Geometric Group Theory

Problem Sheet 3

The starred exercises are optional.

1. Assume that $G = A *_C B$. Show that if G, C are finitely generated then A, B are also finitely generated.

2. Assume that $G = A *_C B$ with A, B, C finitely presented. Show that if the word problem of A, B is decidable and if the membership problem for C in A, B is also decidable then the word problem of G is decidable.

3. (*) The fundamental group of a surface group of genus 2 has a presentation:

$$G = \langle a, b, c, d | [a, b] = [c, d] \rangle$$

where we denote by $[a, b]$ the commutator: $aba^{-1}b^{-1}$. Show that G is an amalgam of two free groups over $\mathbb{Z}$. Deduce that the word problem of G is decidable.

4. Show that if $G = A *_C B$ and $|A : C| \geq 3$, $|B : C| \geq 2$ then G has a free subgroup of rank 2.

5. i) Let G be a finitely generated group such that $G = A *_C B$ where $|A : C| = 2, |B : C| = 2$, and A, B are finite. Show that G has a finite index subgroup isomorphic to $\mathbb{Z}$.

ii) Show that if $G = A *_A$ with A finite then G has a finite index subgroup isomorphic to $\mathbb{Z}$.

6. (*) Show that the group

$$G = \langle x, y | xy^2x^{-1} = y^3 \rangle$$

is not Hopf. (*hint:* consider the homomorphism $x \to x$, $y \to y^2$ and find an element in the kernel).

7. Let G be a finitely presented group. Show that an HNN-extension $G \underset{A}{*}$ is finitely presented if and only if A is finitely generated.

8. Let G be a group acting on a tree T without inversions (i.e. there is no edge e in T such that an element $g \in G$ swaps its endpoints).

Show that if $g \in G$ fixes no vertex of T then there is a line (ie a bi-infinite path) $L \subset T$ such that g acts on L by translations. (*hint:* Consider the vertices for which $d(v, gv)$ is minimum). Show that if $h = aga^{-1}$ then h fixes no vertex of T and acts on $a(L)$ by translations.

Assume that b, c are elements of G such that each one fixes a vertex but there is no vertex of T which is fixed by both b, c. Show that bc does not fix any vertex of T.

9. (*) Show that the product of two free groups of rank 2, $F_2 \times F_2$, cannot be written as a non trivial amalgam over $\mathbb{Z}$. (*hint:* If g, h commute and do not fix a vertex then they translate along the same line. Use this to show that any action on a tree has 'big' edge stabilizers.)

10. (*) Show that the group $G = \mathbb{Z}^2 * \mathbb{Z}^2$ can be written as an HNN-extension over $\mathbb{Z}$. Show that G can be written non-trivially as the fundamental group of a graph of groups with 3 edges.

11. Let $H = \pi_1(G, Y, a_0)$. Show that if Y is not a tree then H admits an epimorphism onto $\mathbb{Z}$.

12. Let $H = \pi_1(G, Y, a_0)$. If H is finitely generated show that there is a finite subgraph Y' of Y such that $H = \pi_1(G, Y', a_0)$.

13. Show that if H is the fundamental group of a finite graph of groups (G, Y) then either H splits over some edge group of Y or $H = G_v$ for some vertex group of Y.

14. Let $H = \pi_1(G, Y, a_0)$. If $v \in V$ consider the normal subgroup of G_v, $N = \langle\langle \alpha_e(G_e) : e \in E(Y) \text{ with } t(e) = v \rangle\rangle$. Show that there is an epimorphism $f : H \to G_v/N$.

15. (*) Show that every finitely generated subgroup of $F_n *_{<c>} F_n$ (F_n free of rank n) is finitely presented. (*hint:* it is enough to show that the subgroup is the fundamental group of a *finite* graph of groups with cyclic edge groups).