C4.4: Hyperbolic Equations

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TA: Isaac Newell
Course Weight: 1.00 unit(s)
Level: M-level
Method of assessment: Written examination
Course Term: Hilary Term 2024
Time & Place:
Lectures: 4:00pm Tuesdays & Wednesdays (L5)
Tutorial Sessions: To be announced

Recommended Prerequisites:

A good background in Multivariate Calculus and Lebesgue Integration is expected (e.g. as covered in the Oxford Prelims & Part A Integration). It would be useful to know some basic Functional Analysis and Distribution Theory; however, this is not strictly necessary as the presentation will be self-contained.

Course Overview:

We introduce analytical and geometric approaches to hyperbolic equations, by discussing model problems from transport equations, wave equations, and conservation laws. These approaches have been applied/extended extensively in recent research and lie at the heart of the Theory of Hyperbolic PDEs.

Learning Outcomes

You will learn the rigorous treatment of hyperbolic equations through analytical and geometric approaches as an introduction to the Theory of Hyperbolic PDEs.

You will see some model problems/methods for hyperbolic equations.

Course Synopsis:

1. **Transport equations and nonlinear first order equations:** Method of characteristics, formation of singularities
2. **Introduction to nonlinear hyperbolic conservation laws:** Discontinuous solutions, Rankine-Hugoniot relation, Lax entropy condition, shock waves, rarefaction waves, Riemann problem, entropy solutions, Lax-Oleinik formula, uniqueness.
3. **Linear wave equations:** The solution of Cauchy problem, energy estimates, finite speed of propagation, domain of determination, light cone and null frames, hyperbolic rotation and Lorentz vector fields, Sobolev inequalities, Klainerman inequality.
4. **Nonlinear wave equations:** local well-posedness, weak solutions

If time permits, we might also discuss parabolic approximation (viscosity method), compactness methods, Littlewood-Paley theory, and harmonic analysis techniques for hyperbolic equations/systems (off syllabus - not required for exam)

Reading List:

We refer to [1], [2, Chapters 2,3,5,7,11,12], and [3] for detailed exposition.

1. Alinhac, S.: Hyperbolic Partial Differential Equations,
   Springer-Verlag: New York, 2009.

2. Evans, L.: Partial Differential Equations. Second edition.
   Graduate Studies in Mathematics, 19.
   American Mathematical Society, 2010.

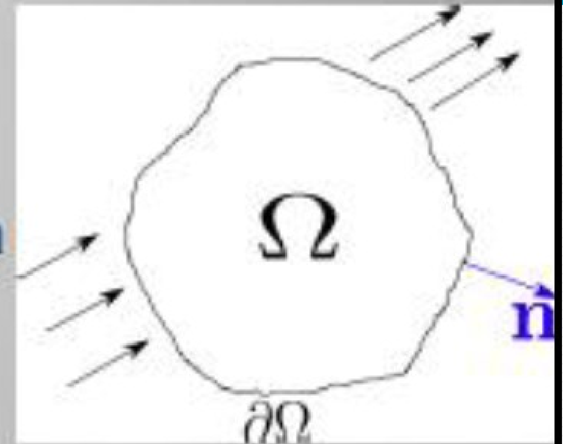
3. John, F.: Partial Differential Equations. Fourth edition.
   Applied Mathematical Sciences, 1.
   Springer-Verlag: New York, 1982

Please note that e-book versions of many books in the reading lists can be found on SOLO and ORLO.

Brief History

- **Analysis of Differential Equations can date back as early as the period when Calculus was invented.**
- **1671: Newton called Fluxional Equations**
- **1676: Leibniz introduced the term**
  Differential Equations
  (Aequatio Differentialis, in Latin)
- **It is fair to say that every subject that uses Calculus involves differential equations.**
- **Many subjects revolve entirely around their underlying PDEs:** Euler equations, Navier-Stokes equations, Maxwell's equations, Boltzmann equation, Schrödinger equation, Einstein equation,...

Conservation Laws:



Rate of change of the total amount of certain quantity contained in a fixed region Ω

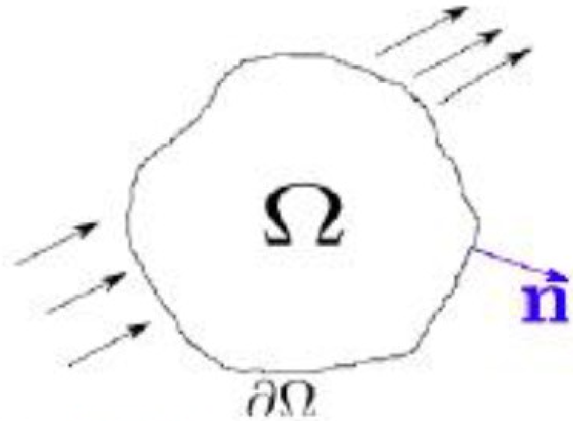
= Flux of this quantity across the boundary $\partial\Omega$ of the region

- The amount of such a quantity in any region can be measured by accounting for how much of it is currently present and how much of it enters or leaves the region in any fixed period of time.

Examples: Three Fundamental Laws of Nature

- **Conservation Laws of Mass and Energy:** Mass and Energy can be neither created nor destroyed.
- **Conservation Law of Momentum:** The total momentum of a closed system of objects remains constant through time.

Conservation Laws



Rate of Change of the Total Amount of Certain Quantity in a Fixed Region Ω = Flux of the Quantity across the Boundary $\partial\Omega$.

Conservation Law via Calculus

$$\Longleftrightarrow \qquad \frac{d}{dt}\int_{\Omega} u\,d\mathbf{x} = -\int_{\partial\Omega} \mathbf{f}\cdot\mathbf{n}\,dS$$

u – Density of the Quantity

$\mathbf{f}$ – Flux of the Quantity

$\mathbf{n}$ – Outward Normal to Ω

dS – Surface Element on $\partial\Omega$

Calculus Manipulations $\Longrightarrow$

$$\partial_t u + \nabla_{\mathbf{x}}\cdot\mathbf{f}(u) = 0$$

Physical Systems with $m \geq 2$ Quantities – Density Functions

$\Longrightarrow$ **Systems of Conservation Laws**:

$$\partial_t\,\mathbf{u} + \nabla\cdot\mathbf{f}(\mathbf{u}) = 0$$

$$\mathbf{u} = (u_1,\cdots,u_m)^\top \qquad \mathbf{f}(\mathbf{u}) = (\mathbf{f}_1(\mathbf{u}),\cdots,\mathbf{f}_d(\mathbf{u}))$$

Euler Equations for Compressible Fluids

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0 & \text{(conservation of mass)} \\ \partial_t(\rho \mathbf{v}) + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} p = 0 & \text{(conservation of momentum)} \\ \partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla_{\mathbf{x}} \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 & \text{(conservation of energy)} \end{cases}$$


Leonhard Euler

Constitutive Relations: $p = p(\rho, e)$

- ρ – density, $\mathbf{v} = (v_1, v_2, v_3)^\top$ – fluid velocity
- p – pressure, e – internal energy

***Govern** the Flows when Convective Motions Dominate Diffusion/Dispersion, …*

e.g. , shock waves in Gases, Elastic Fluids, Shallow Water, ……

Poisson, Challis, Stokes, Kelvin, Rayleigh, Airy, Earnshaw, Riemann, Rankine, Christoffel, Mach, Clausius, Kirchhoff, Gibbs, Hugoniot, Duhem, Hadamard, Jouguet, Zamplen, Weber, Taylor, Becker, Bethe, Weyl, von Neumann, Courant, Friedrichs, ……


George Stokes

Euler Equations for Potential Flow

$$\begin{cases} \partial_t \rho + \nabla_{\mathrm{x}} \cdot (\rho \nabla_{\mathrm{x}} \Phi) = 0, & \text{(Conservation of mass)} \\ \partial_t \Phi + \frac{1}{2}|\nabla_{\mathrm{x}} \Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{B_\infty}{\gamma-1} := \frac{\rho_\infty^{\gamma-1} + \frac{\gamma-1}{2} u_\infty^2}{\gamma-1}, & \text{(Bernoulli's law)} \end{cases}$$

for $\gamma > 1$ or, equivalently, **Nonlinear Wave Equations:**

$$\partial_t \rho(\partial_t \Phi, \nabla_{\mathrm{x}} \Phi, B_\infty) + \nabla_{\mathrm{x}} \cdot \big(\rho(\partial_t \Phi, \nabla_{\mathrm{x}} \Phi, B_\infty) \nabla_{\mathrm{x}} \Phi\big) = 0,$$

with

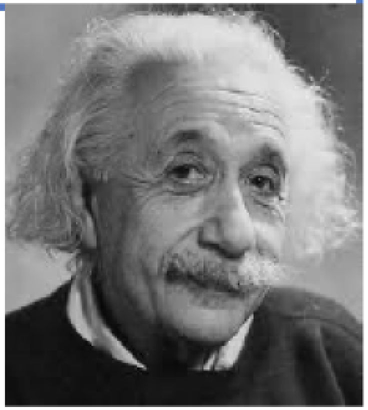
$$\rho(\partial_t \Phi, \nabla_{\mathrm{x}} \Phi, B_\infty) = \big(B_\infty - (\gamma - 1)(\partial_t \Phi + \tfrac{1}{2}|\nabla_{\mathrm{x}} \Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

- Aerodynamics/Gas Dynamics: Fundamental PDE
- The potential flow equations and the full Euler equations coincide or are close each other in many important physical situations

  J. Hadamard: Leçons sur la Propagation des Ondes,

  Hermann: Paris 1903

Conservation Laws and Einstein Equations



$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

$T_{\mu\nu}$ – Stress-energy tensor (Energy-momentum tensor)
$G_{\mu\nu}$ – Einstein tensor (Function of the metric)

These equations, with the geodesic equation, form the core of the mathematical formulation of General Relativity

Structure of the Einstein Equations

$\Longrightarrow$ **Conservation Laws of Energy and Momentum**:

$$\nabla_b T^{ab} = T^{ab}{}_{;b} = 0$$

CALCULUS OF VARIATIONS

A Field of Mathematics that deals with extremizing functionals, as opposed to ordinary calculus which deals with functions:

$$I[w] = \int_{\Omega} L(\nabla_x w(x), w(x), x)dx$$

- Energy or Action Functionals in Physics/Engineering/industry....
- Distance/Metric Functional in Optics (light),
  Geometry (geodesics, minimal surfaces, ...), ...
- Cost Functionals in Optimization (controls, games, image processing, design,
  finance, transportation, ...), ...

POINT: Seek a Minimizer or Critical Point u of $I[\cdot]$:

$$I'[u]=0$$

Great Progress has been made in the recent four decades...

Conservation Laws and Calculus of Variations

- Systems of Euler-Lagrange Equations

- Noether's Theorem:

Any Differentiable Symmetry of the Action of a Physical System Has a Corresponding Conservation Law.

Any Invariance of the Variational Integral I[w] leads to a corresponding Conservation Law for the critical point of I[·]

Scalar Conservation Laws

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u \in \mathbb{R}, \ \mathbf{x} \in \mathbb{R}^n$$

$$\mathbf{f} : \mathbb{R} \to \mathbb{R}^n \tag{1}$$

Nonlinear Wave Equations

$$\partial_t \mathbf{A}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) + \nabla \cdot \mathbf{B}(\mathbf{u}, \mathbf{u}_t, \nabla \mathbf{u}) = 0$$

In this course, we first focus on

Scalar Conservation Laws

Semilinear Wave Equations

Transport Equations:

$$\mathbf{f}(u) = (b_1, b_2, \ldots, b_n)u$$

with constant vector field $(b_1, b_2, \ldots, b_n)$

I. Transport Equations
& Method of Characteristics
for Nonlinear First-Order Equations

1. Transport Equations

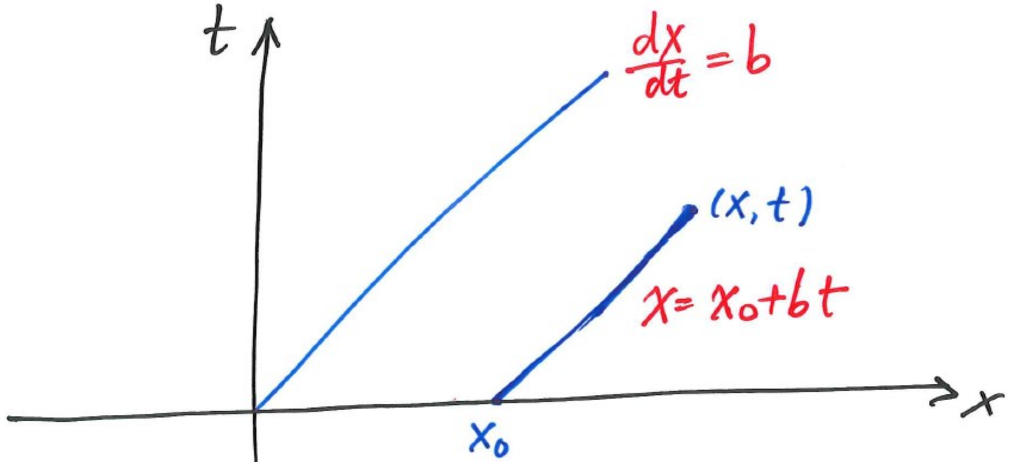
Probably the simplest PDE is

(V) $u_t + b u_x = 0.$ b is a constant.

Along the direction $\frac{dx}{dt} = b$, for any solution $u = u(x,t)$ of (V), we have

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + b u_x = 0.$$

$\Rightarrow$ $u \equiv$ const. along the line with direction $(b, 1) \in \mathbb{R}^{1+1}$!



(*) $u_t + b \cdot Du = 0$ $\quad x \in \mathbb{R}^n,\ t \in \mathbb{R}_+$

$b = (b_1, \cdots, b_n) \in \mathbb{R}^n$

$Du = D_x u = (u_{x_1}, \cdots, u_{x_n})$

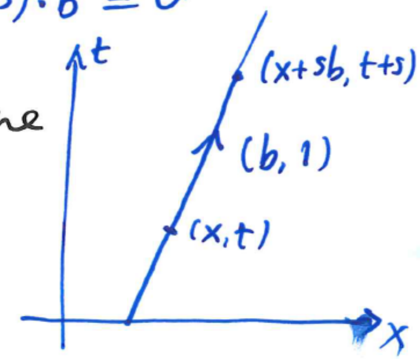
Analysis: Assume $u \in C^1(\mathbb{R}^n \times \mathbb{R}_+)$ is a soln of (*)

$\forall (x,t) \in \mathbb{R}^n \times (0,\infty)$, define

$z(s) = u(x+sb, t+s) = u((x,t) + s(b,1))$

$\frac{dz}{ds} = \frac{\partial u}{\partial t}(x+sb, t+s) + Du(x+sb, t+s) \cdot b \equiv 0.$

↳ u is const. along the line through (x,t) with direction $(b,1) \in \mathbb{R}^{n+1}$

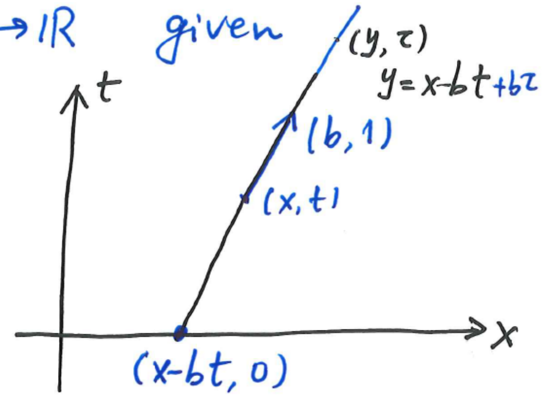


↳ If we know the value of u at any point on each such line, we know its value everywhere in $\mathbb{R}^n \times (0, \infty)$.

Cauchy Problem

$$(*)\begin{cases} u_t + b\cdot Du = 0 & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g(x) : \mathbb{R}^n \to \mathbb{R} & \text{given} \end{cases}$$

Analysis



$$u(x,t) = u(x-bt, 0)$$
$$= g(x-bt)$$
$$\forall x \in \mathbb{R}^n,\ t>0$$

$$\begin{cases} y - x = bs \\ \tau - t = s \end{cases}$$

$\hookrightarrow$ $y = x + b(\tau - t)$
$= (x-bt) + b\tau$

$\hookrightarrow$ Uniqueness

If $g \in C^1 \to$ $u(x,t) = g(x-bt)$ satisfies $(*)$

$\hookrightarrow$ Existence of classical Soln.

If $g \notin C^1 \to$ $\nexists$ C^1 solution.

But we have to accept $u(x,t) = g(x-bt)$ as a solution, which is called a **Weak Solution** of $(*)$.

Nonhomogeneous Problem

$$(**)\begin{cases} u_t + b\cdot Du = f(x,t) & \mathbb{R}^n \times (0,\infty) \\ u|_{t=0} = g(x), & x \in \mathbb{R}^n \end{cases}$$

Analysis: For $z(s) = u(x+bs, t+s)$.

$$\frac{dz(s)}{ds} = u_t(x+bs, t+s) + Du(x+bs, t+s)\cdot b = f(x+bs, t+s)$$

$$\int_{-t}^{0} \frac{dz(s)}{ds}\, ds = \int_{-t}^{0} f(x+bs, t+s)\, ds$$

$$\int_{-t}^{0} \frac{dz(s)}{ds}\, ds = z(0) - z(-t) = u(x,t) - g(x-bt)$$

$$\int_{-t}^{0} f(x+bs, t+s)\, ds = \int_0^t f(x+(s-t)b, s)\, ds$$

$$\Rightarrow \boxed{u(x,t) = g(x-bt) + \int_0^t f(x+(s-t)b, s)\, ds}$$

Verification: $u(x,t)$ solves $(**)$ indeed.

* Method of characteristics

$$u_t + b \cdot Du = 0, \qquad u|_{t=0} = g(x)$$

$$\begin{cases} \dfrac{dx}{dt} = b & \rightarrow \quad x = x_0 + bt \quad \rightarrow \quad x_0 = x - bt \\ \dfrac{du}{dt} = 0 & \rightarrow \quad u = g(x_0) \\ x|_{t=0} = x_0 \\ u|_{t=0} = g(x_0) \end{cases}$$

$$\boxed{u = g(x - bt)}$$

PDE $\Longleftrightarrow$ System of ODEs.

* General $b = b(x,t) \in \mathbb{R}^n$

2. The Method of Characteristics for Nonlinear First Order Equations

We develop the method of characteristics to solve the nonlinear first order PDE

$$F(x, u, Du) = 0 \quad \text{in } U, \qquad u = g \quad \text{on } \Gamma, \tag{2}$$

where $U \subset \mathbb{R}^n$ is an open set, $x \in U$, $\Gamma \subset \partial U$, $g : \Gamma \to \mathbb{R}$ and $F : \overline{U} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are given smooth functions. Writing

$$F = F(x, z, \mathbf{p}) = F(x_1, \cdots, x_n, z, p_1, \cdots, p_n),$$

we use the notation

$$D_x F = (F_{x_1}, \cdots, F_{x_n}), \quad D_z F = F_z, \quad D_{\mathbf{p}} F = (F_{p_1}, \cdots, F_{p_n}).$$

The basic idea of the method is as follows:

- Given $x \in U$, find a curve within U connecting x with a point $x_0 \in \Gamma$.
- Determine u along this curve.
- This usually requires the knowledge of Du along this curve.
- Let $x(s)$ be such a curve and set

$$z(s) = u(x(s)) \quad \text{and} \quad \mathbf{p}(s) = Du(x(s)).$$

  Then $x(s), z(s), \mathbf{p}(s)$ are determined by solving systems of ODEs.

So, the key point is to derive the ODEs governing $x(s), z(s), \mathbf{p}(s)$.

To derive these equations, first

$$\frac{dz}{ds} = \sum_{j=1}^{n} u_{x_j}(x(s))\frac{dx_j}{ds}, \qquad \frac{dp_i}{ds} = \sum_{j=1}^{n} u_{x_i x_j}(x(s))\frac{dx_j}{ds}.$$

In order to eliminate the second derivative $u_{x_i x_j}$, we differentiating the PDE in (2) with respect to x_j to get

$$F_{x_j} + F_z u_{x_j} + \sum_{i=1}^{n} F_{p_i} u_{x_i x_j} = 0.$$

Restricting this equation to the curve $x(s)$, we obtain

$$F_{x_j}(x, z, \mathbf{p}) + F_z(x, z, \mathbf{p})p_j + \sum_{i=1}^{n} F_{p_i}(x, z, \mathbf{p})u_{x_i x_j}(x(s)) = 0.$$

Thus, if we set

$$\frac{dx_i}{ds} = F_{p_i}(x, z, \mathbf{p}),$$

then

$$\frac{dp_i}{ds} = -F_{x_i}(x, z, \mathbf{p}) - F_z(x, z, \mathbf{p})p_i, \quad \frac{dz}{ds} = \sum_{i=1}^{n} p_i F_{p_i}(x, z, \mathbf{p}).$$

We therefore obtain the system of ODEs

$$\begin{cases} \frac{dx}{ds} & = D_{\mathbf{p}} F(x, z, \mathbf{p}), \\ \frac{dz}{ds} & = \mathbf{p} \cdot D_{\mathbf{p}} F(x, z, \mathbf{p}), \\ \frac{d\mathbf{p}}{ds} & = -D_x F(x, z, \mathbf{p}) - D_z F(x, z, \mathbf{p})\mathbf{p}. \end{cases} \tag{3}$$

which is called the characteristic ODEs for (2)

- We still need to determine appropriate initial conditions for the characteristic ODEs (3) using $u = g$ on Γ.
- We use local parametrizations of Γ. Let Γ be locally parametrized by

$$x_i = x_i(\theta_1, \cdots, \theta_{n-1}), \quad i = 1, \cdots, n$$

with parameters $\theta_1, \cdots, \theta_{n-1}$. We will write $x = x(\theta)$ for short.
- Let $x^0 := x(\theta^0)$ be a point on Γ. For the ODEs in (3) it is natural to set $x(0) = x^0$ and $z(0) = z^0 := g(x^0)$. We need to determine $\mathbf{p}(0) = \mathbf{p}^0 := (p_1^0, \cdots, p_n^0)$.
- By the PDE in (2) we have $F(x^0, z^0, \mathbf{p}^0) = 0$.

- Using $u = g$ on Γ, we have $u(x(\theta)) = \tilde{g}(\theta) := g(x(\theta))$. Differentiating with respect to θ_j gives

$$\sum_{i=1}^{n} u_{x_i}(x(\theta))\frac{\partial x_i}{\partial \theta_j} = \tilde{g}_{\theta_j}(\theta), \quad j = 1, \cdots, n-1.$$

By setting $\theta = \theta^0$ we obtain n equations on $\mathbf{p}^0$:

$$\begin{aligned} &\sum_{i=1}^{n} p_i^0 \frac{\partial x_i}{\partial \theta_j}(\theta^0) = \tilde{g}_{\theta_j}(\theta^0), \quad j = 1, \cdots, n-1, \\ &F(x^0, z^0, \mathbf{p}^0) = 0. \end{aligned} \tag{4}$$

In many situations, $\mathbf{p}^0$ can be obtained by solving (4).

Example 1

Consider the problem

$$uu_x + u_y = 2, \qquad u(x,x) = x.$$

Here $F = F(x, y, z, p_1, p_2) = zp_1 + p_2 - 2$. Since $F_x = F_y = 0$, $F_z = p_1$, $F_{p_1} = z$, and $F_{p_2} = 1$, it follows from the characteristic ODEs (3) that

$$\frac{dx}{ds} = z, \quad \frac{dy}{ds} = 1, \quad \frac{dz}{ds} = p_1 z + p_2.$$

Recall that $z = u(x, y)$, $p_1 = u_x(x, y)$ and $p_2 = u_y(x, y)$, we have

$$\frac{dz}{ds} = 2.$$

To include the boundary condition $u(x,x) = x$, we fix any τ, let $(x(s), y(s))$ be the characteristic curve with

$$(x(0), y(0)) = (\tau, \tau).$$

Then $z(0) = \tau$ and thus

$$\begin{cases} \frac{dx}{ds} = z, & x(0) = \tau, \\ \frac{dy}{ds} = 1, & y(0) = \tau, \\ \frac{dz}{ds} = 2, & z(0) = \tau. \end{cases}$$

Solving these equations give

$$y(s) = s + \tau, \qquad z(s) = 2s + \tau, \qquad x(s) = s^2 + \tau s + \tau.$$

Now for any (x, y) we determine s and τ such that $(x, y) = (x(s), y(s))$. It yields

$$s = \frac{y - x}{1 - y} \quad \text{and} \quad \tau = \frac{x - y^2}{1 - y}.$$

Therefore

$$u(x, y) = u(x(s), y(s)) = z(s) = 2s + \tau = \frac{2y - y^2 - x}{1 - y}.$$

This solution makes sense only if $y \neq 1$. ∎

When the PDE in (2) has special structures, the characteristic ODEs can be significantly simplified.

- Consider the first order linear PDE

$$\mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0.$$

Here $F(x, z, \mathbf{p}) = \mathbf{b}(x) \cdot \mathbf{p} + c(x)z$. Since $D_{\mathbf{p}}F = \mathbf{b}(x)$, we have

$$\frac{dx}{ds} = \mathbf{b}(x), \qquad \frac{dz}{ds} = \mathbf{b}(x) \cdot \mathbf{p}(s).$$

Since $\mathbf{p}(s) = Du(x(s)) = -c(x(s))u(x(s)) = -c(x(s))z(s)$, we obtain the simplified characteristic ODEs

$$\frac{dx}{ds} = \mathbf{b}(x), \qquad \frac{dz}{ds} = -c(x)z.$$

The equations on $\mathbf{p}$ are not needed. ■

- Consider the scalar Hamilton-Jacobi equation

$$u_t + f(u_x) = 0,$$

where $f \in C^1(\mathbb{R})$. Here $F = F(t, x, z, q, p) = q + f(p)$ with $p = u_x$ and $q = u_t$. Consequently

$$F_q = 1, \quad F_p = f'(p), \quad F_t = F_x = F_z = 0.$$

Therefore, it follows from the characteristic ODEs (3) that

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = f'(p), \qquad \frac{dz}{ds} = q + pf'(p),$$
$$\frac{dq}{ds} = 0, \qquad \frac{dp}{ds} = 0.$$

Thus we may take $s = t$. Since $q = u_t = -f(u_x) = -f(p)$, we obtain the simplified characteristic ODEs

$$\begin{cases} \frac{dx}{dt} = f'(p), \\ \frac{dz}{dt} = pf'(p) - f(p), \\ \frac{dp}{dt} = 0. \end{cases}$$

These equations imply that

- p are constants along characteristics by the last equation .
- Characteristics are straight lines with velocity $f'(p)$ by the first equation.
- By the second equation, u can be obtained along characteristic lines.

We will use these facts to discuss Hamilton-Jacobi equation later. ∎

3. Formation of Singularities in Solutions of Scalar Conservation Laws

- Consider the initial value problem of the scalar conservation law

$$\begin{aligned} &u_t + f(u)_x = 0, && (x,t) \in \mathbb{R} \times (0,\infty), \\ &u(x,0) = u_0(x), && x \in \mathbb{R}, \end{aligned} \tag{5}$$

where f is a C^1 function. The equation can be write as $u_t + f'(u)u_x = 0$. Here $F = F(t,x,u,q,p) = q + f'(u)p$ with $q = u_t$ and $p = u_x$. Since

$$F_t = F_x = 0, \quad F_q = 1, \quad F_p = f'(u), \quad qu + p = 0,$$

from the characteristic ODEs (3) we have

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = f'(u), \quad \frac{du}{ds} = q + pf'(u) = 0.$$

We can take $s = t$. Thus for (5) the characteristic ODEs become

$$\begin{cases} \frac{dx}{dt} = f'(u), \\ \frac{du}{dt} = 0. \end{cases} \tag{6}$$

From these equation we can conclude

- u are constants along characteristics.
- Characteristics are straight lines with velocity $f'(u)$.

We will use these facts to show the following result.

Lemma 2

The problem (5) cannot have a C^1 solution defined for all $t > 0$ if there exist $x_1 < x_2$ such that $f'(u_0(x_2)) < f'(u_0(x_1))$.

Proof.

- Assume (5) has a C^1 solution defined for all $t > 0$.
- Then u are constants along characteristics and characteristics are straight lines. For characteristic line crossing x-axis at x, its velocity is $f'(u_0(x))$.
- Let l_1, l_2 be the two characteristics lines starting from $(x_1, 0)$ and $x_2, 0)$. Their velocities are $f'(u_0(x_1))$ and $f'(u_0(x_2))$ respectively.

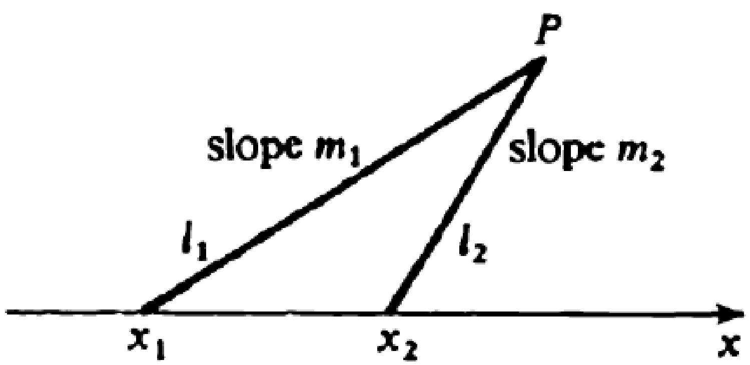


Figure: The plots of l_1 and l_2 whose slopes are $m_1 = 1/f'(u_0(x_1))$ and $m_2 = 1/f'(u_0(x_2))$ respectively,

- Since $f'(u_0(x_2)) < f'(u_0(x_1))$, these two lines must cross at some point P in $t > 0$.
- Along l_i we have $u(x_i, t) = u_0(x_i)$, $i = 1, 2$. Thus u must be discontinuous at P. Contradiction! ∎

Conclusion:

- In general C^1 solutions of (5) can exits for only a finite time no matter how smooth u_0 is.
- In order to allow (5) to admit solutions defined for all $t > 0$, the notion of solution should be generalized to include solutions with "discontinuities".

II. Introduction to Nonlinear Hyperbolic Conservation Laws

1. Weak Solutions & Rankine-Hugoniot Condition

Consider again the initial value problem (5), i.e.

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{7}$$

To motivate the notion of weak solution, assume u is a C^1 solution of (7). Multiplying (7) by any test function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, integrating over $\mathbb{R} \times (0,\infty)$, and using integration by parts, it gives

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x)\varphi dxdt \\ &= \int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x) dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)dx. \end{aligned}$$

Since the last equation makes sense provided that u and u_0 are merely bounded and measurable, it leads to the following definition.

Definition 3

Let $u_0 \in L^\infty(\mathbb{R})$. A function $u \in L^\infty(\mathbb{R} \times (0, \infty))$ is called a weak solution of (7) if

$$\int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x) dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x, 0) dx = 0$$

for all $\varphi \in C_0^\infty(\mathbb{R} \times [0, \infty))$.

Remarks.

(i) If $u \in C^1(\mathbb{R} \times [0, \infty))$ is a classical solution of (7), then u is automatically a weak solution.

(ii) If u is a weak solution of (7) and if u is C^1 in a domain $\Omega \subset \mathbb{R} \times (0,\infty)$, then $u_t + f(u)_x = 0$ in Ω. In fact, for any $\varphi \in C_0^1(\Omega)$ we have by integration by parts that

$$0 = \int_0^\infty \int_{-\infty}^\infty (u\varphi_t + f(u)\varphi_x)dxdt = \int_0^\infty \int_{-\infty}^\infty (u_t + f(u)_x)\varphi dxdt.$$

Since φ is arbitrary, it follows $u_t + f(u)_x = 0$ in Ω.

(iii) If $u_0 \in C(\mathbb{R})$ and $u \in C^1(\mathbb{R} \times [0,\infty))$ is a weak solution of (7), then u is a classical solution. In fact, $u_t + f(u)_x = 0$ in $\mathbb{R} \times (0,\infty)$ by (ii). Thus, by the definition of weak solution and integration by parts, we have

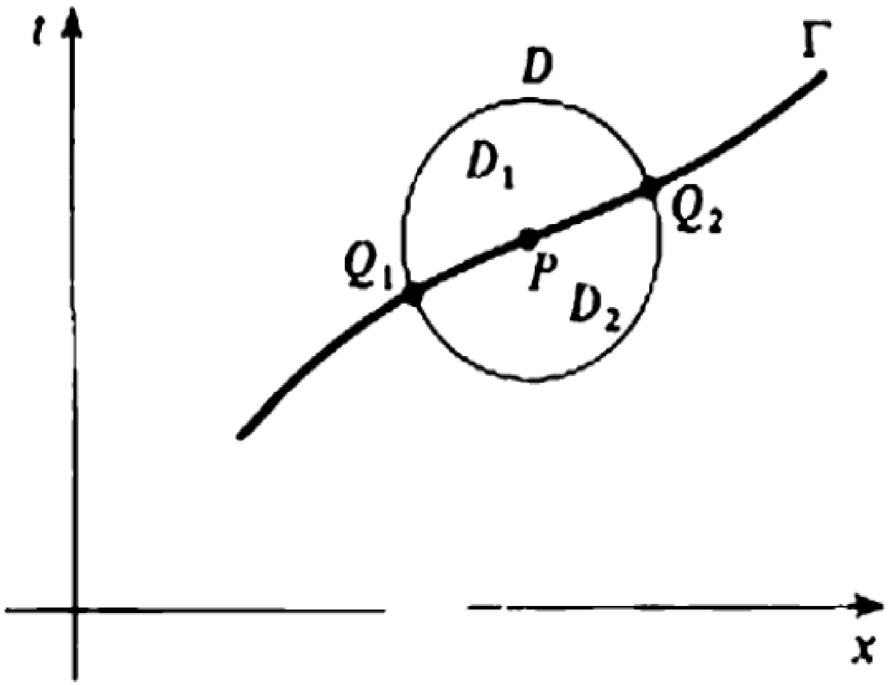
$$0 = \int_{-\infty}^\infty (u(x,0) - u_0(x))\varphi(x,0)dx, \quad \forall \varphi \in C_0^1(\mathbb{R} \times [0,\infty)).$$

Therefore $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$. ■

The notion of weak solution places restrictions on the curve of discontinuity.

- Let Γ be a smooth curve across which u has a jump discontinuity, and u is smooth away from Γ.
- Let $P \in \Gamma$ and let D be a small ball in $t > 0$ centered at P. Assume that the part of Γ in D is given by $x = x(t)$, $a \le t \le b$.
- Γ splits D into two parts: the left part D_1 and the right part D_2. Let

$$u_l := \lim_{\varepsilon \searrow 0} u(x(t) - \varepsilon, t), \qquad u_r := \lim_{\varepsilon \searrow 0} u(x(t) + \varepsilon, t).$$



- For any $\varphi \in C_0^1(D)$, we have

$$0 = \iint_D (u\varphi_t + f(u)\varphi_x)dxdt = \left(\iint_{D_l} + \iint_{D_r}\right)(u\varphi_t + f(u)\varphi_x)dxdt.$$

Since u is C^1 in D_1 and D_2, we have $u_t + f(u)_x = 0$ in D_1 and D_2. Therefore it follows from the divergence theorem that

$$\iint\limits_{D_1} (u\varphi_t + f(u)\varphi_x)dxdt = \iint\limits_{D_1} ((u\varphi)_t + (f(u)\varphi)_x)dxdt$$

$$= \int_{\partial D_1} \varphi(-udx + f(u)dt)$$

$$= \int_{\Gamma} \varphi(-u_l dx + f(u_l)dt).$$

Similarly,

$$\iint\limits_{D_2} (u\varphi_t + f(u)\varphi_x)dxdt = -\int_{\Gamma} \varphi(-u_r dx + f(u_r)dt).$$

Therefore

$$0 = \int_\Gamma \varphi(-[u]dx + [f(u)]dt),$$

where $[u] = u_l - u_r$ and $[f(u)] = f(u_l) - f(u_r)$ denote the jumps across Γ. Let $s := \frac{dx}{dt}$ denote the speed of the curve of discontinuities. Then

$$0 = \int_a^b \varphi(-s[u] + [f(u)])dt.$$

By the arbitrariness of φ, we can conclude that

$$s[u] = [f(u)] \tag{8}$$

at each point on Γ, which is called the *Rankine-Hugoniot condition*.

Proposition 4

If u is a weak solution of (7), then on the curves of discontinuity there must hold the Rankine-Hugoniot condition (8).

We give an example to indicate how to produce weak solutions by the method of characteristics and the Rankine-Hugoniot condition .

Example 5

Consider the initial value problem of Burgers equation

$$u_t + (u^2/2)_x = 0, \quad u(x,0) = u_0(x) := \begin{cases} 1, & x < 0, \\ 1 - x, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$$

- We first use the method of characteristics to find the solution defined for a finite time.
- We know that all characteristics are straight lines and u are constants along characteristics lines.
- Since the flux is $f(u) = u^2/2$, the characteristic line crossing x-axis at x_0 is given by

$$x(t) = x_0 + tu_0(x_0), \qquad x_0 \in \mathbb{R}.$$

and on this line

$$u = u_0(x_0).$$

Since all characteristics starting at $(x_0, 0)$ with $0 \leq x_0 \leq 1$ cross at $(1, 1)$, $u(x, t)$ can not be smooth for $t \geq 1$.

- By the knowledge of characteristics, $u(x,t)$ for $t < 1$ can be determined as follows:
  - $u(x,t) = 1$ for $x < t$ and $u(x,t) = 0$ for $x > 1$.
  - For (x,t) satisfying $0 < t \leq x \leq 1$, the characteristic through it intersects x-axis at $(x_0, 0)$ with $x_0 = (x-t)/(1-t)$. So

$$u(x,t) = u_0(x_0) = 1 - x_0 = 1 - \frac{x-t}{1-t} = \frac{1-x}{1-t}.$$

- Therefore, for $t < 1$ we have

$$u(x,t) = \begin{cases} 1, & x < t, \\ (1-x)/(1-t), & t \leq x \leq 1, \\ 0, & x > 1. \end{cases}$$

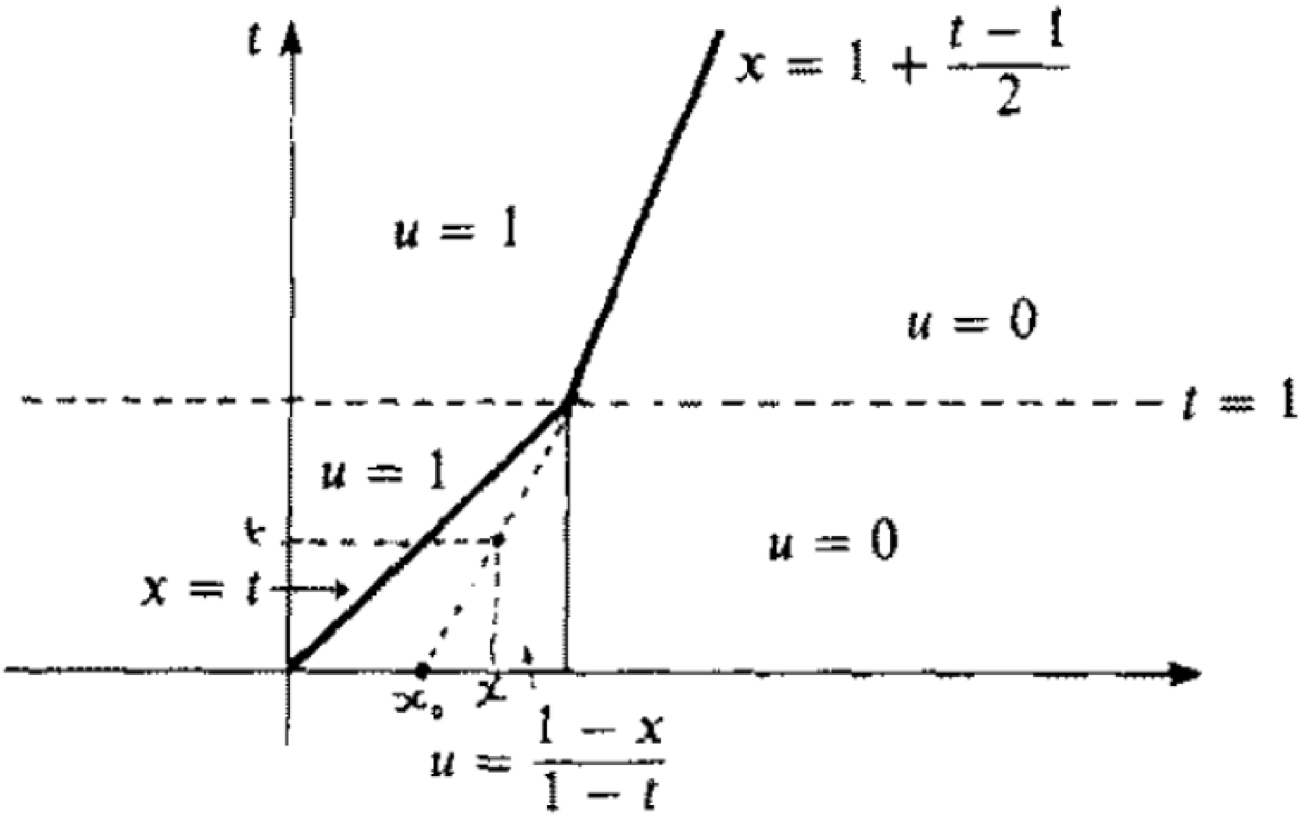
- Next we use the Rankine-Hugoniot condition to define $u(x,t)$ for $t \geq 1$.
  - By the knowledge of characteristics, a curve of discontinuities starting at the point $(1,1)$ is expected with $u = 1$ on the left and $u = 0$ on the right.
  - By the Rankine-Hugoniot condition, the speed of the curve of discontinuities is

$$s(t) = \frac{u_l^2/2 - u_r^2/2}{u_l - u_r} = \frac{1}{2}(u_l + u_r) = \frac{1}{2}.$$

So the curve is given by $x(t) = 1 + (t-1)/2$, $t \geq 1$. Hence, for $t \geq 1$ we have

$$u(x,t) = \begin{cases} 1, & x < 1 + (t-1)/2, \\ 0, & x > 1 + (t-1)/2. \end{cases}$$

The solution u is depicted in the following figure.



- By definition it is easy to check that the above u is a weak solution. ∎

Example 6 (Nonuniqueness of weak solutions)

Consider the initial value problem of Burgers equation

$$u_t + (u^2/2)_x = 0, \qquad u(x,0) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

The method of characteristics determines the solution everywhere in $t > 0$ except in the sector $0 < x < t$. By defining u in $0 < x < t$ carefully, we obtain two functions

$$u_1(x,t) = \begin{cases} 0, & x < t/2, \\ 1, & x > t/2, \end{cases} \quad \text{and} \quad u_2(x,t) = \begin{cases} 0, & x < 0, \\ x/t, & 0 < x < t, \\ 1, & x > t; \end{cases}$$

both turn out to be weak solutions. ■

2. Entropy Conditions

- Example shows that weak solutions of conservation laws are not necessarily unique.
- Criteria should be developed to pick out the "physically relevant" solution.
- Such a criterion is called an entropy condition.
- We motivate the entropy condition for the scalar conservation laws

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{9}$$

where $u_0 \in C^1$ and f is C^2 with $f'' > 0$. Assume that (9) has a smooth solution u (thus $u_0' \geq 0$ by Lemma 2).

- Recall that all characteristics of (9) are straight lines given by

$$(x_0 + f'(u_0(x_0))t, t), \qquad x_0 \in \mathbb{R}.$$

- For any (x, t) with $t > 0$ let x_0 be the crossing point of x-axis and the characteristic through (x, t). Since $u(x, t) = u_0(x_0)$ along the characteristic, we have

$$x = x_0 + t\, f'(u(x, t)), \quad \text{i.e. } x_0 = x - t\, f'(u(x, t)).$$

  So u satisfies the equation $u = u_0(x - t\, f'(u))$.
- Taking derivative with respect to x gives

$$u_x(x, t) = \frac{u_0'(x - t\, f'(u))}{1 + u_0'(x - t\, f'(u))f''(u)t}.$$

- If $u_0'(x - tf'(u)) = 0$, then $u_x(x,t) = 0$; If $u_0'(x - tf'(u)) > 0$, then

$$u_x(x,t) \le \frac{u_0'(x - t\,f'(u))}{u_0'(x - t\,f'(u))f''(u)t} = \frac{1}{f''(u)t} \le \frac{E}{t},$$

where $E = 1/\inf\{f''(u) : |u| \le \|u_0\|_\infty\}$, here we used $|u(x,t)| \le |u_0\|_\infty$.

- Consequently, we have for any $t > 0$, $x \in \mathbb{R}$ and $a > 0$ that

$$\frac{u(x+a,t) - u(x,t)}{a} \le \frac{E}{t}.$$

- This last inequality requires no smoothness of u and thus can be used as a criterion to pick out the “right” weak solution.

Definition 7 (Oleinik)

A weak solution u of the scalar conservation laws is said to satisfy the Oleinik entropy condition if there is a constant E such that

$$\frac{u(x+a,t)-u(x,t)}{a} \leq \frac{E}{t}$$

for all $t > 0$ and $x, a \in \mathbb{R}$ with $a > 0$.

We derive another entropy condition due to Lax which is easier to extend for systems of conservation laws.

- Recall that the characteristics are given by

$$(x_0 + f'(u_0(x_0))t, t), \qquad x_0 \in \mathbb{R}.$$

- Assume that, at some point on a curve C of discontinuities, u has distinct left and right limits u_l and u_r and that a characteristic from left and a characteristic from the right hit C at this point. Then

$$f'(u_l) > s > f'(u_r), \tag{10}$$

  where s denote the speed of the discontinuous curve at that point. We call (10) the Lax entropy condition.

Remark. In case $f'' > 0$, Lax entropy condition can be deduced from Oleinik entropy condition:

- Indeed, by Oleinik entropy condition we always have $u_l \geq u_r$ and thus $u_l > u_r$ on the curve of discontinuities.

- Since $f'' > 0$, f' is strictly increasing and thus $f'(u_l) > f'(u_r)$.
- By Rankine-Hugoniot condition, the speed of discontinuous curve is

$$s = \frac{f(u_l) - f(u_r)}{u_l - u_r} = f'(\xi)$$

for some $\xi \in (u_r, u_l)$. Consequently $f'(u_l) > s > f'(u_r)$ which is the Lax entropy condition.

Definition 8

A curve of discontinuity for u is called a shock curve provided both the Rankine-Hugoniot condition and the entropy condition hold.

Question: *Is it possible to show existence and uniqueness of weak solutions of conservation laws satisfying suitable entropy condition?* We will focus on scalar conservation laws with strictly convex flux.

3. Uniqueness of Entropy Solutions

We will prove the following uniqueness result.

Theorem 9

Consider the initial value problem of the scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where f is a C^2 convex function. If $u, v \in L^\infty(\mathbb{R} \times (0,\infty))$ are two weak solutions satisfying the Oleinik entropy condition, then

$$u = v \quad \textit{in } \mathbb{R} \times (0,\infty)$$

except a set of measure zero.

Proof. Since $u, v \in L^\infty(\mathbb{R} \times (0,\infty))$, it suffices to show that

$$\int_0^\infty \int_{-\infty}^\infty (u-v)\varphi dxdt = 0, \quad \forall \varphi \in C_0^1(\mathbb{R} \times (0,\infty)). \tag{11}$$

By the definition of weak solution, for any $\psi \in C_0^1(\mathbb{R} \times [0,\infty))$ we have

$$\int_0^\infty \int_{-\infty}^\infty (u\psi_t + f(u)\psi_x)\, dxdt + \int_{-\infty}^\infty u_0(x)\psi(x,0)dx = 0,$$
$$\int_0^\infty \int_{-\infty}^\infty (v\psi_t + f(v)\psi_x)\, dxdt + \int_{-\infty}^\infty u_0(x)\psi(x,0)dx = 0.$$

Therefore

$$0 = \int_0^\infty \int_{-\infty}^\infty \{(u-v)\psi_t + (f(u)-f(v))\psi_x\}\, dxdt.$$

By writing

$$f(u) - f(v) = \int_0^1 \frac{d}{d\tau}[f(\tau u + (1-\tau)v)]d\tau = b(u-v),$$

where

$$b(x,t) := \int_0^1 f'(\tau u(x,t) + (1-\tau)v(x,t))d\tau,$$

then it follows

$$0 = \int_0^\infty \int_{-\infty}^\infty (u-v)(\psi_t + b\psi_x)\, dxdt \tag{12}$$

for all $\psi \in C_0^1(\mathbb{R} \times [0,\infty))$.

- If we could solve the linear transport equation

$$\psi_t + b\psi_x = \varphi \tag{13}$$

for any $\varphi \in C_0^1(\mathbb{R} \times (0, \infty))$ to obtain $\psi \in C_0^1(\mathbb{R} \times [0, \infty))$, then we would obtain (11) from (12).

- Unfortunately, (13) may not have a C_0^1 solution ψ because b is not continuous in general.
- To get around this difficulty, we need to use the mollification technique.
- We take a mollifier, i.e. a function $\omega \in C_0^\infty(\mathbb{R}^2)$ with

$$\omega \geq 0, \quad \iint_{\mathbb{R}^2} \omega(x, t)dxdt = 1, \quad \text{supp}(\omega) \subset \{x^2 + t^2 \leq 1\}.$$

- For any $\varepsilon > 0$ set $\omega_\varepsilon(x,t) = \varepsilon^{-2}\omega(x/\varepsilon, t/\varepsilon)$.
- To regularize u and v, we set $u(x,t) = v(x,t) = 0$ for $t < 0$ and define

$$u_\varepsilon = u * \omega_\varepsilon, \qquad v_\varepsilon = v * \omega_\varepsilon$$

where $*$ denotes the convolution, i.e.

$$u * \omega_\epsilon(x,t) = \iint_{\mathbb{R}^2} u(y,s)\omega_\varepsilon(x-y, t-s)dydt.$$

It is well known that both u_ε and v_ε are smooth functions and

$$|u_\varepsilon| \le M \quad \text{and} \quad |v_\varepsilon| \le M, \quad \text{in } \mathbb{R} \times [0,\infty), \tag{14}$$

where $M > 0$ is a constant such that $|u|, |v| \le M$.

- We use the Oleinik entropy condition to show for $\alpha > 0$ that

$$\partial_x u_\varepsilon \le E/\alpha \quad \text{and} \quad \partial_x v_\varepsilon \le E/\alpha, \qquad \forall t \ge \alpha. \tag{15}$$

Let $h(x,t) := u(x,t) - Ex/\alpha$. Then for $a \ge 0$ and $t \ge \alpha$

$$h(x+a,t) - h(x,t) = u(x+a,t) - u(x,t) - \frac{Ea}{\alpha} \le \frac{Ea}{t} - \frac{Ea}{\alpha} \le 0.$$

Thus $x \to (h * \omega_\varepsilon)(x,t)$ is decreasing for each $t \ge \alpha$. Since

$$(h * \omega_\varepsilon)(x,t) = u_\varepsilon(x,t) - \frac{Ex}{\alpha} + \frac{E}{\alpha} \iint_{\mathbb{R}^2} y\omega_\varepsilon(y,s)dyds,$$

we obtain

$$0 \ge \partial_x (h * \omega_\varepsilon) = \partial_x u_\varepsilon - E/\alpha, \quad \forall t \ge \alpha.$$

- Next define

$$b_\varepsilon := \int_0^1 f'(\tau u_\varepsilon + (1-\tau)v_\varepsilon)d\tau.$$

Because of (14) and $f \in C^2$, we have $b_\varepsilon \in C^1$ and there is a constant M_1 independent of ε such that

$$|b_\varepsilon(x,t)| \le M_1, \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{16}$$

- Moreover, for any $\alpha > 0$ there holds

$$\partial_x b_\varepsilon \le C_0 E/\alpha, \qquad \forall t \ge \alpha, \tag{17}$$

where $C_0 := \max\{f''(\xi) : |\xi| \le M\}$. In fact,

$$\partial_x b_\varepsilon = \int_0^1 f''(\tau u_\varepsilon + (1-\tau)v_\varepsilon)\,(\tau\partial_x u_\varepsilon + (1-\tau)\partial_x v_\varepsilon)\,d\tau.$$

Since $f'' \geq 0$, we may use (15) and (14) to derive for $t \geq \alpha$ that

$$\partial_x b_\varepsilon \leq \frac{E}{\alpha} \int_0^1 f''(\tau u_\varepsilon + (1-\tau)v_\varepsilon) d\tau \leq \frac{C_0 E}{\alpha}.$$

- We next prove that $b_\varepsilon \to b$ locally in L^1 as $\varepsilon \to 0$. To see this, using $f \in C^2$ we can write

$$\begin{aligned} & b_\varepsilon(x,t) - b(x,t) \\ & \quad = \int_0^1 \left(f'(\tau u_\varepsilon + (1-\tau)v_\varepsilon) - f'(\tau u + (1-\tau)v) \right) d\tau \\ & \quad = \int_0^1 f''(\xi) \left(\tau(u_\varepsilon - u) + (1-\tau)(v_\varepsilon - v) \right) d\tau, \end{aligned}$$

where ξ is between $\tau u_\varepsilon + (1-\tau)v_\varepsilon$ and $\tau u + (1-\tau)v$.

By (14) we have $|\xi| \leq M$. Therefore

$$|b_\varepsilon(x,t) - b(x,t)| \leq \frac{1}{2} C_0 \left(|u_\varepsilon - u| + |v_\varepsilon - v| \right).$$

Thus for any compact set $K \subset \mathbb{R} \times [0, \infty)$ we have

$$\iint_K |b_\varepsilon - b| dxdt \leq \frac{1}{2} C_0 \iint_K \left(|u_\varepsilon - u| + |v_\varepsilon - v| \right) dxdt$$
$$\to 0 \quad \text{as } \varepsilon \to 0.$$

- For any fixed $\varphi \in C_0^1(\mathbb{R} \times (0, \infty))$, we consider the problem

$$\psi_t^\varepsilon + b_\varepsilon \psi_x^\varepsilon = \varphi, \quad \psi^\varepsilon(x, T) = 0, \tag{18}$$

where $T > 0$ is chosen such that $\varphi = 0$ for $t \geq T$.

By the method of characteristics, the solution of (18) is given by

$$\psi^{\varepsilon}(x,t)=\int_{T}^{t}\varphi(x_{\varepsilon}(s;x,t),s)ds, \tag{19}$$

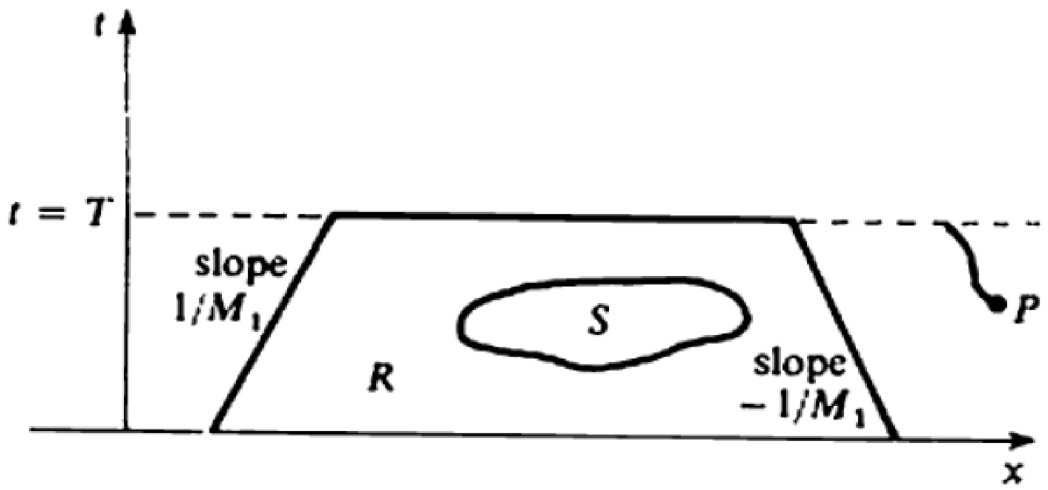
where $x_{\varepsilon}(s):=x_{\varepsilon}(s;x,t)$ is defined by

$$\frac{dx_{\varepsilon}}{ds}=b_{\varepsilon}(x_{\varepsilon},s), \quad x_{\varepsilon}(t)=x.$$

Since $b_{\varepsilon}\in C^1$ satisfies (16), x_{ε} exists for all s and is C^1 with respect to s,x and t. Thus $\psi^{\varepsilon}\in C^1(\mathbb{R}\times[0,\infty))$.

- We show that $\psi^{\varepsilon}\in C_0^1(\mathbb{R}\times[0,\infty))$ and supp(ψ^{ε}) are contained in a compact region independent of ε.

To see this, let $S := \mathrm{supp}(\varphi)$. By the choice of T, S is a compact set contained in $\{(x,t) : 0 < t \leq T\}$. In view of (19), $\psi^\varepsilon(x,t) = 0$ for $t \geq T$.



Next let R be the region bounded by the lines $t = 0$, $t = T$ and two lines with slopes $1/M_1$ and $-1/M_1$ such that $S \subset R$. For any $(x,t) \notin R$ with $t < T$, from (16) it follows that

$$x_\varepsilon(s; x, t) \notin R, \qquad \forall t \leq s \leq T$$

Since

$$\frac{d}{ds}\psi^{\varepsilon}(x_{\varepsilon}(s;x,t),s) = \psi^{\varepsilon}_s + \psi^{\varepsilon}_x \frac{\partial x_{\varepsilon}}{\partial s} = \psi^{\varepsilon}_s + b_{\varepsilon}\psi^{\varepsilon}_x$$
$$= \varphi(x_{\varepsilon}(s;x,t),s) = 0$$

for $t \le s \le T$, we have

$$\psi^{\varepsilon}(x,t) = \psi^{\varepsilon}(x_{\varepsilon}(t;x,t),t) = \psi^{\varepsilon}(x_{\varepsilon}(T;x,t),T) = 0.$$

Therefore $\mathrm{supp}(\psi^{\varepsilon}) \subset R$.

- By using (12) with $\psi = \psi^{\varepsilon}$ and (18) we have

$$0 = \int_0^{\infty}\int_{-\infty}^{\infty}(u-v)\left\{\psi^{\varepsilon}_t + b_{\varepsilon}\psi^{\varepsilon}_x + (b-b_{\varepsilon})\psi^{\varepsilon}_x\right\}dxdt.$$

In view of (18) it follows

$$\int_0^\infty \int_{-\infty}^\infty (u-v)\varphi \, dxdt = \int_0^\infty \int_{-\infty}^\infty (u-v)(b_\varepsilon - b)\psi_x^\varepsilon \, dxdt. \tag{20}$$

To prove (11), it suffices to show that the right hand side of (20) goes to 0 as $\varepsilon \to 0$.

- We need to estimate ψ_x^ε. We first show that for any $\alpha > 0$ there exists C_α such that

$$|\psi_x^\varepsilon| \le C_\alpha, \qquad \forall t \ge \alpha. \tag{21}$$

Since $\psi^\varepsilon = 0$ for $t \ge T$, it suffices to show (21) for $\alpha \le t < T$.

By using (19) we obtain

$$\psi_x^\varepsilon(x,t) = \int_T^t \varphi_x(x_\varepsilon(s,x,t),s)\frac{\partial x_\varepsilon}{\partial x}(s;x,t)ds. \qquad (22)$$

Recall that $x_\varepsilon(t;x;t) = x$, we have $\frac{\partial x_\varepsilon}{\partial x}(t;x,t) = 1$. Let

$$a_\varepsilon(s) := \frac{\partial x_\varepsilon}{\partial x}(s;x,t).$$

Then $a_\varepsilon(t) = 1$ and

$$\begin{aligned}\frac{\partial a_\varepsilon}{\partial s} &= \frac{\partial}{\partial s}\frac{\partial x_\varepsilon}{\partial x} = \frac{\partial}{\partial x}\frac{\partial x_\varepsilon}{\partial s} = \frac{\partial}{\partial x}b_\varepsilon(x_\varepsilon(s;x,t),s)\\ &= \partial_x b_\varepsilon \frac{\partial x_\varepsilon}{\partial x} = (\partial_x b_\varepsilon)a_\varepsilon\end{aligned}$$

Therefore

$$a_\varepsilon(s) = \exp\left(\int_t^s \partial_x b_\varepsilon(x_\varepsilon(\tau; x, t), \tau)d\tau\right).$$

In view of (17), it follows $a_\varepsilon(s) \le e^{C_0 ET/\alpha}$ for $\alpha \le t \le s \le T$. Thus we have from (22) that

$$|\psi_x^\varepsilon(x,t)| \le \int_t^T |\varphi_x| a_\varepsilon(s)ds \le C_\alpha, \qquad \forall \alpha \le t \le T.$$

- We next derive the total variation estimate on ψ^ε: For each $t > 0$ let

$$TV_t(\psi^\varepsilon) := \int_{-\infty}^{\infty} |\psi_x^\varepsilon(x,t)|\, dx$$

denote the total variation of the function $\psi^\varepsilon(\cdot, t)$.

Since the supports of ψ^ε) are contained in a compact region independent of ε, it follows from (21) that for any $\alpha > 0$ there is a constant $\hat{C}_\alpha$ independent of ε such that

$$TV_t(\psi^\varepsilon) \leq \hat{C}_\alpha, \qquad \forall t \geq \alpha.$$

We claim that

$$\exists \beta > 0 \text{ such that } TV_t(\psi^\varepsilon) \leq \hat{C}_\beta \text{ for all } 0 < t \leq \beta. \tag{23}$$

To see this, by using $\varphi \in C_0^1(\mathbb{R} \times (0, \infty))$ we may take $\beta > 0$ such that $\varphi = 0$ for $0 \leq t \leq \beta$. It then follows from (18) that

$$\psi_t^\varepsilon + b_\varepsilon \psi_x^\varepsilon = 0 \quad \text{for } t \leq \beta. \tag{24}$$

Fix $0 \leq t \leq \beta$, let $x_0 < x_1 < \cdots < x_N$ be any partition of $\mathbb{R}$, and set $y_i = x_\varepsilon(\beta; x_i, t)$ for $i = 0, \cdots, N$. Then $y_0 < y_1 < \cdots < y_N$. Since (24) implies that ψ^ε is constant along the characteristic curves $s \to x_\varepsilon(s; x_i, t)$ for $0 \leq s \leq \beta$, we have

$$\psi^\varepsilon(x_i, t) = \psi^\varepsilon(y_i, \beta), \quad i = 0, 1, \cdots, N.$$

Therefore

$$\begin{aligned}\sum_{i=0}^{N-1} |\psi^\varepsilon(x_{i+1}, t) - \psi^\varepsilon(x_i, t)| &\leq \sum_{i=0}^{N-1} |\psi^\varepsilon(y_{i+1}, \beta) - \psi^\varepsilon(y_i, \beta)| \\ &\leq TV_\beta(\psi^\varepsilon).\end{aligned}$$

Taking the supremum over all such partitions gives $TV_t(\psi^\varepsilon) \leq TV_\beta(\psi^\varepsilon) \leq \hat{C}_\beta$.

- Finally we complete the proof by estimating

$$\left| \int_0^\infty \int_{-\infty}^\infty (u - v)(b_\varepsilon - b)\psi_x^\varepsilon dxdt \right| \le I_1 + I_2,$$

where

$$I_1 = \int_0^\alpha \int_{-\infty}^\infty |u - v||b_\varepsilon - b||\psi_x^\varepsilon| dxdt,$$
$$I_2 = \int_\alpha^\infty \int_{-\infty}^\infty |u - v||b_\varepsilon - b||\psi_x^\varepsilon| dxdt.$$

By using (16) and (23) we obtain for $0 < \alpha \le \beta$ that

$$I_1 \le 2M \cdot 2M_1 \int_0^\alpha TV_t(\psi^\varepsilon) dt \le 4MM_1 \alpha \hat{C}_\beta.$$

Thus, for any $\eta > 0$ we can take $0 < \alpha \leq \beta$ such that

$$I_1 \leq 4MM_1\alpha\hat{C}_\beta < \eta/2.$$

For this α, recall that the supports of ψ^ε are contained in a compact region independent of ε, we may use (21) and the local convergence of b_ε to b in L^1 to obtain

$$I_2 \leq \eta/2 \quad \text{for sufficiently small } \varepsilon > 0.$$

Consequently, for small $\varepsilon > 0$ there holds

$$\left| \int_0^\infty \int_{-\infty}^\infty (u - v)(b_\varepsilon - b)\psi_x^\varepsilon dxdt \right| \leq \eta.$$

Since $\eta > 0$ is arbitrary, we can conclude the proof. ∎

4. Riemann Problems

Before giving the general existence result, we consider the scalar conservation law with simple initial values:

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{25}$$

where u_l and u_r are constants. This problem is called Riemann problem. We will determine the unique entropy solution explicitly when $f'' > c_0 > 0$.

- Observing that if $u(x,t)$ is a solution of (25), then, for any $\lambda > 0$, $u_\lambda(x,t) = u(\lambda x, \lambda t)$ is also a solution. It is natural to determine the solution of the form $u(x,t) = v(x/t)$.

We need to consider two cases: $u_l > u_r$ and $u_l < u_r$.

- Case 1. $u_l > u_r$.
  - Since $f'' > 0$, we have $f'(u_l) > f'(u_r)$. Thus any characteristic line starting from the negative x-axis intersects characteristic lines starting from the positive x-axis.
  - Assume that the curve of discontinuities is $s(t)$. We expect that $s(0) = 0$ and $s'(t) = \sigma$ by Rankine-Hugoniot condition, where

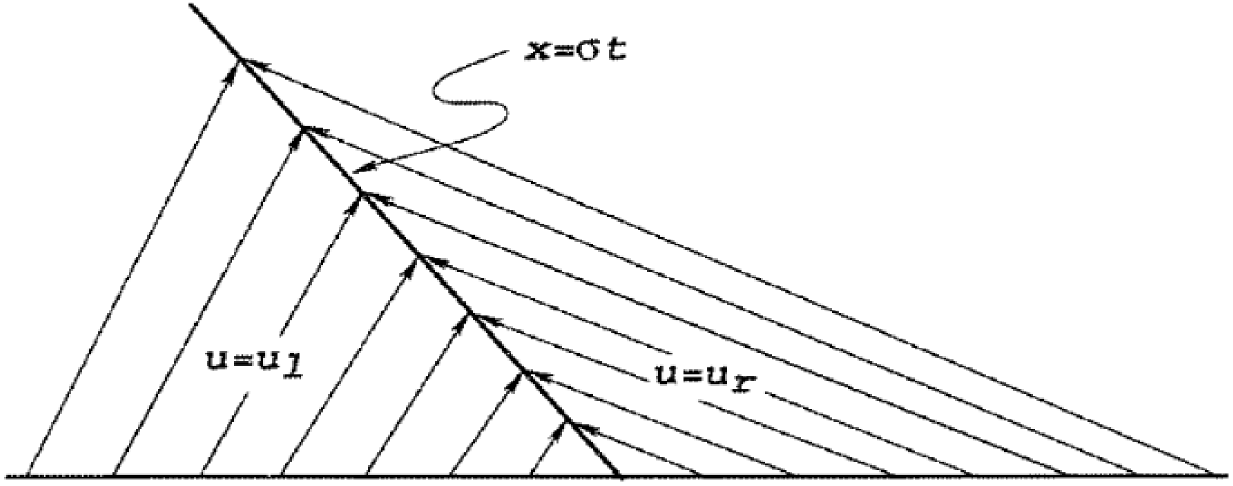
$$f'(u_r) < \sigma := \frac{f(u_l) - f(u_r)}{u_l - u_r} < f'(u_l).$$

    So $s(t) = \sigma t$.

- Therefore we may define

$$u(x,t) = \begin{cases} u_l, & x < \sigma t, \\ u_r, & x > \sigma t. \end{cases} \qquad (26)$$

It is easy to check u is a weak solution. Since $u_l > u_r$, u thus satisfies the Oleinik entropy condition. So, by Theorem 9, u is the unique entropy solution which is called a shock wave.



Shock wave solving Riemann's problem for $u_l > u_r$

Case 2. $u_l < u_r$.

- In this case $f'(u_l) < f'(u_r)$. By the method of characteristics, $u = u_l$ for $x < f'(u_l)t$ and $u = u_r$ for $x > f'(u_r)t$, but u is undetermined in the region $f'(u_l)t < x < f'(u_r)t$.
- In the region $f'(u_l)t < x < f'(u_r)t$, we expect u to be smooth with $u(x,t) = v(x/t)$. Then by $u_t + f(u)_x = 0$ we have

$$v'(x/t)\left(f'(v(x/t)) - x/t\right) = 0.$$

  Assuming v' never vanishes, we find $f'(v(x/t)) = x/t$.
- Since $f'' > c_0 > 0$, $G := (f')^{-1} : \mathbb{R} \to \mathbb{R}$ exists and

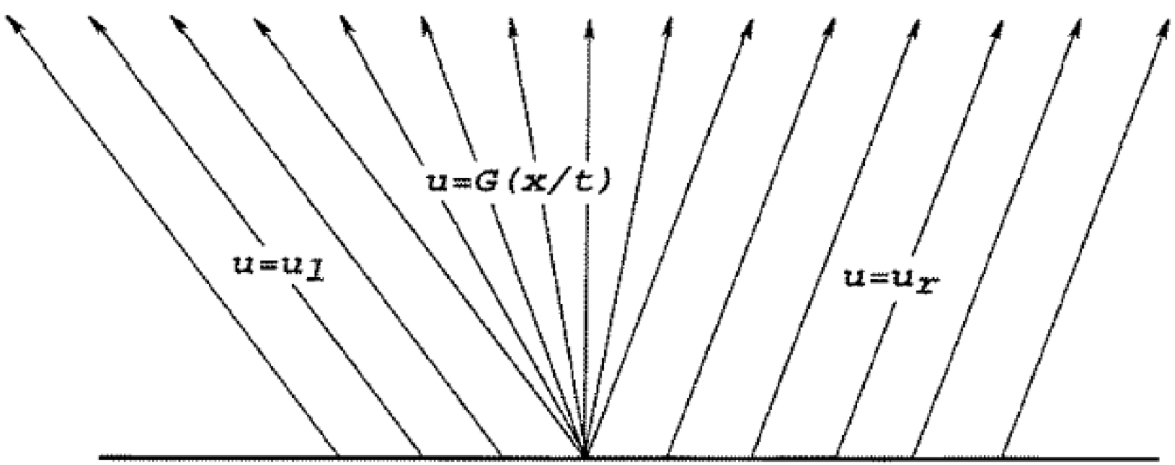
$$|G(x) - G(y)| \leq |x - y|/c_0$$

  for $x, y \in \mathbb{R}$ (see Lemma 14).
- Therefore $v(x/t) = G(x/t)$ for $f'(u_l)t < x < f'(u_r)t$.

- Thus we can define

$$u(x,t) = \begin{cases} u_l, & x < f'(u_l)t, \\ G(x/t), & f'(u_l)t < x < f'(u_r)t, \\ u_r, & x > f'(u_r)t. \end{cases} \tag{27}$$

Then u is continuous in $\mathbb{R} \times (0, \infty)$ and $u_t + f(u)_x = 0$ in each of its region of definition. It is easy to check that u is a weak solution.



Rarefaction wave solving Riemann's problem for $u_l < u_r$

- The Oleinik entropy condition can be directly checked case by case; for instance, if $f'(u_l)t < x < x + a < f'(u_r)t$, then

$$u(x+a,t)-u(x,t) = (f')^{-1}((x+a)/t)-(f')^{-1}(x/t) \leq a/(c_0 t).$$

So, by Theorem 9, u is the unique entropy solution which is called a rarefaction wave.

Summarizing the above discussion we obtain

Theorem 10

Consider the Riemann problem (25), where $f'' \geq c_0 > 0$.

(i) *If $u_l > u_r$, the unique entropy solution is given by the shock wave (26).*

(ii) *If $u_l < u_r$, the unique entropy solution is given by the rarefaction wave (27).*

5. Existence of Entropy Solutions

Consider the initial value problem of the scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{28}$$

We will prove the following existence result.

Theorem 11

Let $u_0 \in L^\infty(\mathbb{R})$ and $f \in C^2(\mathbb{R})$ with $f''(\xi) \geq c_0 > 0$ on $\mathbb{R}$. Then (28) has a unique weak solution $u \in L^\infty(\mathbb{R} \times [0,\infty))$ satisfying the Oleinik entropy condition. Moreover

$$\|u(x,t)\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq \|u_0\|_\infty.$$

- Theorem 11 has several different proofs. We present the one based on the theory of Hamilton-Jacobi equations.
- To motivate it, let $h(x) := \int_0^x u_0(y)dy$ and consider the initial value problem of Hamilton-Jacobi equation

$$\begin{cases} w_t + f(w_x) = 0, & (x,t) \in \mathbb{R} \times (0,\infty), \\ w(x,0) = h(x), & x \in \mathbb{R}. \end{cases} \tag{29}$$

If (29) has smooth solution, we set $u = w_x$. Then $u(x,0) = w_x(x,0) = u_0(x)$. Differentiating the equation in (29) gives

$$u_t = w_{xt} = (w_t)_x = -f(w_x)_x = -f(u)_x.$$

Thus $u = w_x$ is a solution of (28).

- Unfortunately the solution of (29) is not necessarily smooth in general.
- It is necessary to introduce the notion of weak solution of (29).

Definition 12

Consider the problem (29), where h is Lipschitz continuous. A Lipschitz continuous function $w : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is called a weak solution if

(i) $w(x, 0) = h(x)$ for all $x \in \mathbb{R}$;

(ii) $w_t(x, t) + f(w_x(x, t)) = 0$ for a.e. $(x, t) \in \mathbb{R} \times (0, \infty)$.

- When $f \in C^2$ with $f'' \geq c_0 > 0$, we will show that the solution of (29) is given by the Hopf-Lax formula.
- To motivate the formula, assuming (29) has a C^1 solution. Along a characteristic curve $x(t)$ we set $z(t) := w(x(t), t)$ and $p(t) := w_x(x(t), t)$. Then there hold

$$\begin{cases} \frac{dx}{dt} & = f'(p), \\ \frac{dz}{dt} & = pf'(p) - f(p), \\ \frac{dp}{dt} & = 0. \end{cases} \tag{30}$$

Thus along characteristics p are constants. So, characteristics are straight lines with velocity $f'(p)$. To understand the second equation in (30), we introduce the Legendre-Fenchel conjugate

$$f^*(q) = \sup_{p \in \mathbb{R}} \{pq - f(p)\}, \quad q \in \mathbb{R}.$$

Since f is uniformly convex, the maximum is achieved at p satisfying $q = f'(p)$. Thus

$$f^*(q) = pf'(p) - f(p) \qquad \text{with } f'(p) = q.$$

So $\frac{dz}{dt} = f^*(q)$ with $q = f'(p)$. Fix any $(\bar{x}, \bar{t})$ with $\bar{t} > 0$. For a characteristic line through $(\bar{x}, \bar{t})$ that crosses x-axis at $\bar{y}$, its velocity is $(\bar{x} - \bar{y})/\bar{t}$. Thus, along this characteristic,

$$\frac{dz}{dt} = f^*(\frac{\bar{x} - \bar{y}}{\bar{t}}), \qquad z(0) = h(\bar{y}).$$

Therefore

$$w(\bar{x},\bar{t}) = z(\bar{t}) = h(\bar{y}) + \bar{t} f^*(\frac{\bar{x}-\bar{y}}{\bar{t}}) \qquad (31)$$

This formula is problematic since it involves the unknown $\bar{y}$.

- On the othe hand, by the convexity of f we have for any p

$$-w_t = f(w_x) \geq f(p) + f'(p)(w_x - p).$$

So

$$w_t + f'(p) w_x \leq p f'(p) - f(p) = f^*(f'(p)).$$

Consider the straight line $(x(t),t)$ through $(\bar{x},\bar{t})$ with velocity $f'(p)$, let y be the intersection point with x-axis. Then

$$f'(p) = (\bar{x} - y)/\bar{t}$$

and

$$\frac{d}{dt}w(x(t),t) \leq f^*(f'(p)) = f^*(\frac{\bar{x}-y}{\bar{t}}).$$

Therefore

$$w(\bar{x},\bar{t}) \leq h(y) + \bar{t}f^*(\frac{\bar{x}-y}{\bar{t}}). \tag{32}$$

Since $f'' \geq c_0 > 0$, f' is strictly increasing with $f'(-\infty) = -\infty$ and $f'(+\infty) = +\infty$. Thus (32) holds for all $y \in \mathbb{R}$ since we can take y to be any number by adjusting p. Since (31) implies that the equality is achieved at some $\bar{y}$, we expect

$$w(x,t) := \inf_{y\in\mathbb{R}} \left\{ h(y) + t\, f^*(\frac{x-y}{t}) \right\} \tag{33}$$

which is called the Hopf-Lax formula.

- The above argument is not rigorous since it requires $w \in C^1$.
- Our goal is to show that (33) gives a weak solution of (29).

We first give some properties on f^*.

Lemma 13

Let f be a C^1 convex function on $\mathbb{R}$. Then the following hold:

(i) *f^* is convex;*

(ii) *For any $A > 0$ we have*

$$\sup_{q\in\mathbb{R}} \{A|q| - f^*(q)\} \leq \sup\{f(x) : |x| \leq A\};$$

(iii) *For any $x \in \mathbb{R}$ we have $\sup_{q\in\mathbb{R}} \{qx - f^*(q)\} = f(x)$.*

Proof.

(i) f^* is convex because f^* is the supremum of linear functions.

(ii) By the definition of f^* we have

$$f^*(q) = \sup_{y \in \mathbb{R}} \{qy - f(y)\} \geq q\frac{Aq}{|q|} - f(\frac{Aq}{|q|}) = A|q| - f(Aq/|q|).$$

Therefore

$$\sup_{q \in \mathbb{R}} \{A|q| - f^*(q)\} \leq \sup_{q \in \mathbb{R}} \{f(Aq/|q|)\} = \sup \{f(x) : |x| \leq A\}.$$

(iii) Since the definition of f^* implies $f^*(q) \geq qx - f(x)$ for all $q \in \mathbb{R}$, we have

$$\sup_{q \in \mathbb{R}} \{qx - f^*(q)\} \leq f(x).$$

To show the reverse inequality, we note that

$$qx - f^*(q) = qx - \sup_{y\in\mathbb{R}} \{qy - f(y)\} = \inf_{y\in\mathbb{R}} \{q(x-y) + f(y)\}$$

Thus

$$\begin{aligned}\sup_{q\in\mathbb{R}} \{qx - f^*(q)\} &= \sup_{q} \inf_{y} \{q(x-y) + f(y)\} \\ &\geq \inf_{y} \{f'(x)(x-y) + f(y)\}\end{aligned}$$

Since f is convex, we have $f(y) \geq f(x) + f'(x)(y-x)$ and thus

$$f(y) + f'(x)(x-y) \geq f(x), \quad \forall y.$$

So $\sup_{q\in\mathbb{R}} \{qx - f^*(q)\} \geq f(x)$. The proof is complete. ■

Lemma 14

Let $f \in C^2$ be such that $f'' \geq c_0$ for some constant $c_0 > 0$. Then

(i) *$f^* \in C^2$ is strictly convex and $(f^*)' = (f')^{-1}$, where $(f')^{-1}$ denotes the inverse function of f';*

(ii) *$(f^*)'$ is Lipschitz continuous, i.e. for any $p, q \in \mathbb{R}$ there holds*

$$|(f^*)'(p) - (f^*)'(q)| \leq \frac{|p-q|}{c_0}$$

Proof. By the condition on f, f' is strictly increasing with $f'(-\infty) = -\infty$ and $f'(+\infty) = +\infty$, and thus $g := (f')^{-1} : \mathbb{R} \to \mathbb{R}$ exists as a C^1 function with $g'(x) = 1/f''(g(x)) > 0$.

(i) For any $q \in \mathbb{R}$, there always holds $f^*(q) = qx - f(x)$, where x is determined by $q = f'(x)$, i.e. $x = (f')^{-1}(q) = g(q)$. Thus

$$f^*(q) = qg(q) - f(g(q)), \qquad \forall q.$$

This implies that $f^* \in C^1$ and

$$\begin{aligned}(f^*)'(q) &= g(q) + qg'(q) - f'(g(q))g'(q) \\ &= g(q) + qg'(q) - qg'(q) = g(q).\end{aligned}$$

Consequently $(f^*)' = g$ and $f^* \in C^2$ with $(f^*)'' = g' > 0$.

(ii) For any $p, q \in \mathbb{R}$ let $x = (f^*)'(p)$ and $y = (f^*)'(q)$. Then

$$p = f'(x) \quad \text{and} \quad q = f'(y).$$

Since $f'' \geq c_0$, we have

$$f(y) - f(x) - f'(x)(y - x) \geq \frac{1}{2}c_0(y - x)^2,$$
$$f(x) - f(y) - f'(y)(x - y) \geq \frac{1}{2}c_0(x - y)^2.$$

Adding these two inequalities gives

$$c_0(x - y)^2 \leq (f'(x) - f'(y))(x - y) \leq |f'(x) - f'(y)||x - y|$$

This implies that $c_0|x - y| \leq |f'(x) - f'(y)|$, i.e.

$$c_0|(f^*)'(p) - (f^*)'(q)| \leq |p - q|.$$

This completes the proof. ∎

Lemma 15

The function w defined by the Hopf-Lax formula (33) is Lipschitz continuous on $\mathbb{R} \times [0, \infty)$ and $w(x, 0) = h(x)$ for $x \in \mathbb{R}$.

Proof. We use

$$Lip(F) := \sup\{|F(x) - F(y)|/|x - y| : x, y \in \mathbb{R} \text{ and } x \neq y\}$$

to denote the Lipschitz constant of a Lipschitz function F.

- We first show that, for each $t > 0$, $w(\cdot, t)$ is Lipschitz with

$$Lip(w(\cdot, t)) \leq Lip(h).$$

To see this, let $x_1, x_2 \in \mathbb{R}$. We may take $y_1 \in \mathbb{R}$ such that

$$w(x_1, t) = h(y_1) + t\, f^*(\frac{x_1 - y_1}{t}).$$

Then

$$
\begin{aligned}
& w(x_2, t) - w(x_1, t) \\
& = \inf \left\{ h(y) + tf^*(\frac{x_2 - y}{t}) \right\} - h(y_1) - tf^*(\frac{x_1 - y_1}{t}) \\
& \leq h(x_2 - x_1 + y_1) - h(y_1) \leq Lip(h)|x_2 - x_1|.
\end{aligned}
$$

Interchanging the role of x_1 and x_2 we then obtain

$$|w(x_1, t) - w(x_2, t)| \leq Lip(h)|x_1 - x_2|. \tag{34}$$

- We next show that there is a constant $C_0 > 0$ such that

$$|w(x, t) - h(x)| \leq C_0 t, \quad \forall x \in \mathbb{R} \text{ and } t > 0.$$

Indeed, we first have

$$w(x,t) \leq h(x) + t\, f^*(0).$$

Moreover, by using $h(y) \geq h(x) - Lip(h)|x-y|$ we have

$$\begin{aligned}
w(x,t) &= \inf_{y\in\mathbb{R}} \left\{ h(y) + t\, f^*(\frac{x-y}{t}) \right\} \\
&\geq h(x) - \sup_{y\in\mathbb{R}} \left\{ Lip(h)|x-y| - t\, f^*(\frac{x-y}{t}) \right\} \\
&= h(x) - t \sup_{z\in\mathbb{R}} \{ Lip(h)|z| - f^*(z) \} \\
&\geq h(x) - C_1 t,
\end{aligned}$$

where $C_1 := \sup_{|y|\leq Lip(h)} f(y)$ by Lemma 13 (ii).

- We further show that there is a constant C_2 such that

$$|w(x,t_1) - w(x,t_2)| \le C_2(t_2 - t_1) \tag{35}$$

for all $x \in \mathbb{R}$ and $0 < t_1 < t_2$. Indeed, letting $y \in \mathbb{R}$ be such that

$$w(x,t_1) = h(y) + t_1 f^*\left((x-y)/t_1\right),$$

we may use the definition of $w(x,t_2)$ to obtain

$$w(x,t_2) \le h(y) + t_2 f^*\left((x-y)/t_2\right).$$

By writing

$$\frac{x-y}{t_2} = \frac{t_1}{t_2}\frac{x-y}{t_1} + \left(1 - \frac{t_1}{t_2}\right) \cdot 0$$

and using the convexity of f^* we have

$$\begin{aligned} w(x,t_2) &\leq h(y) + t_2\left\{\frac{t_1}{t_2}f^*(\frac{x-y}{t_1}) + \left(1-\frac{t_1}{t_2}\right)f^*(0)\right\} \\ &= h(y) + t_1 f^*(\frac{x-y}{t_1}) + (t_2-t_1)f^*(0) \\ &= w(x,t_1) + (t_2-t_1)f^*(0). \end{aligned}$$

Therefore

$$w(x,t_2) - w(x,t_1) \leq (t_2-t_1)f^*(0), \quad 0 < t_1 < t_2. \tag{36}$$

On the other hand, we may take $z \in \mathbb{R}$ such that

$$w(x,t_2) = h(z) + t_2 f^*((x-z)/t_2).$$

Let $y = \frac{t_1}{t_2}x + (1 - \frac{t_1}{t_2})z$. Since $\frac{x-z}{t_2} = \frac{y-z}{t_1} = \frac{x-y}{t_2-t_1}$, we have

$$w(x,t_2) = h(z) + t_1 f^*(\frac{y-z}{t_1}) + t_2 f^*(\frac{x-z}{t_2}) - t_1 f^*(\frac{y-z}{t_1})$$
$$\geq w(y,t_1) + (t_2 - t_1) f^*(\frac{x-y}{t_2-t_1}).$$

Using (34) we have

$$w(y,t_1) \geq w(x,t_1) - Lip(h)|y-x|.$$

Therefore

$$w(x,t_2) \geq w(x,t_1) - Lip(h)|x-y| + (t_2-t_1)f^*(\frac{x-y}{t_2-t_1}).$$

Consequently

$$w(x,t_2) \geq w(x,t_1) - (t_2 - t_1) \sup_{\eta \in \mathbb{R}} \{Lip(h)|\eta| - f^*(\eta)\}$$

So, by Lemma 13 (ii), we have

$$w(x,t_2) - w(x,t_1) \geq -C_1(t_2 - t_1), \quad 0 < t_1 < t_2.$$

Combining this with (36) we obtain (35).

- Finally, by writing

$$|w(x_1,t_1) - w(x_2,t_2)| \leq |w(x_1,t_1) - w(x_2,t_1)| + |w(x_2,t_1) - w(x_2,t_2)|,$$

we may use (34) and (35) to complete the proof. ■

Theorem 16

The function w defined by the Hopf-Lax formula (33) is Lipschitz continuous, is differentiable a.e. on $\mathbb{R} \times (0, \infty)$ and is a weak solution of (29).

Proof. By Lemma 15, w is Lipschitz on $\mathbb{R} \times [0, \infty)$ with $w(\cdot, 0) = h$. So w is differentiable a.e. in $\mathbb{R} \times (0, \infty)$ by Rademacher's Theorem. It remains only to show that

$$w_t(x, t) + f(w_x(x, t)) = 0$$

for any $(x, t) \in \mathbb{R} \times (0, \infty)$ at which w is differentiable.

- We first choose $z \in \mathbb{R}$ such that

$$w(x, t) = h(z) + t\, f^*((x - z)/t).$$

Fix any $0 < \varepsilon < t$ and set $y = (1 - \frac{\varepsilon}{t})x + \frac{\varepsilon}{t}z$. Then

$$w(y, t - \varepsilon) \leq h(z) + (t - \varepsilon)f^*(\frac{y - z}{t - \varepsilon}).$$

Since $\frac{x-z}{t} = \frac{y-z}{t-\varepsilon}$, we have

$$\begin{aligned} w(x, t) - w(y, t - \varepsilon) &\geq t\, f^*(\frac{x - z}{t}) - (t - \varepsilon)f^*(\frac{x - z}{t}) \\ &= \varepsilon f^*(\frac{x - z}{t}). \end{aligned}$$

Therefore

$$\frac{w(x, t) - w(x + \frac{\varepsilon}{t}(z - x), t - \varepsilon)}{\varepsilon} \geq f^*(\frac{x - z}{t}).$$

- Letting $\varepsilon \searrow 0$ gives

$$\frac{x-z}{t} w_x(x,t) + w_t(x,t) \geq f^*(\frac{x-z}{t}).$$

Consequently, by the definition of f^*,

$$w_t(x,t) + f(w_x(x,t))$$
$$\geq f(w_x(x,t)) + f^*(\frac{x-z}{t}) - \frac{x-z}{t} w_x(x,t) \geq 0.$$

- On the other hand, fix any $q \in \mathbb{R}$ and $\varepsilon > 0$. Then

$$w(x+\varepsilon q, t+\varepsilon) = \inf_{y \in \mathbb{R}} \left\{ h(y) + (t+\varepsilon) f^*(\frac{x+\varepsilon q - y}{t+\varepsilon}) \right\}.$$

- Since $\frac{x+\varepsilon q-y}{t+\varepsilon} = \frac{\varepsilon}{t+\varepsilon} q + \frac{t}{t+\varepsilon}\frac{x-y}{t}$, we may use the convexity of f^* to derive

$$(t+\varepsilon) f^*\left(\frac{x+\varepsilon q-y}{t+\varepsilon}\right) \le \varepsilon f^*(q) + t f^*\left(\frac{x-y}{t}\right).$$

Therefore

$$\begin{aligned} w(x+\varepsilon q, t+\varepsilon) &\le \varepsilon f^*(q) + \inf_{y\in\mathbb{R}} \left\{ h(y) + t f^*\left(\frac{x-y}{t}\right) \right\} \\ &= \varepsilon f^*(q) + w(x,t). \end{aligned}$$

So

$$\frac{w(x+\varepsilon q, t+\varepsilon) - w(x,t)}{\varepsilon} \le f^*(q).$$

Letting $\varepsilon \searrow 0$ gives

$$qw_x(x,t) + w_t(x,t) \le f^*(q), \quad \forall q \in \mathbb{R}.$$

Therefore, by Lemma 13 (iii),

$$-w_t(x,t) \ge \sup_{q\in\mathbb{R}} \{qw_x(x,t) - f^*(q)\} = f(w_x(x,t)),$$

i.e. $w_t(x,t) + f(w_x(x,t)) \le 0$. The proof is thus complete. ∎

We are ready to complete the proof of Theorem 11. To this end, let $h(x) = \int_0^x u_0(y)dy$ and define $w(x,t)$ by the Hopf-Lax formula

$$w(x,t) = \inf_{y\in\mathbb{R}} \left\{ h(y) + t\, f^*\left(\frac{x-y}{t}\right) \right\}.$$

By Theorem 16, w is Lipschitz, is differentiable for a.e. (x, t), and

$$w_t + f(w_x) = 0 \quad \text{a.e. in } \mathbb{R} \times (0, \infty),$$
$$w(x, 0) = h(x), \quad x \in \mathbb{R}.$$

Lemma 17

Let $u := w_x$. Then u is a weak solution of (28).

Proof. Recall that $Lip(w) \leq Lip(h) = \|u_0\|_\infty$, $u \in L^\infty(\mathbb{R} \times (0, \infty))$ with

$$\|u\|_\infty \leq Lip(w) \leq \|u_0\|_\infty.$$

Next for any $\varphi \in C_0^1(\mathbb{R} \times [0, \infty))$ we have

$$0 = \int_0^\infty \int_{-\infty}^\infty (w_t + f(w_x))\varphi_x dxdt. \tag{37}$$

Since w is Lipschitz, $x \to w(x,t)$ is absolute continuous for each $t \geq 0$ and $t \to w(x,t)$ is absolute continuous for each $x \in \mathbb{R}$. So, integration by parts can be used to obtain

$$\begin{aligned}
&\int_0^\infty \int_{-\infty}^\infty w_t \varphi_x dxdt \\
&= -\int_0^\infty \int_{-\infty}^\infty w \varphi_{xt} dxdt - \int_{-\infty}^\infty w(x,0)\varphi_x(x,0)dx \\
&= \int_0^\infty \int_{-\infty}^\infty w_x \varphi_t dxdt + \int_{-\infty}^\infty w_x(x,0)\varphi(x,0)dx.
\end{aligned}$$

Since $w_x(x,0) = u_0(x)$ for a.e. x, we have

$$\int_0^\infty \int_{-\infty}^\infty w_t \varphi_x dxdt = \int_0^\infty \int_{-\infty}^\infty w_x \varphi_t dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)dx.$$

Combining this with (37) gives

$$0 = \int_0^\infty \int_{-\infty}^\infty (w_x \varphi_t + f(w_x)\varphi_x)dxdt + \int_{-\infty}^\infty u_0(x)\varphi(x,0)dx.$$

Thus $u = w_x$ is a weak solution of (28).

- To complete the proof of Theorem 11, it remains only to show that there is a function $\tilde{u}$ with $u = \tilde{u}$ a.e. in $\mathbb{R} \times (0, \infty)$ such that $\tilde{u}$ satisfies the Oleinik entropy condition.
- To this end, we will use, for each (x, t) with $t > 0$, the minimizer of the function

$$\mathcal{F}_{x,t}(y) := h(y) + tf^*(\frac{x-y}{t}) \quad \text{over } \mathbb{R}.$$

The following lemma shows that for each fixed $t > 0$, if $x_1 < x_2$ then the minimizer of $\mathcal{F}_{x_1,t}(y)$ is always on the left of the minimizer of $\mathcal{F}_{x_2,t}(y)$.

Lemma 18

Assume that $f \in C^2$ satisfies $f'' \geq c_0 > 0$. Fix $t > 0$ and $x_1 < x_2$. If $y_1 \in \mathbb{R}$ is such that

$$\min_{y \in \mathbb{R}} \left\{ h(y) + t\, f^*(\frac{x_1 - y}{t}) \right\} = h(y_1) + t\, f^*(\frac{x_1 - y_1}{t}),$$

then

$$h(y_1) + t\, f^*(\frac{x_2 - y_1}{t}) < h(y) + t\, f^*(\frac{x_2 - y}{t}), \quad \forall y < y_1.$$

Proof. Let $\tau = \frac{y_1 - y}{x_2 - x_1 + y_1 - y}$. Then $0 < \tau < 1$ and

$$x_2 - y_1 = \tau(x_1 - y_1) + (1 - \tau)(x_2 - y),$$
$$x_1 - y = (1 - \tau)(x_1 - y_1) + \tau(x_2 - y).$$

By the strict convexity of f^*, see Lemma 14 (i), we have

$$f^*(\frac{x_2 - y_1}{t}) < \tau f^*(\frac{x_1 - y_1}{t}) + (1 - \tau) f^*(\frac{x_2 - y}{t}),$$
$$f^*(\frac{x_1 - y}{t}) < (1 - \tau) f^*(\frac{x_1 - y_1}{t}) + \tau f^*(\frac{x_2 - y}{t}).$$

Adding these two inequalities gives

$$f^*(\frac{x_2 - y_1}{t}) + f^*(\frac{x_1 - y}{t}) < f^*(\frac{x_1 - y_1}{t}) + f^*(\frac{x_2 - y}{t}).$$

Therefore

$$\begin{aligned}
& t\, f^*\left(\frac{x_2 - y_1}{t}\right) + t\, f^*\left(\frac{x_1 - y}{t}\right) + h(y_1) + h(y) \\
& < t\, f^*\left(\frac{x_1 - y_1}{t}\right) + t\, f^*\left(\frac{x_2 - y}{t}\right) + h(y_1) + h(y) \\
& \le t\, f^*\left(\frac{x_1 - y}{t}\right) + h(y) + t\, f^*\left(\frac{x_2 - y}{t}\right) + h(y);
\end{aligned}$$

for the last inequality we used the fact that y_1 is a minimizer. This implies the conclusion. ∎

Now we are able to give the construction of $\tilde{u}$ which is stated in the following result.

Lemma 19

There exists a function $y(x,t)$ defined on $\mathbb{R} \times (0,\infty)$ such that

(i) *for each $t > 0$, $x \to y(x,t)$ is nondecreasing;*

(ii) *for each (x,t) with $t > 0$, $y(x,t)$ is a minimizer of the function*

$$\mathcal{F}_{x,t}(y) := h(y) + tf^*(\frac{x-y}{t}).$$

(iii) *if we set $\tilde{u}(x,t) = (f^*)'(\frac{x-y(x,t)}{t})$, then, for each $t > 0$,*

$$u(x,t) = \tilde{u}(x,t) \quad \text{for a.e. } x.$$

In particular, $u = \tilde{u}$ for a.e. $(x,t) \in \mathbb{R} \times (0,\infty)$.

Proof.

- Fix $t > 0$. For each $x \in \mathbb{R}$ let $y(x,t)$ be the smallest of those points y giving the minimum of $\mathcal{F}_{x,t}(y)$.
- It follows from Lemma 18 that $x \to y(x,t)$ is nondecreasing and thus $y(\cdot,t)$ is continuous for all but at most countably many x.
- At a point x of continuity of $y(\cdot,t)$, $y(x,t)$ is the unique minimizer of $\mathcal{F}_{x,t}(y)$ over $\mathbb{R}$.
- From Theorem 16 it follows for each fixed $t > 0$ that

$$\begin{aligned} x \to w(x,t) &:= \min_{y \in \mathbb{R}} \left\{ h(y) + tf^*\left(\frac{x-y}{t}\right) \right\} \\ &= h(y(x,t)) + tf^*\left(\frac{x-y(x,t)}{t}\right) \end{aligned}$$

is differentiable a.e.

- Since $x \to y(x,t)$ is monotone, it is differentiable a.e. as well. Thus, for a.e. x, $f^*(\frac{x-y(x,t)}{t})$ is differentiable and therefore $x \to h(y(x,t))$ is differentiable as well.
- Consequently for a.e. x

$$u(x,t) = \frac{\partial}{\partial x}\left(h(y(x,t)) + tf^*(\frac{x-y(x,t)}{t})\right)$$
$$= \frac{\partial}{\partial x}(h(y(x,t))) + (f^*)'(\frac{x-y(x,t)}{t})(1-y_x(x,t)).$$

- Since $y(x,t)$ is a minimizer of $\mathcal{F}_{x,t}(y)$ over $\mathbb{R}$, x must be a minimizer of

$$z \to \mathcal{F}_{x,t}(y(z,t)) = h(y(z,t)) + tf^*(\frac{x-y(z,t)}{t}).$$

- Consequently $0 = \frac{\partial}{\partial z}\Big|_{z=x} (\mathcal{F}_{x,t}(y(z,t)))$, i.e.

$$0 = \frac{\partial}{\partial x}(h(y(x,t))) - (f^*)'(\frac{x - y(x,t)}{t})y_x(x,t)$$

We therefore obtain $u(x,t) = (f^*)'(\frac{x-y(x,t)}{t})$ a.e. ■

Theorem 20

Let $f \in C^2$ satisfy $f'' \geq c_0 > 0$, let $u_0 \in L^\infty(\mathbb{R})$ and let $h(x) := \int_0^x u_0(y)dy$. Then the function

$$\tilde{u}(x,t) = (f^*)'(\frac{x - y(x,t)}{t}) \tag{38}$$

defined in Lemma 19 is a weak solution of (28) satisfying the Oleinik entropy condition.

Proof. By condition and Lemma 14, $(f^*)'$ is increasing. Thus, by Lemma 19, we have for any $t > 0$ and $x, a \in \mathbb{R}$ with $a > 0$ that

$$\tilde{u}(x,t) = (f^*)'(\frac{x - y(x,t)}{t}) \geq (f^*)'(\frac{x - y(x+a,t)}{t}).$$

By Lemma 14 (ii), we have

$$\begin{aligned} \tilde{u}(x,t) &\geq (f^*)'(\frac{x + a - y(x+a,t)}{t}) - a/(c_0 t) \\ &= \tilde{u}(x+a,t) - a/(c_0 t). \end{aligned}$$

The proof is complete. ■

Remark. The formula (38) is called the Lax-Oleinik formula. Recall that $(f^*)' = (f')^{-1}$, we have $\tilde{u}(x,t) = (f')^{-1}((x - y(x,t))/t)$.

6. Long-Time Behaviour of Entropy Solutions

We prove a uniform decay estimate for the entropy solution of the scalar conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x) \tag{39}$$

with uniformly convex flux $f(u)$.

Theorem 21

Let $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $f \in C^2$ with $f'' \geq c_0 > 0$. Then the entropy solution of (39) satisfies the estimate

$$|u(x,t)| \leq C/t^{1/2},$$

where C is a constant depending only on c_0 and $\|u_0\|_{L^1}$.

Proof. We use the Lax-Oleinik formula

$$u(x,t) = (f^*)'(\frac{x - y(x,t)}{t}).$$

In order to use the Lipschitz continuity of $(f^*)'$, we take $\sigma \in \mathbb{R}$ such that

$$(f^*)'(\sigma) = 0,$$

i.e. $(f')^{-1}(\sigma) = 0$; we can take $\sigma = f'(0)$. Then

$$\begin{aligned} |u(x,t)| &= \left| (f^*)'(\frac{x - y(x,t)}{t}) - (f^*)'(\sigma) \right| \\ &\leq \frac{1}{c_0} \left| \frac{x - y(x,t)}{t} - \sigma \right|. \end{aligned} \tag{40}$$

To estimate the right hand side, by the definition of $y(x,t)$ we have

$$h(y(x,t)) + tf^*(\frac{x - y(x,t)}{t}) = \min_{y \in \mathbb{R}} \left\{ h(y) + tf^*(\frac{x-y}{t}) \right\}$$
$$\leq h(x - \sigma t) + tf^*(\sigma)$$

where $h(x) = \int_0^x u_0(\eta) d\eta$. Since $f'' \geq c_0 > 0$, we have

$$f^*(\frac{x - y(x,t)}{t}) \geq f^*(\sigma) + (f^*)'(\sigma) \left(\frac{x - y(x,t)}{t} - \sigma \right)$$
$$+ \frac{1}{2} c_0 \left(\frac{x - y(x,t)}{t} - \sigma \right)^2 .$$

Combining these last two inequalities gives

$$\frac{1}{2}tc_0\left(\frac{x-y(x,t)}{t}-\sigma\right)^2 \leq h(x-\sigma t)-h(y(x,t)).$$

Recall the definition of h and $u_0 \in L^1(\mathbb{R})$, we have $|h(x)| \leq \|u_0\|_{L^1}$ for all $x \in \mathbb{R}$. Therefore

$$\frac{1}{2}tc_0\left(\frac{x-y(x,t)}{t}-\sigma\right)^2 \leq 2\|u_0\|_{L^1},$$

i.e.

$$\left|\frac{x-y(x,t)}{t}-\sigma\right| \leq \sqrt{\frac{4\|u_0\|_{L^1}}{c_0 t}}.$$

Combining this with (40) gives the desired estimate. ∎