Exercise sheet 1. Prerequisites: sections 1-5. Week 4.

Q1. Let R be a ring. Show that the Jacobson radical of R coincides with the set $\{x \in R \mid 1 - xy \text{ is a unit for all } y \in R\}$.

Solution. Suppose x lies in the Jacobson radical of R. Suppose for contradiction that $1 - xy$ is not a unit for some $y \in R$. Let $\mathfrak{m}$ be a maximal ideal containing $1 - xy$. We know that $xy \in \mathfrak{m}$ since $x \in \mathfrak{m}$ and thus we conclude that $1 \in \mathfrak{m}$, a contradiction.

Suppose now that $x \in R$ and that $1 - xy$ is a unit for all $y \in R$. Suppose for contradiction that there is a maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$. Then $x \pmod{\mathfrak{m}}$ is a unit in $R/\mathfrak{m}$ and hence there is a $y \in R$ such that $xy \pmod{\mathfrak{m}} = 1 \pmod{\mathfrak{m}}$. In other words, $1 - xy \in \mathfrak{m}$ and so $1 - xy$ is not a unit.

Q2. Let R be a ring.

(i) Show that if $P(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$ is a unit of $R[x]$ then a_0 is a unit of R and a_i is nilpotent for all $i \geq 1$.

(ii) Show that the Jacobson radical and the nilradical of $R[x]$ coincide.

Solution.

(i) Let $Q(x) = b_0 + \cdots + b_tx^t \in R[x]$ be an inverse of $P(x)$. Then $P(0)Q(0) = a_0b_0 = 1$ so that a_0 and b_0 are units. Let $\mathbb{F}_p$ be a prime ideal. Let $j \geq 0$ be the largest integer so that $a_j \pmod{\mathfrak{p}} \neq 0$ and let $l \geq 0$ be the largest integer so that $b_l \pmod{\mathfrak{p}} \neq 0$. If $j > 0$ we have $a_jb_l = 0 \pmod{\mathfrak{p}}$ (since $P(x)Q(x) = 1$), which is not possible because $R/\mathfrak{p}$ is a domain. Hence $j = 0$ and in particular $a_i \in \mathfrak{p}$ for all $i > 0$. Since $\mathfrak{p}$ was arbitrary, we see that a_i lies in the nilradical of R for all $i > 0$.

(ii): We only have to show that any element of the Jacobson radical if $R[x]$ is nilpotent. So let $P(x) \in a_0 + a_1x + \cdots + a_kx^k \in R[x]$ be an element of the Jacobson radical. By Q1, we know that for any $T(x) \in R[x]$, the element $1 - P(x)T(x)$ is a unit. In particular,

$$1 + xP(x) = 1 + a_0x + a_1x^2 + \cdots + a_kx^{k+1}$$

is a unit. By (i), a_i is thus nilpotent for all $i > 0$. In particular $a_0 + a_1x + \cdots + a_kx^k$ is nilpotent (since the radical of a ring is an ideal).

Q3. Let R be a ring and let $N \subseteq R$ be its nilradical. Show that the following are equivalent:

(i) R has exactly one prime ideal.

(ii) Every element of R is either a unit or is nilpotent.

(iii) R/N is a field.

Solution. (i)$\Rightarrow$(ii): Let $\mathfrak{p}$ be the unique prime ideal. Suppose that $r \in R$ is not a unit. Then r is a contained in a maximal ideal, which must coincide with $\mathfrak{p}$. Since $\mathfrak{p}$ is the only prime ideal, the ideal $\mathfrak{p}$ is the nilradical N of R and hence r is nilpotent.

(ii)$\Rightarrow$(iii): Suppose that R/N is not a field. Then either R/N is the zero ring or there is an element $x \in (R/N)^*$, which is not a unit. If R/N is the zero ring, then every element of R is nilpotent (and in fact R is the zero ring). If there is an element $x \in (R/N)^*$, let $x_1 \in R$ be a preimage of x. Then x_1 is not a unit and is not nilpotent. So we have proven the contraposition of (ii)$\Rightarrow$(iii).

(iii)$\Rightarrow$(i): We prove the contraposition. If R has more than one prime ideal then R/N has a non zero prime ideal (since any prime ideal contains N). But this contradicts the fact that R/N is a field.

Q4. Let R be a ring and let $I \subseteq R$ be an ideal. Let $S := \{1 + r \mid r \in I\}$.

(i) Show that S is a multiplicative set.

(ii) Show that the ideal generated by the image of I in R_S is contained in the Jacobson radical of R_S.

(iii) Prove the following generalisation of Nakayama's lemma:

Lemma. *Let M be a finitely generated R-module and suppose that $IM = M$. Then there exists $r \in R$, such that $r - 1 \in I$ and such $rM = 0$.*

Solution. (i): This is clear.

(ii): The ideal I_S generated generated by I in R_S consists of the elements a/b such that $a \in I$ and $b \in S$. By Q1, we thus only have to show that if a/b is such that $a \in I$ and $b \in S$, then $1 - (a/b)(c/d)$ is a unit for all $c \in R$ and $d \in S$. Now $1/b$ and $1/d$ are units of R_S, hence we only have to show that $bd - ac$ is a unit for a, b, c, d as in the previous sentence. Now $bd = (1 + b_1)(1 + d_1) = 1 + b_1 + d_1 + b_1 d_1$ for some $b_1, d_1 \in I$, and thus $bd - ac = 1 + b_1 + d_1 + b_1 d_1 - ac$. Since $b_1 + d_1 + b_1 d_1 - ac \in I$ we see that $bd - ac = 1 + b_1 + d_1 + b_1 d_1 - ac \in S$ and hence is a unit of R_S.

(iii) If $IM = M$ we clearly have $I_S M_S = M_S$. Hence by (ii) and the form of Nakayama's lemma proven in the course, we have $M_S = 0$. Now $m_1, \dots, m_k$ be generators of M. Since M is the kernel of the natural map $M \to M_S$ (since $M_S = 0$), there is an element $s_i \in S$ such that $s_i m_i = 0$ for all i (see the beginning of section 5). Let $s = \prod_i s_i$. Then s annihilates all the m_i and hence M. By construction, $s - 1 \in I$ so we are done.

Q5. Let R be a ring and let M be a finitely generated R-module. Let $\phi : M \to M$ be a surjective homomorphism of R-modules. Prove that ϕ is injective, and is thus an automorphism. [Hint: use ϕ to construct a structure of $R[x]$-module on M and use the previous question.]

Solution. View M as an $R[x]$-module by setting $P(x) \cdot m = P(\phi)(m)$. We have $(x)M = M$ by construction and hence by Q4 (iii), there is a polynomial $Q(x) \in R[x]$ such that $Q(x) - 1 \in (x)$ and $Q(x)M = 0$. Let $m_0 \in \ker(\phi)$. Then $Q(x)(m_0) = m_0$ and hence $m_0 = 0$. Thus ϕ is injective.

Q6. Let R be a ring. Let $\mathcal{S}$ be the subset of the set of ideals of R defined as follows: an ideal I is in $\mathcal{S}$ iff all the elements of I are zero-divisors. Show that $\mathcal{S}$ has maximal elements (for the relation of inclusion) and that every maximal element is a prime ideal. Show that the set of zero divisors of R is a union of prime ideals.

Solution. If $\mathcal{T}$ is a totally ordered subset of $\mathcal{S}$, then the union of its elements is an ideal, and it clearly consists of zero divisors. So every totally ordered subset of $\mathcal{T}$ has upper bounds and thus by Zorn's lemma, the ordered set $\mathcal{T}$ has maximal elements. Note that we may refine this reasoning as follows. Let $I \in \mathcal{S}$. Consider the subset $\mathcal{S}_I$ of $\mathcal{S}$, which consists of ideals containing I. By a completely similar reasoning, the subset $\mathcal{S}_I$ has maximal elements for the relation of inclusion. We contend that if $J \in \mathcal{S}_I$ is a maximal element, then it is also maximal in $\mathcal{S}$. Indeed, suppose that $J' \supseteq J$ for some ideal $J' \in \mathcal{S}$. Then $J' \in \mathcal{S}_I$ and hence $J' = J$. Now note that

$$\{\text{zero-divisors of } R\} = \cup_{r \in R,\, r \text{ a zero-div.}} (r) \subseteq \cup_{r \in R,\, r \text{ a zero-div.}} J(r)$$

where $J(r)$ a maximal element of $\mathcal{S}$ containing the ideal (r). Since $J(r)$ also consists of zero-divisors, we conclude that

$$\{\text{zero-divisors of } R\} = \cup_{r \in R,\, r \text{ a zero-div.}} J(r)$$

Hence we only have to prove that the maximal elements of $\mathcal{S}$ are prime ideals.

Let I be a maximal element of $\mathcal{S}$. Let $x, y \in R\backslash I$ and suppose for contradiction that $xy \in I$. Then we have

$$((x) + I)((y) + I) \subseteq I$$

By maximality of I, there are elements $a \in (x)+I$ and $b \in (y)+I$, which are not zero divisors. Hence $ab \in I$ so that ab is a zero divisor, which is contradiction (note that the set of non zero divisors is a multiplicative set). So we must have $x \in I$ or $y \in I$, so I is prime.

Q7. Let R be a ring. Consider the inclusion relation on the set $\mathrm{Spec}(R)$. Show that there are minimal elements in $\mathrm{Spec}(R)$.

Solution. Let $\mathcal{T}$ be a totally ordered subset of $\mathrm{Spec}(R)$ for the relation $\supseteq$. Note that the maximal elements for the relation $\supseteq$ are the minimal elements for the inclusion relation (which is $\subseteq$). Let $I := \cap_{\mathfrak{p} \in \mathcal{T}}$. Then I is an ideal. We claim that I is prime.

To see this, let $x, y \in R$ and suppose for contradiction that $x, y \in R\backslash I$ and that $xy \in I$. By assumption there are prime ideals $\mathfrak{p}_x, \mathfrak{p}_y \in \mathcal{T}$ such that $x \notin \mathfrak{p}_x$ and $y \notin \mathfrak{p}_y$. Suppose without restriction of generality that $\mathfrak{p}_x \supseteq \mathfrak{p}_y$ (recall that $\mathcal{T}$ is totally ordered). We have $xy \in \mathfrak{p}_y$ and thus either x or y lies in $\mathfrak{p}_y$. This contradicts the fact that $x, y \notin \mathfrak{p}_y$. The ideal I thus lies in $\mathrm{Spec}(R)$ and it is a lower bound for $\mathcal{T}$. We may thus apply Zorn's lemma to conclude that there are minimal elements in $\mathrm{Spec}(R)$.