Exercise sheet 3. Prerequisites: sections 1-10. Week 8

Q1. Let R be a subring of a ring T. Suppose that T is integral over R. Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}_1, \mathfrak{q}_2$ be prime ideals of T such that $\mathfrak{q}_1 \cap R = \mathfrak{q}_2 \cap R = \mathfrak{p}$ and $\mathfrak{q}_1 \neq \mathfrak{q}_2$. Show that we have $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$ and $\mathfrak{q}_2 \not\subseteq \mathfrak{q}_1$.

Solution. By symmetry, we only have to show that $\mathfrak{q}_1 \not\subseteq \mathfrak{q}_2$. Suppose for contradiction that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$. The ring $R/\mathfrak{p}$ is can be viewed as a subring of $T/\mathfrak{q}_1$ and by assumption we have $\mathfrak{q}_2 \pmod{\mathfrak{q}_1} \cap R/\mathfrak{p} = (0)$. We may thus assume wrog that R and T to be domains and that $\mathfrak{q}_1$ and $\mathfrak{p}$ are zero ideals. Now let $e \in \mathfrak{q}_2\backslash\{0\}$ and let $P(x) \in R[x]$ be a non zero monic polynomial such that $P(e) = 0$. Since T is a domain, we may assume that the constant coefficient of $P(x)$ is non zero (otherwise replace $P(x)$ by $P(x)/x^k$ for a suitable $k \geq 1$). But then $P(0)$ is a linear combination of positive powers of e (since $P(e) = 0$), so $P(0) \in R \cap \mathfrak{q}_2 = (0)$. This is a contradiction, since $P(0) \neq 0$.

Q2. Let R be a ring. Show that the two following conditions are equivalent:

(i) R is a Jacobson ring.

(ii) If $\mathfrak{p} \in \mathrm{Spec}(R)$ and $R/\mathfrak{p}$ contains an element b such that $(R/\mathfrak{p})[b^{-1}]$ is a field, then $R/\mathfrak{p}$ is a field.

Here we write $(R/\mathfrak{p})[b^{-1}]$ for the localisation of $R/\mathfrak{p}$ at the multiplicative subset $1, b, b^2, \ldots$.

Solution.

(i) $\Rightarrow$ (ii) : If R is a Jacobson, then so is $R/\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence (ii) follows from Lemma 10.2.

(ii) $\Rightarrow$ (i) : Note first that R is a Jacobson ring iff any prime ideal of R is the intersection of the maximal ideals containing it (this is straightforward). Now suppose that R is not Jacobson. Then there is a prime ideal $\mathfrak{p}$ of R and an element $e \notin \mathfrak{p}$ such that e is in the Jacobson radical of $\mathfrak{p}$. In other words, $e \pmod{\mathfrak{p}} \neq 0$ and $e \pmod{\mathfrak{p}}$ lies in the Jacobson radical of $R/\mathfrak{p}$. Now let $\mathfrak{q}$ be an ideal maximal among the prime ideals of $R/\mathfrak{p}$, which do not contain $e \pmod{\mathfrak{p}}$. The ideal $\mathfrak{q}$ is prime, because it corresponds to a maximal ideal of $(R/\mathfrak{p})[(e \pmod{\mathfrak{p}})^{-1}]$ by Lemma 5.6, and it is not maximal, since $e \pmod{\mathfrak{p}}$ lies in the intersection of all the maximal ideals of $R/\mathfrak{p}$. The ring $(R/\mathfrak{p})/\mathfrak{q}$ has by construction the property that any of its non zero prime ideals contains $(e \pmod{\mathfrak{p}}) \pmod{\mathfrak{q}}$. In particular, the ring $((R/\mathfrak{p})/\mathfrak{q})[((e \pmod{\mathfrak{p}}) \pmod{\mathfrak{q}})^{-1}]$ is a field, because it is a domain and its only prime ideal is the zero ideal. On other hand, $((R/\mathfrak{p})/\mathfrak{q})$ is a not field, since $\mathfrak{q}$ is not maximal. Now if we let $q : R \to R/\mathfrak{p}$ be the quotient map, we have $((R/\mathfrak{p})/\mathfrak{q}) \simeq R/q^{-1}(\mathfrak{q})$ and thus this contradicts (ii). We have thus proven the contraposition of the implication (ii) $\Rightarrow$ (i).

Q3. Let k be field and let R be a finitely generated algebra over k. Show that the two following conditions are equivalent:

(i) $\mathrm{Spec}(R)$ is finite.

(ii) R is finite over k.

Solution. (i) $\Rightarrow$ (ii) : Suppose that $\mathrm{Spec}(R)$ is finite. By Noether's normalisation lemma, there is an injection $k[x_1, \ldots, x_d] \to R$, which makes R into a finite $k[x_1, \ldots, x_d]$-algebra. Since the corresponding map of spectra $\mathrm{Spec}(R) \to \mathrm{Spec}(k[x_1, \ldots, x_d])$ is surjective by Theorem 8.8, this implies that $\mathrm{Spec}(k[x_1, \ldots, x_d])$ is finite. In particular, $k[x_1, \ldots, x_d]$ has only finitely many maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$. Since $k[x_1, \ldots, x_d]$ is a Jacobson ring by Theorem 10.5, we have $\cap_i \mathfrak{m}_i = \mathfrak{r}((0)) = 0$ and so we may deduce from the Chinese remainder theorem that $k[x_1, \ldots, x_d] \simeq \oplus_i R/\mathfrak{m}_i$. Since $k[x_1, \ldots, x_d]$ is a domain, this implies that $t = 1$. In particular, $k[x_1, \ldots, x_d]$ is field, which is only possible if $d = 0$ (otherwise, x_1 is a non unit). Hence R is finite over k.

(ii) $\Rightarrow$ (i) : This follows from Proposition 8.12.

Q4. Let k be an algebraically closed field. Let $P_1, \dots, P_d \in k[x_1, \dots, x_d]$. Suppose that the set

$$\{(y_1, \dots, y_d) \in k^d \mid P_i(y_1, \dots, y_d) = 0 \, \forall i \in \{1, \dots, d\}\}$$

is finite. Show that

$$\mathrm{Spec}(k[x_1, \dots, x_d]/(P_1, \dots, P_d))$$

is finite.

Solution. From Corollary 9.5 and Corollary 9.3, we deduce that $\mathfrak{r}(((P_1, \dots, P_d))$ is the intersection of finitely many maximal ideals of $k[x_1, \dots, x_d]$, say $\mathfrak{m}_1, \dots \mathfrak{m}_t$. From the Chinese remainder theorem, we deduce that

$$k[x_1, \dots, x_d]/\mathfrak{r}((P_1, \dots, P_d)) \simeq \prod_i k[x_1, \dots, x_d]/\mathfrak{m}_i \simeq \prod_i k,$$

In particular, $\mathrm{Spec}(k[x_1, \dots, x_d]/\mathfrak{r}((P_1, \dots, P_d)))$ is finite. Now we have

$$\mathrm{Spec}(k[x_1, \dots, x_d]/\mathfrak{r}((P_1, \dots, P_d))) \simeq \mathrm{Spec}(k[x_1, \dots, x_d]/(P_1, \dots, P_d))$$

(see the remark after Lemma 4.4) so the conclusion follows.

Q5. Let R be a ring and let R_0 be the prime ring of R (see the preamble of the notes for the definition). Suppose that R is a finitely generated R_0-algebra. Suppose also that R is a field. Prove that R is a finite field.

Solution. Since R_0 is contained in a field, it is a domain and so R_0 is either a finite field or it is isomorphic to $\mathbb{Z}$. Suppose first that R_0 is a finite field. Then R is a finite field extension of a finite field by the weak Nullstellensatz and hence R is a finite field. Now suppose that $R \simeq \mathbb{Z}$. Then R contains the fraction field $\mathbb{Q}$ of $\mathbb{Z}$ and R is a finitely generated $\mathbb{Q}$-algebra, which is a field. By the weak Nullstellensatz again, we conclude that R is a finite field extension of $\mathbb{Q}$. From Corollary 10.3, we deduce that $\mathbb{Z} \simeq \mathbb{Q}$ (note that $\mathbb{Z}$ is a Jacobson ring), which is a contradiction. So R_0 must be finite field and so R is a finite field.

Q6. Let k be a field and let $\mathfrak{m}$ be a maximal ideal of $k[x_1, \dots, x_d]$. Show that there are polynomials $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_d(x_1, \dots, x_d)$ such that $\mathfrak{m} = (P_1, \dots, P_d)$.

Solution. By induction on $d \geq 1$. If $d = 1$ then this follows from the fact that $k[x_1]$ is a PID. We suppose that the statement holds for $d - 1$. Let $K = k[x_1, \dots, x_d]/\mathfrak{m}$. By the weak Nullstellensatz, this is a finite field extension of k. Let $\phi : k[x_1, \dots, x_d] \to K$ be the natural surjective homomorphism of k-algebras. Let $L = \phi(k[x_1, \dots, x_{d-1}])$. This is a domain and by Lemma 8.9, L is a field, since it contains k and is contained inside an integral extension of k. Let $\psi : k[x_1, \dots, x_{d-1}] \to L$ be the surjective homomorphism of k-algebras arising by restricting ϕ. The map ψ induces a surjective homomorphism of k-algebras

$$\Psi : k[x_1, \dots, x_d] \simeq (k[x_1, \dots, x_{d-1}])[x_d] \to L[x_d]$$

and there is a surjective homomorphism of L-algebras

$$\Lambda : L[x_d] \to K,$$

which sends x_d to $\phi(x_d)$. By construction, we have $\phi = \Lambda \circ \Psi$. In particular, we have $\mathfrak{m} := \Psi^{-1}(\Lambda^{-1}(0))$. Since $L[x_d]$ is a PID and $\phi(x_d)$ is algebraic over k, we have $\Lambda^{-1}(0) = (P(x_d))$ for some non zero polynomial $P(x_d) \in L[x_d]$. Now let $P_d(x_1, \dots, x_d) \in (k[x_1, \dots, x_{d-1}])[x_d]$ be a preimage by Ψ of $P(x_d)$.

We claim that $\mathfrak{m} = (\ker(\Psi), P_d)$. To see this, note that $\Psi((\ker(\Psi), P_d)) = (P(x_d))$ and so we have $(\ker(\Psi), P_d) \subseteq \mathfrak{m}$. On the other hand, if $e \in \mathfrak{m}$ then $\Psi(e) \in (P(x_d))$ and thus there is an element $e' \in (P_d)$ such that $\Psi(e) = \Psi(e')$ (since Ψ is surjective). In particular, we have $e - e' \in \ker(\Psi)$, so that $e \in (\ker(\Psi), P_d)$.

Now by the inductive assumption, $\ker(\Psi)$ is generated by polynomials

$$P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_{d-1}(x_1, \dots, x_{d-1})$$

and so $\mathfrak{m}$ is generated by $P_1(x_1), P_2(x_1, x_2), P_3(x_1, x_2, x_3), \dots, P_d(x_1, \dots, x_d)$.

Q7. Let R be a domain. Show $R[x]$ is integrally closed if R is integrally closed.

Here are some hints for this exercise. Let K be the fraction field of R.

(i) Show first that it suffices to show that $R[x]$ is integrally closed in $K[x]$ (ie that the integral closure of $R[x]$ in $K[x]$ is $R[x]$).

(ii) Consider $Q(x) \in K[x]$ and suppose that $Q(x)$ is integral over $R[x]$. Show that $Q(x) + x^t$ satisfies an integral equation with coefficients in $R[x]$, whose constant coefficient is a monic polynomial, if t is sufficiently large.

(iii) Conclude.

Solution.

Suppose that R is integrally closed in its fraction field K. The fraction field of $R[x]$ is $K(x) = \mathrm{Frac}(K[x])$. Let $Q(x) \in K(x)$ and suppose that $Q(x)$ is integral over $R[x]$. Then $Q(x)$ is in particular integral over $K[x]$ and we saw that in the solution of Q4 that $K[x]$ is integrally closed, since it is a PID. So we deduce that $Q(x) \in K[x]$.

Now let

$$Q^n + P_{n-1}Q^{n-1} + \dots + P_0 = 0$$

be a non trivial integral equation for Q over $R[x]$ (so that $P_i \in R[x]$ and $n \geq 1$). Let t be a natural number, which is strictly larger than the degrees of all the P_i and of Q. Let $T = Q - x^t$. The polynomial T is monic by construction and we have

$$(T + x^t)^n + P_{n-1}(T + x^t)^{n-1} + \dots + P_0 = 0$$

so that T satisfies an integral equation of the type

$$T^n + H_{n-1}T^{n-1} + \dots + H_0 = 0$$

where

$$H_0 = P_0 + x^t P_1 + x^{2t} P_2 + \dots + x^{tn}.$$

Now note that H_0 is a monic polynomial, because $tn > ti + \deg(P_i)$ for all $i \in \{0, \dots, n-1\}$. Finally, note that in view of the penultimate equation, we have

$$T(T^{n-1} + H_{n-1}T^{n-2} + \dots + H_1) = -H_0$$

and by Q5 of sheet 2, we have $T \in R[x]$ (because $H_0 \in R[x]$ and H_0 and T are monic). Since $x^t \in R[x]$ we see that we also have $Q \in R[x]$, which is what was to be proven.