Question 1. *Solution.* (a) [4 marks BK] $P_n \to P$ weakly if for every $f \in C_b(E)$, we have $\int_E f(x)P_n(dx) \to \int_E f(x)P(dx)$ as $n \to \infty$, which equivalent to any of the followings:

In terms of $U_\rho(E)$: for every $f \in U_\rho(E)$, $\int_E f(x)P_n(dx) \to \int_E f(x)P(dx)$ as $n \to \infty$.

In terms of closed sets. For every closed set F, $\limsup_{n\to\infty} P_n(F) \le P(F)$.

In terms of open sets. Fro every open subset G, $\liminf_{n\to\infty} P_n(G) \ge P(G)$.

(b) [8 = 2+6 marks BK-New] (i) $X_n \to X$ in probability if for every $\delta > 0$,

$$\mathbb{P}\left[\rho(X_n, X) > \delta\right] \to 0 \quad \text{as } n \to \infty.$$

$X_n \to X$ in distribution if the laws of X_n converges to the law of X, that is for every bounded and continuous function f, $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ as $n \to \infty$.

(ii) Suppose $X_n \to X$ in probability, and f is bounded and uniformly continuous on E, so that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ as long as $\rho(x, y) < \delta$. Therefore

$$\begin{aligned}\mathbb{E}\left|f(X_n) - f(X)\right| &= \mathbb{E}\left[|f(X_n) - f(X)| : \rho(X_n, X) < \delta\right] + \mathbb{E}\left[|f(X_n) - f(X)| : \rho(X_n, X) \ge \delta\right] \\ &\le \varepsilon + \mathbb{E}\left[|f(X_n) - f(X)| : \rho(X_n, X) > \frac{1}{2}\delta\right] \\ &\le \varepsilon + 2C\mathbb{P}\left[\rho(X_n, X) > \frac{1}{2}\delta\right]\end{aligned}$$

where $C > 0$ such that $|f(x)| \le C$ for all $x \in E$ (as f is bounded). Letting $n \to \infty$ to obtain that

$$\limsup_{n\to\infty} \mathbb{E}\left|f(X_n) - f(X)\right| \le \varepsilon$$

for every $\varepsilon > 0$, and therefore one must have

$$\limsup_{n\to\infty} \mathbb{E}\left|f(X_n) - f(X)\right| = 0.$$

Since $\mathbb{E}\left|f(X_n) - f(X)\right| \ge 0$, hence $\lim_{n\to\infty} \mathbb{E}\left|f(X_n) - f(X)\right| = 0$. It follows that, for every $f \in U_\rho(E)$ we have

$$\left|\mathbb{E}\left[f(X_n)\right] - \mathbb{E}\left[f(X)\right]\right| \le \mathbb{E}\left|f(X_n) - f(X)\right| \to 0$$

as $n \to \infty$. Hence, by (a) (i), $X_n \to X$ in distribution.

(c) [13=2+11 marks BK-Seen] A family $\mathscr{L}$ of probability measures on $(E, \mathscr{B}(E))$ is tight, if for every $\varepsilon > 0$ there is a compact subset $K_\varepsilon \subset E$, such that

$$P\left[K_\varepsilon\right] \ge 1 - \varepsilon \quad \text{for every } P \in \mathscr{L}.$$

If P is a probability measure on $(E, \mathscr{B}(E))$, we show that $\{P\}$ is tight. In fact for every $\varepsilon > 0$, since (E, ρ) is a separable metric space, there is a dense countable subset $\{x_1, x_2, \cdots\}$ of E. Therefore for every $n = 1, 2, \cdots$,

$$\bigcup_{i=1}^{\infty} \bar{B}_{x_i}\left(2^{-n}\right) = E,$$

where $\bar{B}_x(r)$ denotes the closed ball centered at x with radius r. Therefore

$$1 = P(E) = P\left[\bigcup_{i=1}^{\infty} \bar{B}_{x_i}\left(2^{-n}\right)\right] = \lim_{k\to\infty} P\left[\bigcup_{i=1}^{k} \bar{B}_{x_i}\left(2^{-n}\right)\right]$$

for every n. By definition, for every n, there is $k_n \in \mathbb{N}$ such that

$$P\left[\bigcup_{i=1}^{k_n} \bar{B}_{x_i}\left(2^{-n}\right)\right] > 1 - \frac{\varepsilon}{2^n}.$$

Let $K_\varepsilon = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{k_n} \bar{B}_{x_i}(2^{-n})$. Then

$$P(E\setminus K_\varepsilon) = P\left[\bigcup_{n=1}^{\infty}\left(E\setminus\bigcup_{i=1}^{k_n}\bar{B}_{x_i}\left(2^{-n}\right)\right)\right] \le \sum_{n=1}^{\infty} P\left(E\setminus\bigcup_{i=1}^{k_n}\bar{B}_{x_i}\left(2^{-n}\right)\right)$$
$$< \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n} = \varepsilon$$

and therefore $P[K_\varepsilon] > 1-\varepsilon$. We claim K_ε is compact. For every $\delta > 0$, we may choose n such that $\frac{1}{2^n} < \delta$. Then

$$\bigcup_{i=1}^{k_n} B_{x_i}(\delta) \supset \bigcup_{i=1}^{k_n}\bar{B}_{x_i}\left(2^{-n}\right) \supset K_\varepsilon$$

which implies that K_ε is totally bounded. Since K_ε is closed, and E is complete, therefore K_ε is compact. According to definition, $\{P\}$ is tight.

Question 2. *Solution.* (a) [15 =3+2+5+5 marks BK-Seen] (i) $I: E \to [0,\infty]$ is a rate function if $\{x\in E : I(x)\le c\}$ is closed for every c, or equivalently, I is lower semi-continuous. Such I is a good rate function if $\{x\in E : I(x)\le c\}$ is compact for every $c\ge 0$.

$\{X^\varepsilon : \varepsilon\in(0,1)\}$ satisfies the weak large deviation principle with rate function I, if for every open subset $G\subset E$

$$\liminf_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in G] \ge -\inf_G I,$$

(LDP lower bound), and for every compact subset $K\subset E$

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[K] \le -\inf_K I.$$

$\{X^\varepsilon : \varepsilon\in(0,1)\}$ satisfies the large deviation principle with rate function I if the previous LDP lower bounds for open sets $G\subset E$ hold, and for every closed subset $S\subset E$ we have

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in S] \le -\inf_S I.$$

(ii) $\{X^\varepsilon : \varepsilon\in(0,1)\}$ is exponentially tight, if for every $c>0$ there is a compact subset $K_c\subset E$ such that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\notin K_c] < -c.$$

(iii) Suppose $\{X^\varepsilon : \varepsilon\in(0,1)\}$ is exponentially tight, and satisfies the weak LDP. We first show that I is a good rate function. If $c\ge 0$, we show that $I_c = \{x\in E : I(x)\le c\}$ is compact. In fact, since $\{X^\varepsilon : \varepsilon\in(0,1)\}$ is exponentially tight, so there is a compact subset K_c, $\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\notin K_c]\le -c$. By LDP lower bound applying to the open set $E\setminus K_c$ we have

$$-\inf_{E\setminus K_c} I \le \liminf_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in E\setminus K_c] = \limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\notin K_c] < -c.$$

That is $\inf_{E\setminus K_c} I > c$, which implies that $I_c\subset K_c$, since I is a rate function, so I_c is closed, together with the fact that K_c is compact, we deduce that I_c is compact too. To prove that $\{X^\varepsilon : \varepsilon\in(0,1)\}$ satisfies LDP, we only need to show the upper bound for any closed subset $S\subset E$. For every $c>0$ we may choose a compact subset K_c such that $\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\notin K_c]\le -c$. Then

$$P[X^\varepsilon\in S] \le P[X^\varepsilon\in K_c\cap S] + P[X^\varepsilon\in E\setminus K_c]$$

so that

$$\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in S] \le \max\left\{\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in K_c\cap S], \limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in E\setminus K_c]\right\}$$
$$\le \max\left\{\limsup_{\varepsilon\downarrow 0}\varepsilon\log P[X^\varepsilon\in K_c\cap S], -c\right\}$$
$$\le \max\left\{-\inf_{K_c\cap S} I, -c\right\} \le \max\left\{-\inf_S I, -c\right\}.$$

That is for every c we have

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left[X^{\varepsilon} \in S\right] \leq \max\left\{-\inf_{S} I, -c\right\}$$

so by letting $c \uparrow \infty$ we may conclude that

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left[X^{\varepsilon} \in S\right] \leq -\inf_{S} I$$

which proves the LDP upper bound.

(iv) Let G be an open subset, and S be a closed subset. Since f is continuous so $f^{-1}(G)$ is open and $f^{-1}(S)$ is closed. Thus, since X^{ε} satisfies LDP with a good rate function I, we have

$$\liminf_{\varepsilon\downarrow 0} P\left[Y^{\varepsilon} \in G\right] = \liminf_{\varepsilon\downarrow 0} P\left[X^{\varepsilon} \in f^{-1}(G)\right] \geq -\inf_{f^{-1}(G)} I$$

and

$$\limsup_{\varepsilon\downarrow 0} P\left[Y^{\varepsilon} \in S\right] = \limsup_{\varepsilon\downarrow 0} P\left[X^{\varepsilon} \in f^{-1}(S)\right] \leq -\inf_{f^{-1}(S)} I.$$

Let

$$\varphi(x) = \inf\left\{I(y) : f(y) = x\right\} = \inf\left\{I(y) : y \in f^{-1}(x)\right\}$$

with a convention that $\inf \emptyset = \infty$. Let A be a subset of E. Suppose $\inf_{f^{-1}(A)} I = \infty$, then for every $I(y) = \infty$ for every $y \in f^{-1}(A)$, that is $I(y) = \infty$ for $y \in f^{-1}(x)$ and $x \in A$, so that $\inf_{A} \varphi = \infty$. Suppose $l = \inf_{f^{-1}(A)} I < \infty$. Then

$$\inf_{f^{-1}(A)} I = \inf_{x\in A} \inf_{y\in f^{-1}(x)} I(y) = \inf_{x\in A} \varphi(x).$$

We therefore have

$$\liminf_{\varepsilon\downarrow 0} P\left[Y^{\varepsilon} \in G\right] \geq -\inf_{G} \varphi$$

and

$$\limsup_{\varepsilon\downarrow 0} P\left[Y^{\varepsilon} \in S\right] \leq -\inf_{S} \varphi.$$

(b) [10=1+2+7 marks BK+New] (i) Let H^1 denote the Cameron-Martin space of all continuous function h on $[0,1]$ such that $h(0) = 0$ and its derivative $\dot{h} \in L^2[0,1]$. The functional

$$I(x) = \frac{1}{2}\left\|x\right\|_{H^1}^2 = \frac{1}{2}\int_0^1 \left|\dot{x}(t)\right|^2 dt$$

for $x \in H^1$, otherwise $I(x) = \infty$.

Schilder's Theorem : $(\sqrt{\varepsilon}B_t)_{t\in[0,1]}$ satisfies the large deviation principle with the good rate function I, that is,

$$\liminf_{\varepsilon\downarrow 0} \varepsilon \log P\left[\sqrt{\varepsilon}B_{\cdot} \in G\right] \geq -\inf_{G} I$$

for every open subset G of $C_0([0,1], \mathbb{R}^d)$, and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon \log P\left[\sqrt{\varepsilon}B_{\cdot} \in S\right] \leq -\inf_{S} I$$

for every closed subset G of $C_0([0,1], \mathbb{R}^d)$. Where the continuous function space $C_0([0,1], \mathbb{R}^d)$ is equipped with its uniform norm (and the induced metric).

(ii) Let us consider the stochastic differential equation

$$dX_t = -X_t dt + dB_t, \quad X_0 = 0,$$

which has the unique strong solution given by

$$X_t = e^{-t}\left[\int_0^t e^s dB_s\right]$$

which can be rewritten as, by using integration by parts,

$$X_t = -e^{-t}\int_0^t e^s B_s ds + B_t.$$

Therefore we define the mapping from $E = C_0([0,1],\mathbb{R})$ to itself which sends $x \in E$ to $f(x) = w$ where

$$w(t) = -e^{-t}\int_0^t e^s x(s) ds + x(t).$$

Clearly $w \in E$, and if $x, y \in E$, so that

$$f(x)(t) - f(y)(t) = -e^{-t}\int_0^t e^s (x(s) - y(s)) ds + (x(t) - y(t)).$$

and therefore

$$\|f(x) - f(y)\| \leq 2\|x - y\|$$

where $\|x\|$ is the uniform norm, that is, $\|x\| = \sup_{t\in[0,1]}|x(t)|$, this shows that $f : E \to E$ is continuous. By definition, $X^{\varepsilon} = f(\sqrt{\varepsilon}B)$, so according to (i), X^{ε} satisfies LDP with the rate function given by

$$\psi(x) = \inf\left\{I(h) : h \in H^1 \text{ and } f(h) = x\right\}$$

otherwise $\psi(x)$. Moreover $f(h) = x$ and $h \in H^1$, if and only if

$$dx(t) = -x(t)dt + \dot{h}(t)dt, \quad x(0) = 0.$$

If there is no such $h \in H^1$, then $\psi(x) = \infty$.

Question 3. *Solution.* (a) [9=3+6 marks BK-New] (i) Kolmogorov's criterion. Suppose the following two conditions are satisfied:

$$\lim_{L\to\infty}\inf_n \mathbb{P}\left[\left|X_0^{(n)}\right| \leq L\right] = 1$$

and there are $\alpha > 0$, $\beta > 0$ and a constant $C > 0$ such that

$$\mathbb{E}\left[\left|X_t^{(n)} - X_s^{(n)}\right|^{\alpha}\right] \leq C|t-s|^{1+\beta}$$

for all $s, t \geq 0$, then $\left\{X^{(n)} : n = 1, 2, \cdots\right\}$ is relatively compact, that is, the family of the laws of $\left\{X^{(n)} : n = 1, 2, \cdots\right\}$, which are probability measures on the continuous path space $C([0,\infty),\mathbb{R}^d)$, is relatively compact with respect to the Prohorov's metric. Hence, there is a sub-sequence $\left(X^{(n_k)}\right)$ which converges weakly.

(ii) We may apply Kolmogorov's criterion. Since $X_0^{(n)} = 0$ so that $\mathbb{P}\left[\left|X_0^{(n)}\right| \leq L\right] = 1$ for every $L > 0$, hence the first condition in (i) is satisfied. Since (σ_n) is bounded so we may assume that $|\sigma_n| \leq C$, so that

$$\left|X_t^{(n)} - X_s^{(n)}\right| \leq C|B_t - B_s| + M|t-s|.$$

Now

$$\left|X_t^{(n)} - X_s^{(n)}\right|^4 \leq 8C^4|B_t - B_s|^4 + 8M^4|t-s|^4$$

so that

$$\begin{aligned}\mathbb{E}\left[\left|X_t^{(n)} - X_s^{(n)}\right|^4\right] &\leq 8C^4\mathbb{E}|B_t - B_s|^4 + 8M^4|t-s|^4 \\ &\leq C_T|t-s|^2\end{aligned}$$

for all $0 \leq s, t \leq T$, where C_T depends only on T, so according to Kolmogorov's criterion, $\left\{X^{(n)} : n = 1, 2, \cdots\right\}$ is relatively compact.

Solution. (b) [16= 6+10 marks Seen+New] (i) Cramér's large deviation principle. Suppose

$$M_\mu(\lambda) = \int_{\mathbb{R}^d} e^{\lambda \cdot x} \mu(dx)$$

exists for every $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}^d$, and

$$I_\mu(x) = \sup_\lambda \left\{ \lambda \cdot x - \log M_\mu(\lambda) \right\}.$$

Let μ_n be the distribution of $\frac{1}{n}(X_1 + \cdots + X_n)$ where X_i are i.i.d. with the same distribution μ, for $n = 1, 2, \cdots$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(S) \leq -\inf_S I_\mu$$

for every closed subset $S \subset \mathbb{R}^d$, and

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(G) \geq -\inf_G I_\mu$$

for every open subset $G \subset \mathbb{R}^d$.

If $\mu \sim N(0, \mathbf{1})$, then

$$M_\mu(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda \cdot x} e^{-\frac{x^2}{2}} dx = e^{\frac{|x|^2}{2}}$$

so that

$$\phi(x) = \sup_\lambda \left\{ \lambda \cdot x - \frac{\lambda^2}{2} \right\}.$$

While the quadratic form $\lambda \cdot x - \frac{\lambda^2}{2}$ takes its maximum at $\lambda = x$, so that

$$\phi(x) = \frac{|x|^2}{2}, \qquad \text{for } x \in \mathbb{R}^d.$$

(ii) According to (i), Cramér's large deviation principle, μ_n, with the same distribution of $\frac{1}{\sqrt{n}}\xi$, $\xi \sim N(0, \mathbf{1})$, with a rate function $\phi(x) = \frac{|x|^2}{2}$ for $x \in \mathbb{R}^d$, Since in this case μ_n is a normal distribution $N(0, \frac{1}{n}\mathbf{1})$, which coincides with the distribution of

$$\frac{1}{\sqrt{n}}\xi = \left(\frac{1}{\sqrt{n}}\xi_1, \cdots, \frac{1}{\sqrt{n}}\xi_d \right)$$

and therefore, according to Cramér's theorem

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{\sqrt{n}}\xi \in S \right] \leq -\inf_S \phi$$

for every closed subset $S \subset \mathbb{R}^d$, and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{\sqrt{n}}\xi \in G \right] \geq -\inf_G \phi$$

for every open subset $G \subset \mathbb{R}^d$. Now consider

$$f(Y_1, \cdots, Y_d) = Y_1^4 + \cdots + Y_d^4$$

which is a continuous mapping from $\mathbb{R}^d$ to $\mathbb{R}$. Since

$$f(\frac{1}{\sqrt{n}}\xi) = \frac{1}{n^2}\left(\xi_1^4 + \cdots + \xi_d^4 \right)$$

so that ν_n is the distribution of $f(\frac{1}{\sqrt{n}}\xi)$. Hence, according to Varadhan's contraction principle, ν_n satisfies the large deviation principle

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_n(S) = \frac{1}{n} \log \mathbb{P}\left[f(\frac{1}{\sqrt{n}}\xi) \in S \right] \leq -\inf_S \varphi$$

for every closed subset $S \subset \mathbb{R}^d$, and

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_n(G) = \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[f(\frac{1}{\sqrt{n}}\xi) \in G\right] \geq -\inf_G \varphi$$

for every open subset $G \subset \mathbb{R}^d$, where

$$\varphi(z) = \inf\left\{\frac{|x|^2}{2} : x = (x_1, \cdots, x_d) \text{ s.t. } x_1^4 + \cdots + x_d^4 = z\right\}$$

with the convention that $\inf \emptyset = \infty$. Here we have used Varadhan's contraction principle.

Varadhan's Contraction Principle. If (X_n) a sequence of random variables valued in a Polish space E which satisfies large deviation principle with rate function I, and suppose $f : E \to E'$ is a continuous mapping from E to another Polish space E', then $(f(X_n))$ satisfies the large deviation principle with rate function

$$I'(x') = \{I(x) : x \in E \text{ such that } f(x) = x'\}.$$