Geometric Group Theory

Problem Sheet 2

The starred exercises are optional.

1. Let $\langle S|R\rangle$ be a finite presentation of a group G.

i. Explain how to enumerate all words on S representing the identity in G.

ii. Explain how to enumerate all finite presentations of G.

2. Let $\langle S|R\rangle$ be a finite presentation of a finite group G. Give an algorithm to solve the word problem for this presentation.

3. Show that if a finitely presented group $G = \langle S|R\rangle$ has a solvable word problem and the finitely presented group $H = \langle S'|R'\rangle$ is isomorphic to some subgroup of G then H also has a solvable word problem. Note that here we *do not assume* that we are given an injective homomorphism $f : H \to G$.

4. If H is a finitely generated subgroup of G then the *membership problem* for H asks whether there is an algorithm to decide if $g \in G$ lies in H. Show that the membership problem is solvable for cyclic subgroups of F_n (the free group of rank n). In other words there is an algorithm such that given $u, w \in F_n$ decides whether $u \in < w >$.

5. (*) Show that the following presentations are presentations of the trivial group:

i) $\langle a, b, c|aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2\rangle$

ii) $\langle a, b|a^n = b^{n+1}, aba = bab\rangle$

iii) $\langle a, b|ab^na^{-1} = b^{n+1}, ba^nb^{-1} = a^{n+1}\rangle$.

6. (*) An infinite finitely generated group is called just infinite if all its quotients are finite groups. Show that every infinite finitely generated group has a quotient that is just infinite.

7. i. Show that G is residually finite if and only if for every $g \in G$ there is some finite index subgroup H of G, such that $g \notin H$.

ii. Show that if G has a finite index subgroup which is residually finite then G itself is residually finite.

8. Let G be a residually finite group. Show that if G has finitely many conjugacy classes of elements of finite order then G has a torsion free finite index subgroup.

9. (*) Give an example of a residually finite group which is not Hopf.

10. If H is a subgroup of the free group F_n of index $|F_n : H| = r$ show that H is a free group of rank $r(n-1)+1$. (*hint:* look closely at the proof that H is free).

11. If $g \neq 1$ is an element of F_n show that the normalizer of $< g >$ in F_n is a cyclic group.(*hint:* if u is in the normalizer then $< u, g >$ is free.)

12. (*) We say that a subgroup H of G is *separable* if it is equal to the intersection of all finite index subgroups of G containing it.

Show that every cyclic subgroup of F_n (the free group of rank n) is separable.

Hint: It is enough to show that given u, v there is a homomorphism f to a finite group such that $f(u) \notin < v >$. Imitate now the proof in the notes that F_n is residually finite.

13. Determine the center of the group $\langle a, b | a^2 = b^3 \rangle$.

14. Show that a finite group H acting on a tree T either fixes a vertex of T or fixes a geometric edge of T (ie $H \cdot e \subset \{e, \bar{e}\}$ for some edge e). Deduce that any finite subgroup of an amalgam $A *_C B$ is contained in a conjugate of A or B.

15. (*) Show that if A, B are residually finite then $A * B$ is also residually finite.