The Basel Problem

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The Basel Problem asks for the exact value of the infinite sum $\sum_{k=1}^{\infty}\frac{1}{k^2}$. It was first posed by Pietro Mengoli in 1644 and, despite much effort, was only solved ninety years later by Leonhard Euler in 1734, who found that

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}. \tag{1}$$

His original proof was not rigorous by today's standards, but since then many rigorous proofs have been found. Here we use an approach that relies only on ideas and results from *Analysis I & II*.

The material here is **non-examinable** and can be read after one has completed Section 9 of the lecture notes on the *Mean Value Theorem*.

We start with a trigonometric identity.

Lemma 1. *For any $n\geq 1$ and $x\in\mathbb{R}\setminus\{k\pi : k\in\mathbb{Z}\}$,*

$$\frac{1}{\sin^2 x}=\frac{1}{4^n}\sum_{k=0}^{2^n-1}\frac{1}{\sin^2\left(\frac{x+k\pi}{2^n}\right)}. \tag{2}$$

Proof. We use standard trigonometric identities to deduce that

$$\frac{1}{\sin^2 x}=\frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)^2}=\frac{1}{4\sin^2\frac{x}{2}}+\frac{1}{4\cos^2\frac{x}{2}}=\frac{1}{4}\left(\frac{1}{\sin^2\left(\frac{x}{2}\right)}+\frac{1}{\sin^2\left(\frac{x+\pi}{2}\right)}\right).$$

This is the $n=1$ case. Now by induction, assuming the result for $n-1$,

$$\begin{aligned}\frac{1}{\sin^2 x}&=\frac{1}{4^{n-1}}\sum_{k=0}^{2^{n-1}-1}\frac{1}{\sin^2\left(\frac{x+k\pi}{2^{n-1}}\right)}\\&=\frac{1}{4^{n-1}}\sum_{k=0}^{2^{n-1}-1}\frac{1}{4}\left(\frac{1}{\sin^2\left(\frac{x+k\pi}{2^n}\right)}+\frac{1}{\sin^2\left(\frac{x+k\pi+2^{n-1}\pi}{2^n}\right)}\right)=\frac{1}{4^n}\sum_{k=0}^{2^n-1}\frac{1}{\sin^2\left(\frac{x+k\pi}{2^n}\right)}.\end{aligned}$$ □

Remark 2. In fact it is true that

$$\frac{1}{\sin^2 x}=\frac{1}{n^2}\sum_{k=0}^{n-1}\frac{1}{\sin^2\left(\frac{x+k\pi}{n}\right)}$$

for all $n\geq 1$, not just when n is a power of 2. [Exercise: see if you can prove this!]

Lemma 3. *For all $x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$*

$$\frac{1}{\sin^2 x} = \sum_{k=-\infty}^{\infty} \frac{1}{(x-k\pi)^2}. \tag{3}$$

Remark 4. When we have a doubly infinite sequence (a_n) defined for all $n \in \mathbb{Z}$, the statement that

$$\sum_{k=-\infty}^{\infty} a_k = L$$

is interpreted to mean that both $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_{-k}$ converge and L is their sum. Equivalently, $\forall \varepsilon > 0 \colon \exists N \colon \forall n, m \geq N \colon |\sum_{k=-m}^{n} a_k - L| < \varepsilon$. [Exercise: prove these statements are equivalent.]

Proof. By using the fact that $\sin^2 x$ is periodic with period π we can replace the terms $k = 2^{n-1}+1, \dots, 2^n - 1$ with $k = -(2^{n-1}-1), \dots, -1$ giving (after renaming k to $-k$)

$$\frac{1}{\sin^2 x} = \frac{1}{4^n} \sum_{k=-2^{n-1}}^{2^{n-1}-1} \frac{1}{\sin^2\left(\frac{x-k\pi}{2^n}\right)}.$$

Now the idea is to replace the terms $\frac{1}{\sin^2 z}$ in the sum with $\frac{1}{z^2}$, so it helps to know how the difference behaves. As $x \to 0$ we have

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \frac{x^2 - (x - \frac{x^3}{6} + O(x^5))^2}{x^2(x - O(x^3))^2} = \frac{\frac{x^4}{3} + O(x^6)}{x^4 + O(x^6)} = \frac{\frac{1}{3} + O(x^2)}{1 + O(x^2)} \to \frac{1}{3}. \tag{4}$$

(Or apply *L'Hôpital* multiple times.) Thus we can extend this function to a continuous function on $[-2,2]$, say. Thus by the *Boundedness Theorem* there is some M such that, for all $x \in [-2,2] \setminus \{0\}$,

$$\left|\frac{1}{\sin^2 x} - \frac{1}{x^2}\right| \leq M.$$

But then, assuming x is fixed and n large enough so that all $\frac{x-k\pi}{2^n} \in [-2,2]$ (which we can do as we rearranged the sum so that $|k| \leq 2^{n-1}$),

$$\begin{aligned}\left|\frac{1}{\sin^2 x} - \sum_{k=-2^{n-1}}^{2^{n-1}-1} \frac{1}{(x-k\pi)^2}\right| &= \left|\frac{1}{4^n} \sum_{k=-2^{n-1}}^{2^{n-1}-1} \left(\frac{1}{\sin^2\left(\frac{x-k\pi}{2^n}\right)} - \frac{1}{\left(\frac{x-k\pi}{2^n}\right)^2}\right)\right| \\ &\leq \frac{1}{4^n} \sum_{k=-2^{n-1}}^{2^{n-1}-1} M = \frac{2^n M}{4^n} = \frac{M}{2^n}.\end{aligned}$$

Now both sums $\sum_{k=0}^{\infty} \frac{1}{(x-k\pi)^2}$ and $\sum_{k=1}^{\infty} \frac{1}{(x-(-k)\pi)^2}$ converge by comparison with $\sum \frac{1}{k^2}$. Thus taking the limit as $n \to \infty$ we deduce that (3) holds by sandwiching. □

We are now essentially done. Substitute $x = \frac{\pi}{2}$ to obtain

$$1 = \sum_{k=-\infty}^{\infty} \frac{1}{(\frac{\pi}{2} - k\pi)^2} = \frac{4}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)^2} = \frac{8}{\pi^2} \sum_{k>0 \text{ odd}} \frac{1}{k^2}.$$

But

$$\sum_{k>0 \text{ odd}} \frac{1}{k^2} = \sum_{k>0} \frac{1}{k^2} - \sum_{k>0 \text{ even}} \frac{1}{k^2} = \sum_{k>0} \frac{1}{k^2} - \sum_{k>0} \frac{1}{(2k)^2} = \frac{3}{4} \sum_{k>0} \frac{1}{k^2}.$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{4}{3} \sum_{k>0 \text{ odd}} \frac{1}{k^2} = \frac{4}{3} \cdot \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

This solves the Basel Problem. However, in some sense it would be nicer to evaluate (3) at $x = 0$ since then we don't need to split into even and odd terms. Unfortunately both sides are now undefined. But we do have for all $0 < |x| \le 2$, say,

$$\frac{1}{\sin^2 x} - \frac{1}{x^2} = \sum_{k \neq 0} \frac{1}{(x - k\pi)^2}.$$

Now the RHS converges uniformly on $[-2, 2]$ by the *Weierstrass M-test*: take $M_k = \frac{1}{k^2}$ for large k. Thus the RHS is a *Uniform Limit of Continuous Functions*, so is continuous, and in particular, continuous at $x = 0$. But the limit of the LHS is $\frac{1}{3}$ as $x \to 0$ by (4). Hence

$$\frac{1}{3} = \sum_{k \neq 0} \frac{1}{(-\pi k)^2} = \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

and the Basel Problem is again solved.

What about sums of other powers? Can we find closed form expressions for

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^s}$$

for other values of s and use them to calculate $\sum \frac{1}{k^s}$?

One idea would be to repeatedly differentiate both sides of (3), with the RHS being differentiated term by term to get higher negative powers of $x - k\pi$. But can we differentiate an infinite sum of functions term by term? The answer in general is of course 'No'. But the following theorem allows us to do this in many cases.

Theorem 5. *Suppose I is an interval, $u_k \colon I \to \mathbb{R}$ are differentiable and*

(a) $\sum u_k(x_0)$ *converges for some $x_0 \in I$,*

(b) $\sum u_k'(x)$ *converges uniformly on I.*

Then $\sum u_k(x)$ converges to a differentiable function with

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} u_k(x) = \sum_{k=1}^{\infty} u_k'(x).$$

Remark 6. We need $\sum u_k'$ to converge uniformly here, $\sum u_k$ converging uniformly does not help. Also condition (a) is just needed to 'anchor' the sum at some point, since otherwise we could just take $u_k(x) = a_k$ to be a constant with $\sum a_k$ diverging as a counterexample.

Remark 7. You will see this theorem again in *Analysis III* with a much simpler proof (but with the added assumption that u_k' are all continuous). The idea there is to just integrate $\sum u_k'$ and use the fact (which will be proved in *Analysis III*) that the integral of a uniform sum is the sum of the integrals.

Proof. First we show that $\sum u_k(x)$ converges for every $x \in I$. Write $f_n(x) := \sum_{k=1}^n u_k(x)$ for the partial sums and note that $f_n'(x) = \sum_{k=1}^n u_k'(x)$ as the derivative of a *finite* sum is just the sum of the derivatives.

Now $f_n' = \sum_{k=1}^n u_k'$ converges uniformly as $n \to \infty$, so by the *Cauchy Convergence Criterion for Uniform Convergence*, for all $\varepsilon > 0$ there exists some N such that for $n, m \geq N$, and for all $x \in I$ we have $|f_n'(x) - f_m'(x)| < \varepsilon$. But by the *Mean Value Theorem* applied to $f_n - f_m$ we have

$$\big|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))\big| = |(x - x_0)(f_n'(\xi) - f_m'(\xi))| \leq \varepsilon|x - x_0|$$

for some $\xi \in I$ between x_0 and x. (Note that *uniform* convergence of $\sum u_k'$ was needed here — we needed to apply the bound at some unknown ξ.) But by (a) and the usual *Cauchy Convergence Criterion*, we have some N' such that if $n, m \geq N'$ $|f_n(x_0) - f_m(x_0)| < \varepsilon$. Thus for $n, m \geq \max\{N, N'\}$, $|f_n(x) - f_m(x)| < \varepsilon(|x - x_0| + 1)$. As $|x - x_0| + 1$ here is just a constant, we see that for any fixed x, $(f_n(x))$ is Cauchy and so f_n converges.

Thus we have a well-defined limit function $f(x) := \sum_{k=1}^\infty u_k(x)$. Write

$$g(x) := \sum_{k=1}^{\infty} u_k'(x) = \lim_{n\to\infty} f_n'(x)$$

for our conjectured derivative of f, noting that by (b) this does indeed converge. Fix $x_1 \in I$ and now consider differentiability of f at x_1. Using the MVT again we see that for $m, n \geq N$

$$\Big|\frac{f_n(x) - f_n(x_1)}{x - x_1} - \frac{f_m(x) - f_m(x_1)}{x - x_1}\Big| = \Big|f_n'(\xi) - f_m'(\xi)\Big| < \varepsilon.$$

for some ξ between x and x_1. Taking a limit as $n \to \infty$ gives

$$\Big|\frac{f(x) - f(x_1)}{x - x_1} - \frac{f_m(x) - f_m(x_1)}{x - x_1}\Big| \leq \varepsilon$$

for $m \geq N$. Now choose $m \geq N$ sufficiently large so that

$$|f_m'(x_1) - g(x_1)| < \varepsilon$$

(recall $f_m' \to g$). Then (by definition of f_m') choose $\delta > 0$ so that for $0 < |x - x_1| < \delta$

$$\Big|\frac{f_m(x) - f_m(x_1)}{x - x_1} - f_m'(x_1)\Big| < \varepsilon.$$

Then, for $0 < |x - x_1| < \delta$, the triangle inequality gives

$$\begin{aligned}\Big|\tfrac{f(x)-f(x_1)}{x-x_1} - g(x_1)\Big| &\leq \Big|\tfrac{f(x)-f(x_1)}{x-x_1} - \tfrac{f_m(x)-f_m(x_1)}{x-x_1}\Big| + \Big|\tfrac{f_m(x)-f_m(x_1)}{x-x_1} - f_m'(x_1)\Big| + |f_m'(x_1) - g(x_1)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.\end{aligned}$$

Hence $f'(x_1)$ exists and is equal to $g(x_1)$. □

As an example, let's calculate $\sum \frac{1}{k^4}$. We note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sin^2 x} = \frac{-2\cos x}{\sin^3 x} = \sum_{k=-\infty}^{\infty} \frac{-2}{(x-k\pi)^3}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{1}{\sin^2 x} = \frac{6\cos^2 x + 2\sin^2 x}{\sin^4 x} = \sum_{k=-\infty}^{\infty} \frac{6}{(x-k\pi)^4}.$$

Both are valid for $x \in (0,\pi)$, say, as the sums on the right converge uniformly (by the *M-test* with $M_k = \frac{1}{k^3}$ or $\frac{1}{k^4}$ when k is large). Substituting $x = \frac{\pi}{2}$ into this last one gives

$$2 = \sum_{k=-\infty}^{\infty} \frac{6}{(\frac{\pi}{2}-k\pi)^4} = \frac{16\cdot 6}{\pi^4}\sum_{k=-\infty}^{\infty} \frac{1}{(2k-1)^4} = \frac{16\cdot 6\cdot 2}{\pi^4}\sum_{k>0 \text{ odd}} \frac{1}{k^4}.$$

Now using

$$\sum_{k>0 \text{ odd}} \frac{1}{k^4} = \sum_{k>0}\frac{1}{k^4} - \sum_{k>0}\frac{1}{(2k)^4} = \frac{15}{16}\sum_{k>0}\frac{1}{k^4},$$

gives

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{16}{15}\cdot\frac{\pi^4}{6\cdot 16} = \frac{\pi^4}{90}.$$

[Alternatively one can subtract $1/x^4$ from both sides and take limits $x \to 0$.]

Unfortunately the same trick does not work for $\sum \frac{1}{k^3}$ as we get a sum over all odd (or non-zero) integers of $\frac{1}{k^3}$ and the terms with $-k$ cancel the ones with k. Nevertheless, it should be clear that we can repeat this process for all *even* powers and in fact $\sum \frac{1}{k^s}$ will always be a rational number times π^s when s is even and $s \geq 2$.

We remark that identities such as (3) can be proved very easily using next year's course on *Complex Analysis*. You have seen in *Analysis I & II* that one can often prove tricky identities by constructing a function F based on them and showing that $F' = 0$. The identities then follow from the *Constancy Theorem*. Similar, but much more powerful techniques exist in *Complex Analysis*. For example, if you can construct an F (say by taking the difference of the two sides in (3)) that is (complex) differentiable and just *bounded* on all of $\mathbb{C}$ (or even $\mathbb{C}$ with some isolated points removed), then F is a constant.

Finally, we remark that the *Riemann zeta function*

$$\zeta(s) := \sum_{k=1}^{\infty}\frac{1}{k^s},$$

which converges for all real $s > 1$, has many amazing connections with number theory, some of which you may see in future courses.