$$(f \circ \varphi^{-1})(g \circ \varphi^{-1})' = (\tilde{f} \circ \varphi^{-1})(\tilde{g} \circ \varphi^{-1})' \text{ on } V \subseteq \mathbb{C}.$$

Equivalently, whenever z is a local coordinate on C, we have $f(z)\frac{dg}{dz}(z) = \tilde{f}(z)\frac{d\tilde{g}}{dz}(z)$.

Think of $f\,dg$ as $f(z)\frac{dg}{dz}(z)\,dz$ locally.

We say that a meromorphic differential $f\,dg$ has a zero of order m at $p \in C$, or a pole of order n at $p \in C$ if whenever z is a local coordinate on C with p at $z = a$, then $f(z)\frac{dg}{dz}(z)$ has a zero of order m or a pole of order n at $z = a$

This is independent of choice of coordinates, and also invariant under $f\,dg = \tilde{f}\,d\tilde{g}$.

Thus we can define the <u>divisor</u> $(f\,dg)$ of $f\,dg$ to be $\sum_{i=1}^{k} m_i p_i - \sum_{j=1}^{\ell} n_j q_j$, where $f\,dg$ has zeroes at $p_1, \dots, p_k$ of order $m_1, \dots, m_k$, and poles at $q_1, \dots, q_\ell$ of order $n_1, \dots, n_\ell$.

A divisor D on C is called a <u>canonical divisor</u> if $D = (f\,dg)$ for some meromorphic differential $f\,dg$.

<u>Lemma 5.2.</u> Let $f\,dg$ and $\tilde{f}\,d\tilde{g}$ be meromorphic differentials on a compact, connected Riemann surface C. Then there is a unique meromorphic $h : C \longrightarrow \mathbb{CP}^1$ with

$$\tilde{f}\,d\tilde{g} = (hf)\,dg, \quad \text{and also}$$
$$(\tilde{f}\,d\tilde{g}) = (f\,dg) + (h) \text{ in divisors.}$$

Hence, all canonical divisors are linearly equivalent.

<u>Proof.</u> Let z be a local coordinate on an open subset U in C. Define h locally on C by $h(z) = \dfrac{\tilde{f}(z)\,\frac{d\tilde{g}}{dz}(z)}{f(z)\,\frac{dg}{dz}(z)}$.

Then h is meromorphic on U. If $\tilde{z}$ is an alternative coordinate on $\tilde{U} \subset C$ then on $U \cap \tilde{U}$ we have

$$\frac{\tilde{f}\,\frac{d\tilde{g}}{dz}}{f\,\frac{dg}{dz}} = \frac{\tilde{f}\,\frac{d\tilde{g}}{d\tilde{z}}\,\frac{d\tilde{z}}{dz}}{f\,\frac{dg}{d\tilde{z}}\,\frac{d\tilde{z}}{dz}} = \frac{\tilde{f}\,\frac{d\tilde{g}}{d\tilde{z}}}{f\,\frac{dg}{d\tilde{z}}},$$

so the definition of h is independent of the choice of local coordinates. The rest is easy. $\square$

The next result is proved in Kirwan Prop. 6.3) and Hitchin Prop. 25 for algebraic curves in $\mathbb{CP}^2$.

<u>Proposition 5.3.</u> Let C be a compact, connected Riemann surface of genus g, and K be a canonical divisor on C. Then $\deg K = 2g-2$.

§5.3. Hyperplane divisors

Let C be a nonsingular curve in $\mathbb{CP}^2$, of degree d.
Let L be a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$, $L \neq C$.
Define a divisor H on C by
$H = \sum_{p \in C \cap L} I_p(C, L)\, p$. We call H a hyperplane divisor, as it is the intersection with a hyperplane in $\mathbb{CP}^2$.
By strong Bézout, $\deg H = d = \deg C$.
Let H, H' be hyperplane divisors from lines $ax+by+cz=0$ and $a'x+b'y+c'z=0$ in $\mathbb{CP}^2$.
Then $\dfrac{ax+by+cz}{a'x+b'y+c'z}$ is a meromorphic function on C,
and $\left(\dfrac{ax+by+cz}{a'x+b'y+c'z}\right) = H - H'$. So $H \sim H'$,
and all hyperplane divisors are linearly equivalent.

Proposition 5.4. Let C be a nonsingular curve of degree d and genus g in $\mathbb{CP}^2$, and $H_1, \dots, H_m$ be hyperplane divisors, $m \geqslant d$. Then

$$\ell(H_1 + \dots + H_m) \geqslant \deg(H_1 + \dots + H_m) + 1 - g.$$

Sketch proof. Let H_i come from the line $a_i x + b_i y + c_i z = 0$.
Consider functions $f_Q = \dfrac{Q(x,y,z)}{\prod_{i=1}^{m}(a_i x + b_i y + c_i z)}$ on C, Q homogeneous of degree m.

Then $f_Q \in \mathcal{L}(H_1 + \cdots + H_m)$. Also $f_Q = f_{Q'}$ iff $Q - Q' = P \cdot R$, C defined by $P(x,y,t)$ of degree d, R homogeneous of degree $m-d$. Counting dimensions shows the vector subspace of such f_Q in $\mathcal{L}(H_1 + \cdots + H_m)$ has dimension $\underbrace{\deg(H_1 + \cdots + H_m)}_{md} + 1 - \underbrace{g}_{\frac{1}{2}(d-1)(d-2)}$. $\square$

§5.4. The Riemann–Roch Theorem

Here is the main result of this part of the course.

Theorem 5.5. (Riemann–Roch) Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$, of genus g. Suppose D is a divisor, and K a canonical divisor, on C. Then
$$\ell(D) - \ell(K - D) = \deg D + 1 - g.$$

N.B. Proof not examinable.

Proof. Start with four lemmas.

Lemma 5.6. For any divisor D on C and any $p \in C$,
$$0 \leq \ell(D+p) - \ell(D) \leq 1.$$

Proof. Let $D = \sum_{q \in C} n_q q$. Then $\mathcal{L}(D)$ is the vector space of meromorphic $f : C \to \mathbb{CP}^1$ with poles of order $\leq n_q$ at q / zeroes of order $\geq -n_q$ at q. $\mathcal{L}(D+p)$ is the same, but allowing pole order $\leq n_p + 1$ /

zero order $\geq -1-n_p$ at p.

So $\mathcal{L}(D) \subseteq \mathcal{L}(D+p)$, giving $\ell(D) \leq \ell(D+p)$.

If z is a local coordinate on C with $z=0$ at p, the linear map $\mathcal{L}(D+p) \to \mathbb{C}$ taking f to the coefficient of z^{-1-n_p} in f has kernel $\mathcal{L}(D)$. $\square$

Hence $\ell(D+p) \leq \ell(D)+1$.

<u>Lemma 5.7.</u> For any divisor D, $p \in C$, and canonical divisor K we have

$$0 \leq \ell(D+p) - \ell(K-D-p) - \ell(D) + \ell(K-D) \leq 1.$$

<u>Proof.</u> By Lemma 5.6 we have $0 \leq \ell(D+p) - \ell(D) \leq 1$ and $0 \leq \ell(K-D) - \ell(K-D-p) \leq 1$.

So we only have to exclude the possibility that

$$\ell(D+p) - \ell(D) = 1 \text{ and } \ell(K-D) - \ell(K-D-p) = 1.$$

Then $\mathcal{L}(D) \subsetneq \mathcal{L}(D+p)$ and $\mathcal{L}(K-D-p) \subsetneq \mathcal{L}(K-D)$, so $\exists\ f \in \mathcal{L}(D+p) \setminus \mathcal{L}(D)$ and $g \in \mathcal{L}(K-D) \setminus \mathcal{L}(K-D-p)$.

Let ω be a meromorphic differential with $(\omega) = K$. Consider the meromorphic differential $fg\omega$. Adding up the degrees of zeroes/poles of f, g, ω shows that $fg\omega$ has a nontrivial simple pole at p, and no other poles. But by a version of Cauchy's

Residue Theorem for compact Riemann surfaces, the sum of residues of poles of a meromorphic differential is zero. But $fg\omega$ has one pole at p with nonzero residue, a contradiction. $\square$

<u>Lemma 5.8.</u> Let D be a divisor on C. Then there exist $p_1, \dots, p_k \in C$ and hyperplane divisors $H_1, \dots, H_m$ with $D + p_1 + \dots + p_k = H_1 + \dots + H_m$.

<u>Proof.</u> (a) If $D = -p$, have $D + p_1 = 0$ with $k = 1$, $p_1 = p$, $m = 0$.

(b) If $D = p$, choose a line L in $\mathbb{CP}^2$ through p and distinct from T_pC, so $I_p(C, L) = 1$. The corresponding divisor is $H_1 = p + \sum_{p \neq q \in C \cap L} I_q(C, L)\, q$.

Then $D + p_1 + \dots + p_k = H_1$, with $p_1, \dots, p_k$ the $q \in C \cap L \setminus \{p\}$ with multiplicity $I_q(C, L)$.

The general case is a linear combination of (a), (b). $\square$

<u>Lemma 5.9.</u> For any D, K we have

$$\ell(D) - \ell(K - D) \geq \deg D - g + 1. \qquad (*_1)$$

<u>Proof.</u> By Lemma 5.8 can choose $p_1, \dots, p_k$, $H_1, \dots, H_m$ with $D + p_1 + \dots + p_k = H_1 + \dots + H_m$. Can make m larger by adding extra p_i, so suppose $m \geq d$, $md \geq 2g - 2$, $d = \deg C$.

Proposition 5.4 gives

$$\ell(H_1+\dots+H_m) \geq \deg(H_1+\dots+H_m) + 1-g.$$

Also $\deg(K-(H_1+\dots+H_m)) = 2g-2-md < 0$,

so $\ell(K-(H_1+\dots+H_m)) = 0$, by Proposition 5.1(ii),

and $\ell(H_1+\dots+H_m) - \ell(K-(H_1+\dots+H_m)) \geq \deg(H_1+\dots+H_m) + 1-g$.

Hence $\ell(D+P_1+\dots+P_k) - \ell(K-D-P_1-\dots-P_k) \geq \deg D + k+1-g. \quad (*_2)$

But Lemma 5.7 gives

$$[\ell(D+P_1+\dots+P_i) - \ell(K-D-P_1-\dots-P_i)] - [\ell(D+P_1+\dots+P_{i-1}) - \ell(K-D-P_1-\dots-P_{i-1})] \leq 1 \quad (*_3)$$

Subtracting $(*_3)$ for $i=1,\dots,k$ from $(*_2)$ gives $(*_1)$. $\square$

We can now prove Riemann–Roch:

Lemma 5.9 for D and $K-D$ gives

$$\ell(D) - \ell(K-D) \geq \deg D - g + 1$$

$$\begin{aligned} \ell(K-D) - \ell(D) &\geq \deg(K-D) - g+1 \\ &= (2g-2) - \deg D - g+1 \\ &= -(\deg D - g+1), \end{aligned}$$ by

Proposition 5.3. This proves Theorem 5.5. $\square$

§5.5. Applications of Riemann-Roch

Definition A *holomorphic differential* on a nonsingular curve C is a meromorphic differential with no poles. They form a vector space $HD(C)$, whose dimension $\dim HD(C)$ is called the *geometric genus* of C.

If ω is any meromorphic differential with divisor $K=(\omega)$, then $f\in\mathcal{L}(K)$ iff $f\omega$ is a holomorphic differential. Hence $HD(C)\cong\mathcal{L}(K)$ and $\dim HD(C)=\ell(K)$.

Theorem 5.10. The vector space of holomorphic differentials on a nonsingular algebraic curve C has dimension g, the genus of the curve.

Proof. The dimension is $\ell(K)$. By Riemann-Roch with $D=0$, $\ell(0)-\ell(K)=1-g$. But $\mathcal{L}(0)$ is the holomorphic functions on C, which are constant, so $\mathcal{L}(0)\cong\mathbb{C}$ and $\ell(0)=1$. $\square$

Theorem 5.11. Let C be a nonsingular algebraic curve in $\mathbb{CP}^2$. Then every meromorphic function f on C is of the form $Q(x,y,z)/\prod_{i=1}^{m}(a_ix+b_iy+c_iz)$ for Q homogeneous of degree m, some $m\geqslant 0$.

Proof. Let $D = -(f)$. By Lemma 5.8, can choose $p_1, \dots, p_k \in C$ and hyperplane divisors $H_1, \dots, H_m$ with $D + p_1 + \dots + p_k = H_1 + \dots + H_m$. Can make m larger by adding extra p_i so suppose $m \geq d = \deg C$ and $md > 2g-2$. Then

$$f \in \mathcal{L}(D) \subset \mathcal{L}(D + p_1 + \dots + p_k) = \mathcal{L}(H_1 + \dots + H_m).$$

But Proposition 5.4 shows the subspace of

$$\frac{Q(x,y,z)}{\prod_{i=1}^{m}(a_i x + b_i y + c_i z)}$$

in $\mathcal{L}(H_1 + \dots + H_m)$ has dimension $\deg(H_1 + \dots + H_m) + 1 - g$.

Also Riemann-Roch gives

$$\ell(H_1 + \dots + H_m) - \ell(K - (H_1 + \dots + H_m)) = \deg(H_1 + \dots + H_m) + 1 - g,$$

and $\ell(K - (H_1 + \dots + H_m)) = 0$ as

$$\deg(K - (H_1 + \dots + H_m)) = 2g - 2 - md < 0.$$

Therefore $\mathcal{L}(H_1 + \dots + H_m)$ is all of the form

$$\frac{Q(x,y,z)}{\prod_{i=1}^{m}(a_i x + b_i y + c_i z)},$$

and f is of this form. $\square$

Can understand the <u>holomorphic</u> geometry of curves, and other complex algebraic objects, solely in terms of algebra, and polynomials.

§5.6. The group law on a cubic

Let C be a nonsingular cubic. Then C has at least one point of inflection by Prop 3.16 (c). (In fact C has exactly 9.) C has genus 1 by Theorem 4.7, so C is homeomorphic to T^2. §3.9 shows we can write C as $y^2 z = x(x-z)(x-\lambda z)$, $\lambda \in \mathbb{C} \setminus \{0,1\}$.

§4.5 shows $\mathbb{C}/\Lambda$ isomorphic to a cubic, $\Lambda \subset \mathbb{C}$ a lattice.

Observe: $\mathbb{C}/\Lambda$ is an abelian group.

In fact every nonsingular cubic has the structure of an abelian group – classical result.

Theorem 5.12. Let C be a nonsingular cubic in $\mathbb{CP}^2$ and e a point of inflection on C. Then there is a unique abelian group structure on C with identity e, such that if $p, q, r \in C$ then $p+q+r=e$ (using addition on C) iff p, q, r are the 3 points of intersection (with multiplicity) with a line in $\mathbb{CP}^2$.

Proof. First show that the condition above induces unique inverse and addition maps:

Let $p \in C$, and let L be the line through p, e if $p \neq e$, and $T_e C$ if $p = e$. Then (with multiplicity), $L \cap C$ is 3 points p, e, q in C. Define $-p = q$, the additive inverse in C.

We have $-(-p) = p$, using this same line L.

Let $p, q \in C$, and let L be the line through p, q if $p \neq q$, and T_pC if $p = q$. Then $L \cap C$ is 3 points p, q, r (with multiplicity). Define $p+q = -r$. This defines addition on C. It is immediate, considering the line through e, p, that $e+p = p+e = p$ and $p+(-p) = e$, and $p+q = q+p$.

Also (given associativity), $p+q+r = e$ iff p, q, r lie on a line L in $\mathbb{CP}^2$.

It remains only to prove that addition is associative, i.e. $p+(q+r) = (p+q)+r$. We do this using Riemann-Roch.

Define $a, b, c, d \in C$ by

$a = p+q$, $\quad b = a+r = (p+q)+r$,

$c = q+r$, $\quad d = p+c = p+(q+r)$.

As $p, q, -a$ are collinear, there exists $\alpha x + \beta y + \gamma z$ vanishing at $p, q, -a$, and $\exists$ $\alpha' x + \beta' y + \gamma' z$ vanishing at $e, a, -a$, so $f = \dfrac{\alpha' x + \beta' y + \gamma' z}{\alpha x + \beta y + \gamma z}$ is meromorphic on C with divisor $(f) = e + a - p - q$.

Similarly we make meromorphic g with $(g) = e + b - a - r$.

Hence $(fg) = 2e + b - p - q - r$.

Similarly we find meromorphic $f'g'$ with

$$(f'g') = 2e + d - p - q - r.$$

Therefore both fg and $f'g'$ lie in $\mathcal{L}(p+q+r-2e)$.

But by Riemann–Roch,

$$\ell(p+q+r-2e) - \ell(K-(p+q+r)+2e) = 1+1-1 = 1,$$

and $\ell(K-(p+q+r)+2e) = 0$ as $\deg(K-(p+q+r)+2e) = -1 < 0$, since $g=1$. Hence $\mathcal{L}(p+q+r-2e) \cong \mathbb{C}$, and fg is proportional to $f'g'$, so $b=d$ and

$$(p+q)+r = p+(q+r). \qquad \square$$

Here is an alternative point of view on the group structure: divisors on C form an abelian group. Also, linear equivalence classes of divisors form an abelian group. Let G be the group of linear equivalence classes $[D]$ of divisors D with $\deg D = 0$.

Then using Riemann–Roch can show that each such $[D]$ may be written as $[p-e]$ for some unique $p \in C$.

So we identify G with C.

Points of inflection of C are $p \in C$ with $p+p+p=0$, since T_pC intersects C at p with multiplicity 3.

Thus the points of inflection are the 8 points of order 3, plus the identity, in $C \cong (\mathbb{R}/\mathbb{Z})^2$.