B5.6: Nonlinear Dynamics, Bifurcations and Chaos

- Lecturer: Radek Erban
- Lectures: Tuesdays 11am, Thursdays 11am (except of W7) & Wednesday 11am in W6
- Prerequisites: This course builds on ten Prelims and Part A courses. Students taking this course should have mastered the material in Part A courses on Differential Equations and Complex Analysis, and Prelims courses covering Probability, Computational Mathematics, Introductory Calculus, Multivariable Calculus, Fourier Series and PDEs, Geometry, Dynamics and Constructive Mathematics.

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- Classes: The course is accompanied by four Problem Sheets (labelled 1, 2, 3, and 4), which will be discussed in your classes, and by Problem Sheet 0.

  Three classes, covering Problem Sheets 1, 2 and 3, are scheduled in Hilary Term. Your last class will be in Trinity Term and will cover Problem Sheet 4, your vacation work.
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- MSc students: the first 8 lectures are part of core course A2 Mathematical Methods II

Introduction: main questions of course B5.6

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

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$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

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We want to find $\mathbf{x}$ as a function of t and sketch the phase plane or phase space.
What is the behaviour of $\mathbf{x}(t)$ as $t \to \infty$?
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Linear example (Question 1(a) on Problem Sheet 0):

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \qquad \text{for} \qquad M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_0 = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

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Closed form formula for solutions [Prelims Probability and Calculus courses]:

$$\mathbf{x}_k = 3^k \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + (-2)^k \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + 2^k \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

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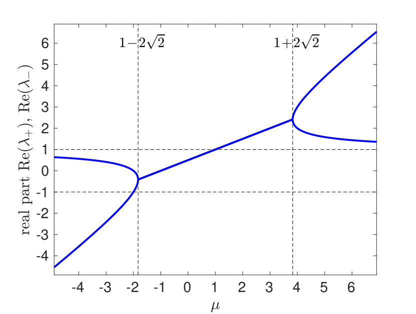

eigenvalues of M are

$$\lambda_\pm = \frac{1 + \mu \pm \sqrt{\left(\mu - 1 + 2\sqrt{2}\right)\left(\mu - 1 - 2\sqrt{2}\right)}}{2}$$

general solution $\lambda_+^k \mathbf{v}_+ + \lambda_-^k \mathbf{v}_-$

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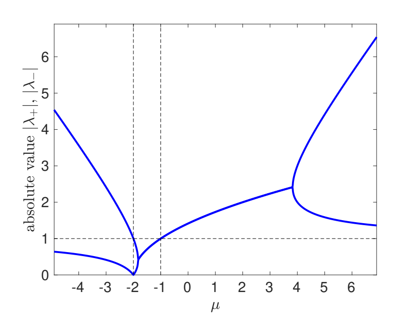

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$$\lim_{k\to\infty} \|\mathbf{x}_k\| = \infty \text{ for } \mu \in (-1, \infty)$$

$$\lim_{k\to\infty} \|\mathbf{x}_k\| = 0 \text{ for } \mu \in (-2, -1)$$

Nonlinear example: logistic map

Discrete-time dynamical system: Let $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0, 1]$, $\Theta = [0, 4]$ and $F(x, \mu) = \mu\, x\, (1 - x)$. Let $x_0 = 0.7 \in \Omega$, $\mu \in \Theta$ and $x_k \in [0, 1]$, $k = 0, 1, 2, \ldots,$ be defined iteratively by $x_{k+1} = F(x_k; \mu)$, *i.e.*

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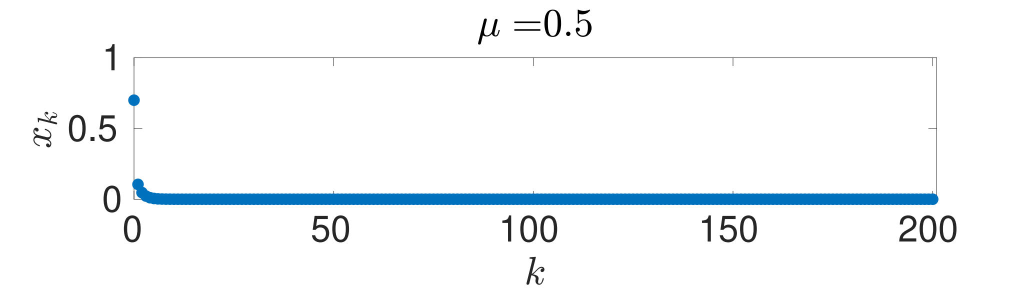

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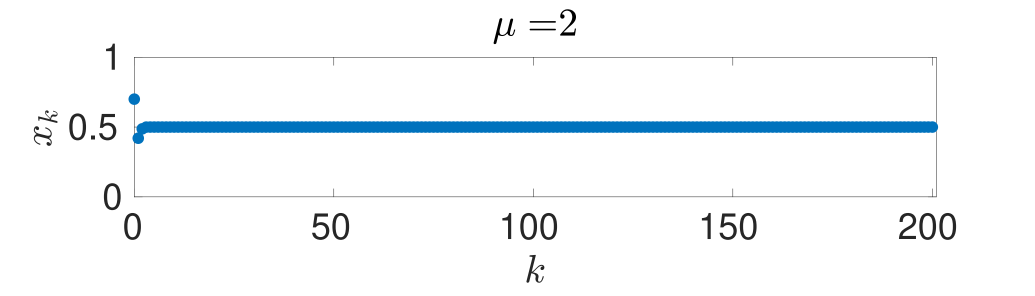

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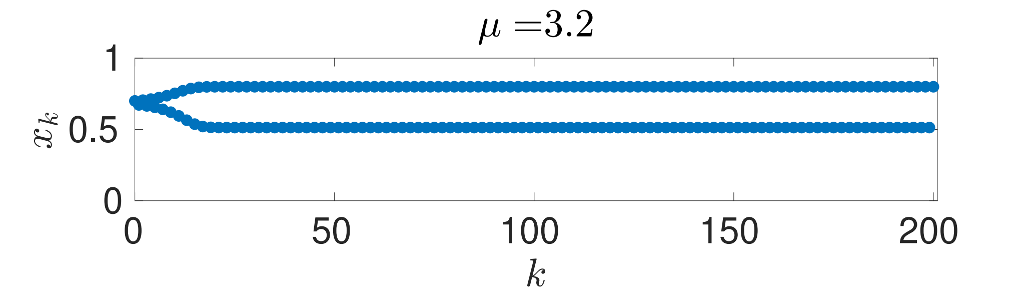

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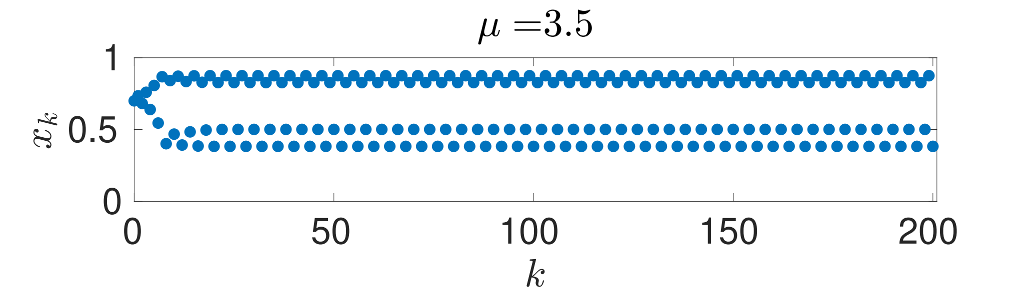

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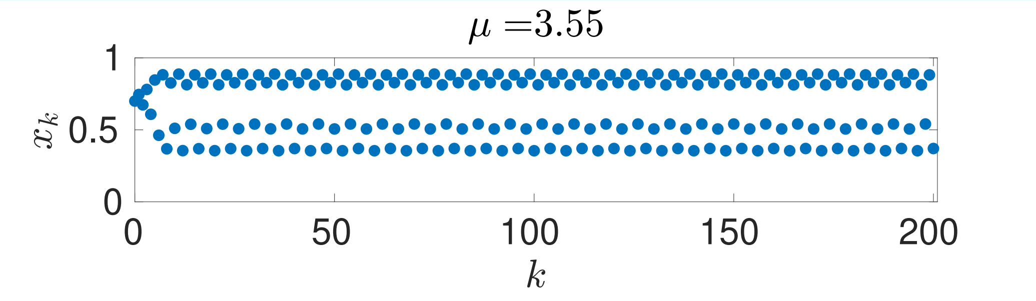

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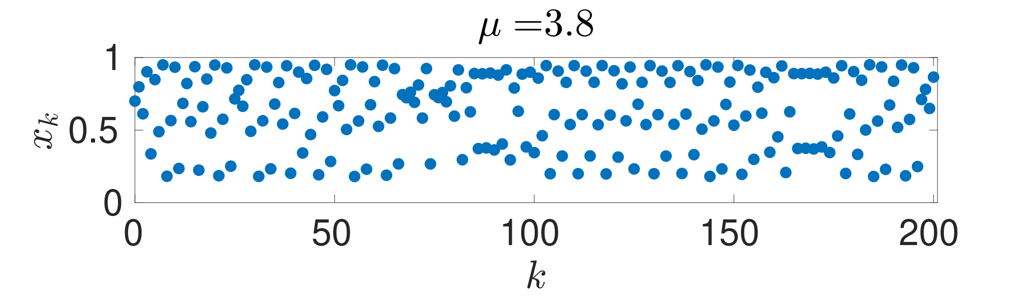

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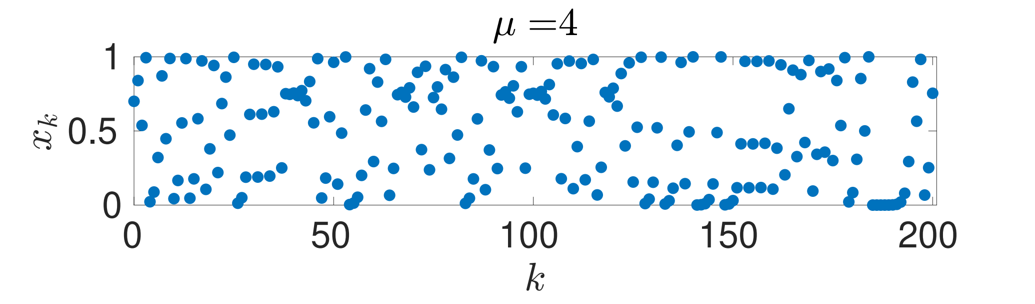

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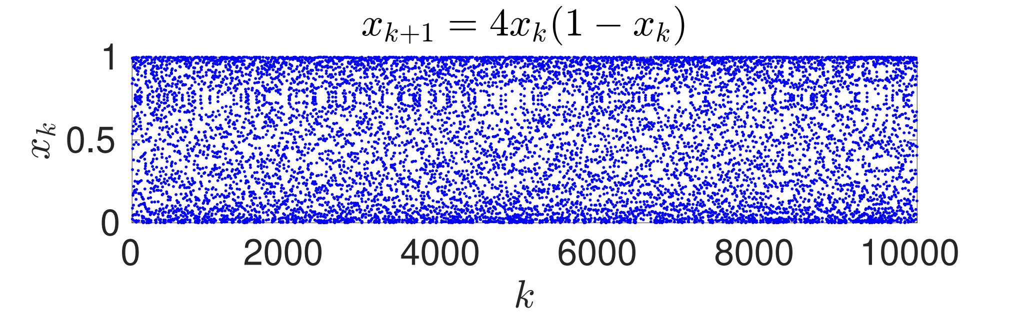

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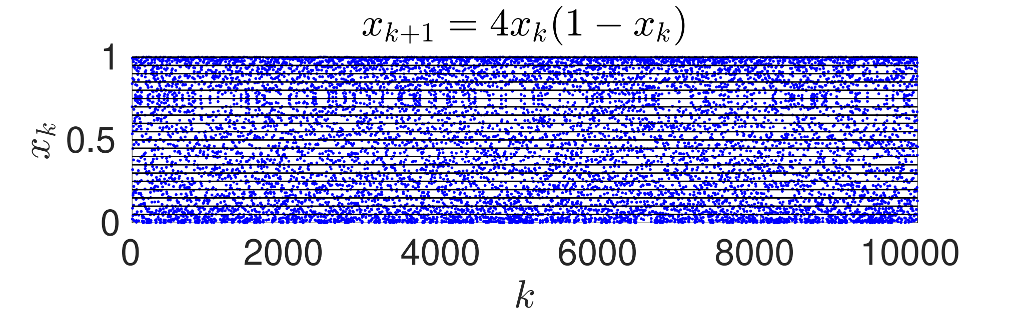

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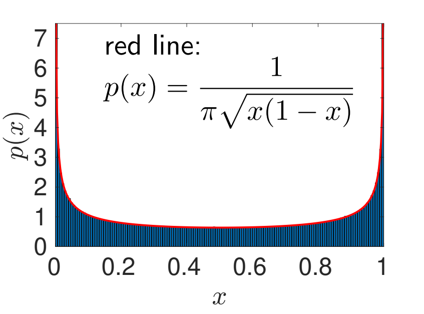
Histogram of values x_k, for $k = 0,1,2,\ldots,10^6$ (blue bars): $x_{k+1} = 4\,x_k\,(1-x_k)$



Problem Sheet 0 Question 4:

Let X_k be a continuous random variable on interval $[0,1]$ with the probability density function $p(x)$. Then the random variable $X_{k+1} = F(X_k) = 4\,X_k\,(1-X_k)$ has the same probability density function $p(x)$.

[Prelims Probability and Calculus]

Prelims Probability and Calculus: Problem Sheet 0 Question 4

Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p:[0,1]\to[0,\infty)$ given by $p(x)=1/(\pi\sqrt{x(1-x)})$. Let $F:[0,1]\to[0,1]$ be defined by $F(x)=4\,x\,(1-x)$. Then the cummulative distribution function of $F(X)$ is

$$\begin{aligned}\mathbb{P}(F(X)<x) &= \mathbb{P}\left(X<\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+\mathbb{P}\left(X>\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\\ &= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathsf{d}z+\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^{1} p(z)\,\mathsf{d}z\end{aligned}$$

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$$\begin{aligned}
\mathbb{P}(F(X) < x) &= \mathbb{P}\left(X < \frac{1}{2}\left(1-\sqrt{1-x}\right)\right) + \mathbb{P}\left(X > \frac{1}{2}\left(1+\sqrt{1-x}\right)\right) \\
&= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathrm{d}z + \int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^1 p(z)\,\mathrm{d}z \\
&= 1 + \frac{2}{\pi}\left(\sin^{-1}\sqrt{\frac{1}{2}\left(1-\sqrt{1-x}\right)} - \sin^{-1}\sqrt{\frac{1}{2}\left(1+\sqrt{1-x}\right)}\right) \\
&= 1 - \frac{2}{\pi}\sin^{-1}\left(\sqrt{1-x}\right) \qquad \text{for} \quad x \in [0,1]
\end{aligned}$$

Prelims Probability and Calculus: Problem Sheet 0 Question 4

Let X be a continuous random variable on interval $[0,1]$ with the probability density function $p:[0,1]\to[0,\infty)$ given by $p(x)=1/(\pi\sqrt{x(1-x)})$. Let $F:[0,1]\to[0,1]$ be defined by $F(x)=4\,x\,(1-x)$. Then the cummulative distribution function of $F(X)$ is

$$
\begin{aligned}
\mathbb{P}(F(X)<x) &= \mathbb{P}\left(X<\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+\mathbb{P}\left(X>\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\\
&= \int_0^{\frac{1}{2}\left(1-\sqrt{1-x}\right)} p(z)\,\mathrm{d}z+\int_{\frac{1}{2}\left(1+\sqrt{1-x}\right)}^{1} p(z)\,\mathrm{d}z\\
&= 1+\frac{2}{\pi}\left(\sin^{-1}\sqrt{\frac{1}{2}\left(1-\sqrt{1-x}\right)}-\sin^{-1}\sqrt{\frac{1}{2}\left(1+\sqrt{1-x}\right)}\right)\\
&= 1-\frac{2}{\pi}\sin^{-1}\left(\sqrt{1-x}\right)\qquad \text{for}\quad x\in[0,1]
\end{aligned}
$$

Consequently, the probability density function of $F(X)$ is:

$$
\frac{\mathrm{d}}{\mathrm{d}x}\,\mathbb{P}(F(X)<x)=-\frac{2}{\pi}\,\frac{\mathrm{d}}{\mathrm{d}x}\,\sin^{-1}\left(\sqrt{1-x}\right)=\frac{1}{\pi\sqrt{x(1-x)}}=p(x)
$$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x};\boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

We want to find $\mathbf{x}$ as a function of t and sketch the phase plane or phase space.
What is the behaviour of $\mathbf{x}(t)$ as $t \to \infty$?
How do our answers depend on the initial value $\mathbf{x}_0$?
How does the behaviour of $\mathbf{x}(t)$ depend on parameters $\boldsymbol{\mu}$?

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Linear example (Question 1(b) on Problem Sheet 0):

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x} \qquad \text{for} \qquad M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}(0) = \begin{pmatrix} 8 \\ 1 \\ 3 \end{pmatrix}$$

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Closed form solution formula [Prelims Calculus and Part A Differential Equations courses]:

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} + e^{2t} \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

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Linear example (Question 2(b) on Problem Sheet 0): $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ for $M = \begin{pmatrix} 1 & 2 \\ -1 & \mu \end{pmatrix}$

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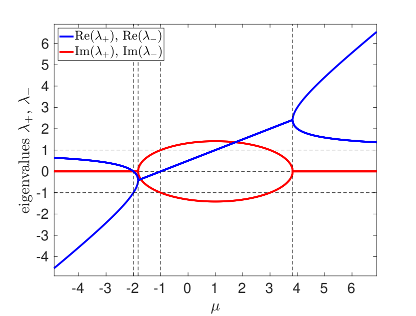

eigenvalues of M are

$$\lambda_\pm = \frac{1 + \mu \pm \sqrt{\left(\mu - 1 + 2\sqrt{2}\right)\left(\mu - 1 - 2\sqrt{2}\right)}}{2}$$

$[0, 0]$ is the only critical point

[Part A Differential Equations 1]

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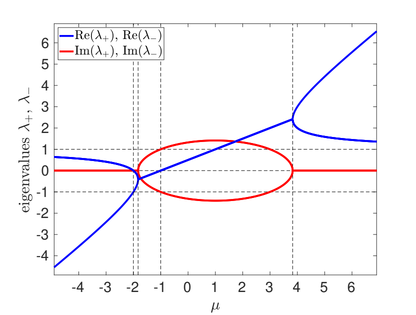

[Part A Differential Equations 1]

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$[0, 0]$ is the only critical point which is

- saddle for $\mu < -2$
- stable node for $-2 < \mu < 1 - 2\sqrt{2}$
- stable spiral for $1 - 2\sqrt{2} < -1$
- unstable spiral for $-1 < \mu < 1 + 2\sqrt{2}$
- unstable node for $\mu > 1 + 2\sqrt{2}$

B5.6 covers nonlinear dynamics (linear systems were in Prelims/Part A)

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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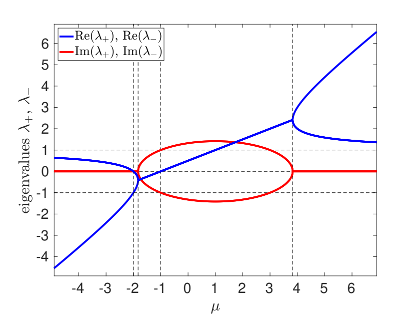

[Part A Differential Equations 1]

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- unstable node for $\mu > 1 + 2\sqrt{2}$

center for $\mu = -1$, stable/unstable inflected node for $\mu = 1 \pm 2\sqrt{2}$

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \mu\, y + y^2(1-x) - x^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mu\, x - x\, y\,(1+x) + y - y^3$$

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Let $\mu \in (-1,1)$ be a parameter. Consider a planar autonomous ODE system given by:

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Part A Differential Equations 1: linearized system next to the critical point $[x_c, y_c]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = M \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 1 - y_c^2 - 3x_c^2 & -\mu + 2\, y_c\,(1-x_c) \\ \mu - y_c - 2\, x_c\, y_c & -x_c\,(1+x_c) + 1 - 3\, y_c^2 \end{pmatrix}$$

$[0,0]$: unstable spiral $\quad M = \begin{pmatrix} 1 & -\mu \\ \mu & 1 \end{pmatrix} \quad$ eigenvalues: $1 \pm \mu i$

$\left[\sqrt{1-\mu^2}, \mu\right]$: stable node $\quad M = \begin{pmatrix} -2+2\mu^2 & \mu - 2\mu\sqrt{1-\mu^2} \\ -2\mu\sqrt{1-\mu^2} & -2\mu^2 - \sqrt{1-\mu^2} \end{pmatrix}$ eigenvalues: $-2, -\sqrt{1-\mu^2}$

$\left[-\sqrt{1-\mu^2}, \mu\right]$: saddle $\quad M = \begin{pmatrix} -2+2\mu^2 & \mu + 2\mu\sqrt{1-\mu^2} \\ 2\mu\sqrt{1-\mu^2} & -2\mu^2 + \sqrt{1-\mu^2} \end{pmatrix}$ eigenvalues: $-2,\ \sqrt{1-\mu^2}$

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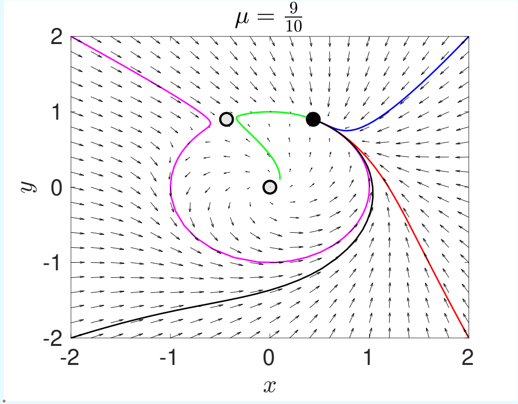

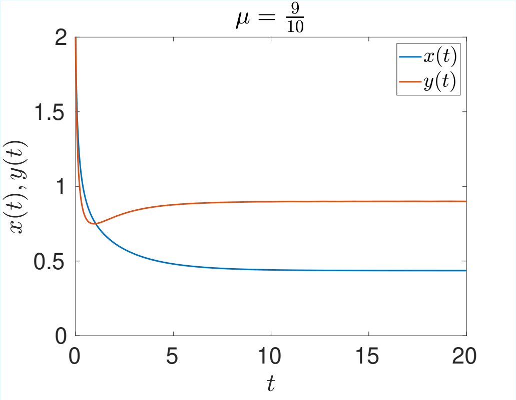

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu > 1$ be a parameter. Consider a planar autonomous ODE system given by:

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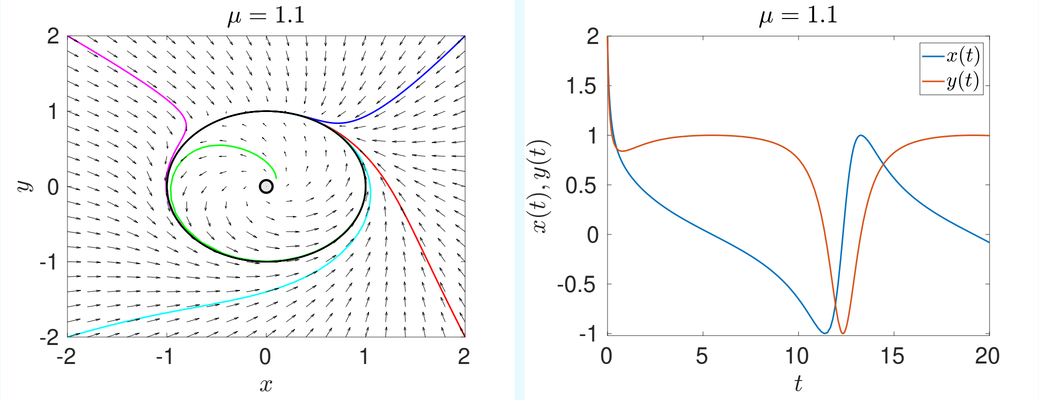

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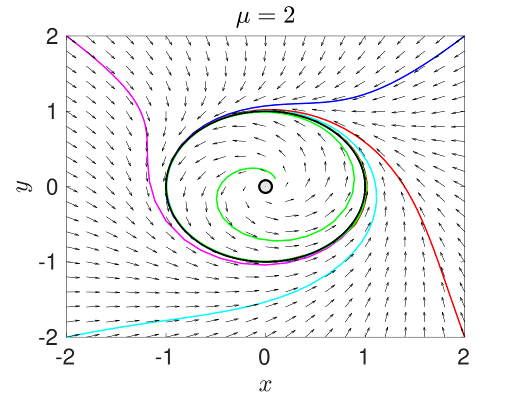

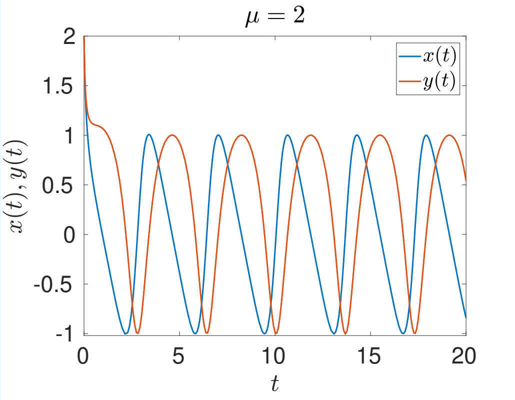

Nonlinear example: Problem Sheet 0 Question 5

Let $\mu \in \mathbb{R}$ be a parameter. Consider a planar autonomous ODE system given by:

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Prelims Calculus: We transform the ODEs to polar coordinates by using variables $r(t)$ and $\theta(t)$, where $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. We obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(1-r^2)$$

We conclude that $r(t) \to 1$ as $t \to \infty$ for any initial condition satisfying $r(0) > 0$.

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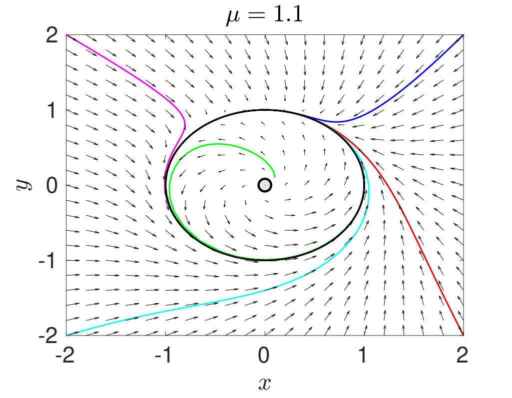
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mu - y = \mu - r\sin(\theta)$$

If $\mu > 1$, then $\mathrm{d}\theta/\mathrm{d}t > \mu - 1 > 0$.



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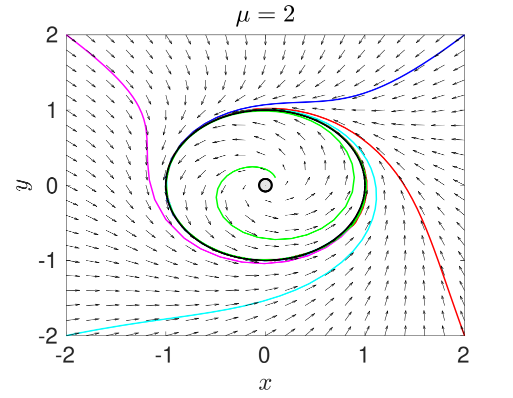

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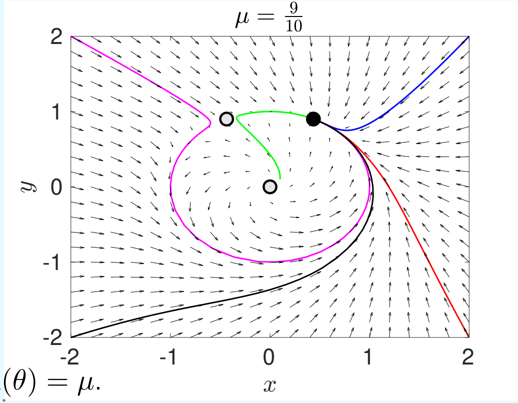

If $|\mu| < 1$, then $\mathrm{d}\theta/\mathrm{d}t = 0$ for $r = 1$ and $\sin(\theta) = \mu$.

ODEs and Chaos

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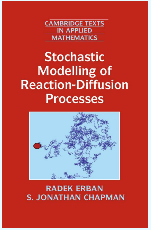
- In course B5.6, we will focus on chaotic solutions of ODEs for $n = 3$, but chaos is common for $n \gg 3$. Examples are discussed in course B5.1 Stochastic Modelling of Biological Processes. [video of molecular dynamics simulation of ions in water]



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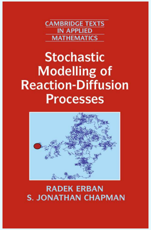

- B5.6: we will consider relatively simple ODEs (small n, polynomials): (1) good for developing general theory; (2) there are also interesting applications

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, $\ldots$, X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

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The time evolution of concentration $x_1(t)$ is given by the ODE $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \sum_{j=1}^{\ell} c_j \, r_j,$

where r_j is the rate of the j th reaction and c_j is the change in the number of molecules of X_1 corresponding to the occurrence of one j-th reaction, i.e. it is the difference between the number (stoichiometric coefficient) in front of X_1 on the right hand side of the reaction and the corresponding stoichiometric coefficient on the left hand side. The rate $r_j \equiv r_j(t)$ is computed as a product of the rate constant and the concentrations of the reactants (mass action kinetics).

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, $\ldots$, X_n which are subject to ℓ chemical reactions.

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Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

$X_1 + X_2 \xrightarrow{k_1} X_3$	$X_3 \xrightarrow{k_2} 2X_1$	$2X_3 \xrightarrow{k_3} \emptyset$	$\emptyset \xrightarrow{k_4} X_1$	$X_1 + 2X_3 \xrightarrow{k_5} X_2$
$c_1 = -1$	$c_2 = 2$	$c_3 = 0$	$c_4 = 1$	$c_5 = -1$
$r_1 = k_1 \, x_1 \, x_2$	$r_2 = k_2 \, x_3$	$r_3 = k_3 \, x_3^2$	$r_4 = k_4$	$r_5 = k_5 \, x_1 \, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 \; + \; 2 \, k_2 \, x_3 \; + \; k_4 \; - \; k_5 \, x_1 \, x_3^2$$

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, ..., X_n which are subject to ℓ chemical reactions.

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$c_1 = -1$	$c_2 = 2$	$c_3 = 0$	$c_4 = 1$	$c_5 = -1$
$r_1 = k_1 \, x_1 \, x_2$	$r_2 = k_2 \, x_3$	$r_3 = k_3 \, x_3^2$	$r_4 = k_4$	$r_5 = k_5 \, x_1 \, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1 \, x_1 \, x_2 + 2 \, k_2 \, x_3 + k_4 - k_5 \, x_1 \, x_3^2$$

The units of $x_i(t)$ are usually moles (or number of molecules) per unit of volume, k_1 and k_3 have units of $[\mathrm{m}^3 \, \mathrm{sec}^{-1}]$, k_2 is in $[\mathrm{sec}^{-1}]$, k_4 is in $[\mathrm{m}^{-3} \, \mathrm{sec}^{-1}]$ and k_5 is in $[\mathrm{m}^6 \, \mathrm{sec}^{-1}]$, but we will assume that $x_i(t)$ and all parameters are dimensionless.

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, …, X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \dots, n$.

The time evolution of concentration $x_2(t)$ is given by the ODE $\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = \sum_{j=1}^{\ell} c_j\, r_j,$

Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

$X_1 + X_2 \xrightarrow{k_1} X_3$	$X_3 \xrightarrow{k_2} 2X_1$	$2X_3 \xrightarrow{k_3} \emptyset$	$\emptyset \xrightarrow{k_4} X_1$	$X_1 + 2X_3 \xrightarrow{k_5} X_2$
$c_1 = -1$	$c_2 = 0$	$c_3 = 0$	$c_4 = 0$	$c_5 = 1$
$r_1 = k_1\, x_1\, x_2$	$r_2 = k_2\, x_3$	$r_3 = k_3\, x_3^2$	$r_4 = k_4$	$r_5 = k_5\, x_1\, x_3^2$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1\, x_1\, x_2 + 2\, k_2\, x_3 + k_4 - k_5\, x_1\, x_3^2$$

similary for $x_2(t)$: $\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -k_1\, x_1\, x_2 + k_5\, x_1\, x_3^2$

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species X_1, X_2, ..., X_n which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \dots, n$.

The time evolution of concentration $x_3(t)$ is given by the ODE $$\frac{\mathrm{d}x_3}{\mathrm{d}t} = \sum_{j=1}^{\ell} c_j\, r_j,$$

Example: system of $n = 3$ chemical species which are subject to $\ell = 5$ reactions:

$$X_1 + X_2 \xrightarrow{k_1} X_3 \qquad X_3 \xrightarrow{k_2} 2X_1 \qquad 2X_3 \xrightarrow{k_3} \emptyset \qquad \emptyset \xrightarrow{k_4} X_1 \qquad X_1 + 2X_3 \xrightarrow{k_5} X_2$$

$$c_1 = 1 \qquad c_2 = -1 \qquad c_3 = -2 \qquad c_4 = 0 \qquad c_5 = -2$$

$$r_1 = k_1\, x_1\, x_2 \qquad r_2 = k_2\, x_3 \qquad r_3 = k_3\, x_3^2 \qquad r_4 = k_4 \qquad r_5 = k_5\, x_1\, x_3^2$$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -k_1\, x_1\, x_2 + 2\, k_2\, x_3 + k_4 - k_5\, x_1\, x_3^2$$

similary for $x_2(t)$: $$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -k_1\, x_1\, x_2 + k_5\, x_1\, x_3^2$$

similary for $x_3(t)$: $$\frac{\mathrm{d}x_3}{\mathrm{d}t} = k_1\, x_1\, x_2 - k_2\, x_3 - 2\, k_3\, x_3^2 - 2\, k_5\, x_1\, x_3^2$$

Chemical reaction networks

Consider a well-stirred (well-mixed) chemical system with n chemical species $X_1, X_2, \ldots, X_n$ which are subject to ℓ chemical reactions.

Let $x_i(t)$ be the concentration of the chemical species X_i, $i = 1, 2, \ldots, n$.

The time evolution of concentration $x_1(t)$ is given by the ODE $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \displaystyle\sum_{j=1}^{\ell} c_j \, r_j,$

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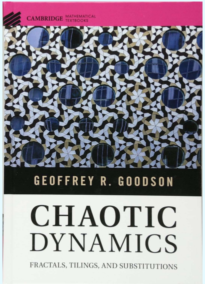
other examples: Questions 3, 4 and 6 on Problem Sheet 1

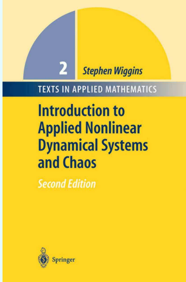
Course B5.6: ODEs with relatively small n and simple right hand sides (often polynomials). They appear in applications as (i) models of (bio)chemical systems; or (ii) they can also be constructed in experiments (synthetic biology, DNA computing).

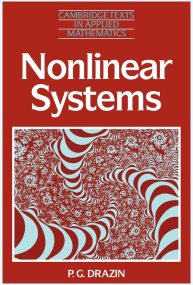
Polynomials can also approximate more complicated right hand sides of ODEs (stable manifold, center manifold, bifurcations). Let us go back to some theory.

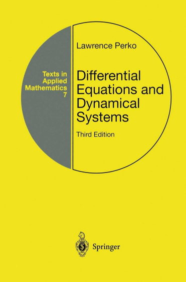
Theory and Reading List

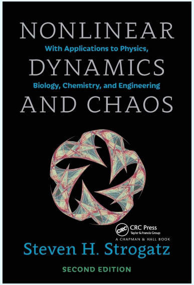
The B5.6 course material could be introduced with different levels of mathematical rigour, ranging from the ‘definition-theorem-proof approach’ to an example-based course covering dynamical systems appearing in applications. There are 6 books in the Reading List:

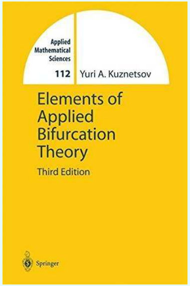












Theory and Reading List

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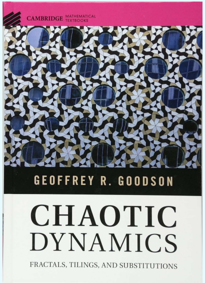

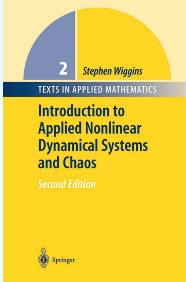

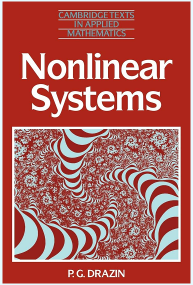

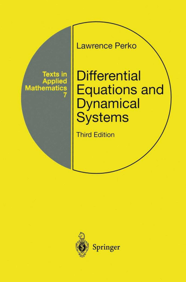

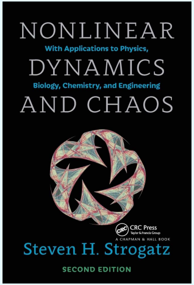

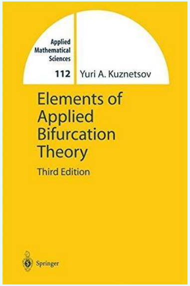

Our lectures will provide enough background theory for understanding the questions on your problem sheets and exams, but you could also use one of these books for supplementary reading about the topics covered by this course. Students interested in building further theory with more proofs could like [Wiggins] or [Perko], or [Kuznetsov] (for bifurcations), or [Goodson] (for maps), while [Drazin] or [Strogatz] could be more appreciated by students interested in applications.

The flow defined by an ODE

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0$$

Then we define the *flow* $\phi_t : \Omega \to \Omega$ by

$$\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0) = \mathbf{x}(t)$$

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$$\phi_t(\mathbf{x}_0)=\phi(t,\mathbf{x}_0)=\mathbf{x}(t)$$

Example: Question 1(b) on Problem Sheet 0 for general initial condition $\mathbf{x}_0\in\mathbb{R}^3$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}=M\mathbf{x} \qquad \text{for} \qquad M=\begin{pmatrix}2&1&-1\\1&-1&2\\-1&1&2\end{pmatrix}$$

Then

$$\phi_t(\mathbf{x}_0)=\phi(t,\mathbf{x}_0)=\exp[Mt]\,\mathbf{x}_0=\left(\sum_{j=0}^{\infty}\frac{M^j t^j}{j!}\right)\mathbf{x}_0$$

where we have used the definition of the matrix exponential: $\exp[A]=\sum_{j=0}^{\infty}\frac{A^j}{j!}$

The flow defined by an ODE

Considering the linear system of ODEs given by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where matrix $M \in \mathbb{R}^{n\times n}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi_t = \exp[Mt]$.

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In particular, the properties of the matrix exponential imply that the flow ϕ_t satisfies

(a) $\phi_0 = I$ (b) $\phi_s \circ \phi_t = \phi_{s+t}$ (c) $\phi_t \circ \phi_{-t} = \phi_{-t} \circ \phi_t = I$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

The flow defined by an ODE

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where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

For linear systems, the properties (a)–(c) mean:

(a) $\phi_0(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$;
(b) $\phi_s(\phi_t(\mathbf{x})) = \phi_{s+t}(\mathbf{x}))$ for all $s, t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$;
(c) $\phi_t(\phi_{-t}(\mathbf{x})) = \phi_{-t}(\phi_t(\mathbf{x})) = \mathbf{x}$ for all $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

The flow defined by an ODE

Considering the linear system of ODEs given by $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where matrix $M \in \mathbb{R}^{n\times n}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi_t = \exp[Mt]$.

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Nonlinear ODE system: $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$

Part A Differential Equations 1: Picard's existence theorem implies the global existence and uniqueness of solutions for $\mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}^m)$ which satisfies the global Lipschitz condition $|\mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) - \mathbf{f}(\mathbf{y}; \boldsymbol{\mu})| \leq C|\mathbf{x} - \mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^m$.
Then $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined for all $t \in \mathbb{R}$ and ϕ_t satisfies the properties (a)–(c).

The flow defined by an ODE

Considering the linear system of ODEs given by $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where matrix $M \in \mathbb{R}^{n\times n}$, the flow $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ is given by $\phi_t = \exp[Mt]$.

In particular, the properties of the matrix exponential imply that the flow ϕ_t satisfies

(a) $\phi_0 = I$ (b) $\phi_s \circ \phi_t = \phi_{s+t}$ (c) $\phi_t \circ \phi_{-t} = \phi_{-t} \circ \phi_t = I$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

Note: Assuming the global Lipschitz condition could exclude some interesting ODEs. Our assumptions on $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$ and $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$ could be relaxed. In some cases, we would only get the local existence of solutions to the nonlinear ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

ϕ_t would not be defined for all $t \in \mathbb{R}$ and ϕ_t would only satisfy properties (a)–(c) where it is defined.

Let us illustrate this with an example with $n = 1$.

The flow defined by an ODE: nonlinear example

Consider the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2$ (it does not satisfy the global Lipschitz condition).

The flow defined by an ODE: nonlinear example

Consider the ODE $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x^2$ (it does not satisfy the global Lipschitz condition).

Given the initial condition $x(0) = x_0 \in \mathbb{R}$, we can solve this ODE to obtain

$$x(t) = \frac{x_0}{1 - t\,x_0} \qquad \text{for} \quad t \in I(x_0),$$

where $I(x_0)$ is the *maximal interval of existence* given by $I(0) = \mathbb{R}$,

$$I(x) = \left(-\infty, \frac{1}{x}\right) \quad \text{for } \; x > 0, \qquad \text{and} \qquad I(x) = \left(\frac{1}{x}, \infty\right) \quad \text{for } \; x < 0.$$

The flow defined by an ODE: nonlinear example

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In particular, the flow ϕ_t is defined as the mapping $\phi : Q \to \mathbb{R}$, where

$$Q = \{(t, x) \,|\, x \in \mathbb{R} \text{ and } t \in I(x)\} \qquad \text{and} \qquad \phi_t(x) = \phi(t, x) = \frac{x}{1 - tx}.$$

Problem Sheet 1 Question 7: We can rescale time to get a topologically equivalent ODE system which has $I(x) = \mathbb{R}$.

In general, the time along trajectories can be rescaled without affecting the phase portrait. In what follows, we will assume that ϕ_t is defined for all $t \in \mathbb{R}$ and $\phi \in C^1(\mathbb{R} \times \Omega)$ for any considered parameter values $\boldsymbol{\mu} \in \Theta$.

Equilibrium points, flow, trajectory - summary

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- the *flow* of the ODE is the map $\phi_t : \Omega \to \Omega$ such that $\phi_t(\mathbf{x}_0) = \phi(t, \mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$, where $\mathbf{x}(t; \mathbf{x}_0) \in \Omega$ is the solution with the initial condition $\mathbf{x}(0) = \mathbf{x}_0 \in \Omega$
- an *orbit* or *trajectory* based on $\mathbf{x}_0$ is the curve $\Gamma_{\mathbf{x}_0} \subset \Omega$ defined by

$$\Gamma_{\mathbf{x}_0} = \left\{ \mathbf{x}(t; \mathbf{x}_0) \,\middle|\, t \in I(x_0) \right\} ,$$

  where $I(x_0)$ is the maximum interval of existence (WLOG we assume $I(x_0) = \mathbb{R}$)
- $S \subset \Omega$ is an *invariant set* if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$

Equilibrium points: stability

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- $\mathbf{x}_c$ is *stable* if

  $\forall \varepsilon > 0 \, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$ and $t \geq 0$ we have $\phi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}_c)$

  where the open ball of radius r is defined by $B_r(\mathbf{x}_c) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{x}_c\| < r \right\}$
- $\mathbf{x}_c$ is *asymptotically stable* if (i) it is stable; and

  (ii) $\exists \delta > 0$ such that $\phi_t(\mathbf{x}_0) \to \mathbf{x}_c$ for all $\mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$

Equilibrium points: stability, Lyapunov function

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- $\mathbf{x}_c$ is *stable* if

  $\forall \varepsilon > 0\, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$ and $t \geq 0$ we have $\phi_t(\mathbf{x}) \in B_\varepsilon(\mathbf{x}_c)$

  where the open ball of radius r is defined by $B_r(\mathbf{x}_c) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{x}_c\| < r \right\}$
- $\mathbf{x}_c$ is *asymptotically stable* if (i) it is stable; and

  (ii) $\exists \delta > 0$ such that $\phi_t(\mathbf{x}_0) \to \mathbf{x}_c$ for all $\mathbf{x}_0 \in B_\delta(\mathbf{x}_c)$
- Lyapunov function: $V \in C^1(A)$, where $A \subset \Omega \subset \mathbb{R}^n$ is open and $\mathbf{x}_c \in A$

  $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \mathbf{x}_c$ and $V(\mathbf{x}_c) = 0$

  if $\mathrm{d}V/\mathrm{d}t \leq 0$ for all $\mathbf{x} \in A \setminus \{\mathbf{x}_c\}$, then $\mathbf{x}_c$ is stable

  if $\mathrm{d}V/\mathrm{d}t < 0$ for all $\mathbf{x} \in A \setminus \{\mathbf{x}_c\}$, then $\mathbf{x}_c$ is asymptotically stable

Problem Sheet 1 Question 5: proving stability by finding a suitable Lyapunov function

Equilibrium points: linearization

Given $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

- $\mathbf{x}_c$ is an *equilibrium point* or *critical point* or *fixed point* if $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = 0$
- linearization at $\mathbf{x}_c$ is given by

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$$

where M is the Jacobian matrix

$$M = D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_c) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_c) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{x}_c) & \frac{\partial f_n}{\partial x_2}(\mathbf{x}_c) & \cdots & \frac{\partial f_n}{\partial x_n}(\mathbf{x}_c) \end{pmatrix}$$

- equilibrium point $\mathbf{x}_c$ is called
  hyperbolic: if none of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have zero real part
  sink: if all of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have negative real part
  source: if all of the eigenvalues of the matrix $D\mathbf{f}(\mathbf{x}_c)$ have positive real part
  saddle: if it is a hyperbolic equilibrium point and $D\mathbf{f}(\mathbf{x}_c)$ has at least one eigenvalue with a positive real part and at least one with a negative real part

Invariant manifolds

stable manifold theorem:

- the nonlinear system has locally similar behaviour close to a hyperbolic critical point
- it shows the existence of two invariant manifolds: stable manifold, unstable manifold

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We know what invariant means: set $S \subset \Omega$ is *invariant* if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$

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Invariant manifolds

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What is a manifold? Wiggins [page 29], Perko [page 107], Kuznetsov [page 598]

- linear settings: a linear vector subspace of $\mathbb{R}^n$
- nonlinear settings: a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph

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- linear settings: a linear vector subspace of $\mathbb{R}^n$
- nonlinear settings: a surface embedded in $\mathbb{R}^n$ which can be locally represented as a graph
- there is also the center manifold (invariant manifold that appears in the center manifold theorem), but we will start with the stable manifold theorem

Stable manifold theorem: linear systems

Consider the linear system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ where $M \in \mathbb{R}^{n\times n}$ and none of the eigenvalues of $M \in \mathbb{R}^{n\times n}$ have zero real part.

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Assume that M is diagonalizable (semi-simple) and denote its eigenvalues and eigenvectors by $\quad \lambda_j = a_j + \mathrm{i}\, b_j \quad$ and $\quad \mathbf{w}_j = \mathbf{u}_j + \mathrm{i}\, \mathbf{v}_j$,
where $a_j, b_j \in \mathbb{R}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^n$, for $j = 1, 2, \ldots, n$. Then we define

stable subspace: $E^s = \text{span}\{\mathbf{u}_j, \mathbf{v}_j \,|\, a_j < 0\}$
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Example (Question 1(b) on Problem Sheet 0): $\lambda_1 = -2$, $\lambda_2 = 2$ and $\lambda_3 = 3$

$$M = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \qquad \mathbf{w}_1 = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \qquad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{w}_3 = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

Then we have $E^s = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$, $\qquad E^u = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \right\}$

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Denote the eigenvalues and generalized eigenvectors of M by

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remarks: (1) if λ is an eigenvalue of matrix $M \in \mathbb{R}^{n\times n}$ of algebraic multiplicity $m \le n$, then for $k = 1, 2, \ldots, m$, any nonzero solution $\mathbf{v}$ of $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ is called a generalized eigenvector of M

(2) if some eigenvalues of $M \in \mathbb{R}^{n\times n}$ have zero real part, we also define *center subspace*: $E^c = \mathrm{span}\{\mathbf{u}_j, \mathbf{v}_j \,|\, a_j = 0\}$

examples: Question 1 on Problem Sheet 1

Stable manifold theorem: linear systems

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Moreover, the solution is given by $\mathbf{x}(t) = \sum_{j=1}^{n} c_j\, e^{\lambda_j t}\mathbf{w}_j$ which implies:

if $\mathbf{x}(t) = \mathbf{x}_0 \in E^s$, then $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ and $\lim_{t\to-\infty} \|\mathbf{x}(t)\| = \infty$

if $\mathbf{x}(t) = \mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

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if $\mathbf{x}(t) = \mathbf{x}_0 \in E^u$, then $\lim_{t\to\infty} \|\mathbf{x}(t)\| = \infty$ and $\lim_{t\to-\infty} \mathbf{x}(t) = \mathbf{0}$

Question 2 on Problem Sheet 1: this is also true for non-diagonalizable matrix M

(the nonlinear system has locally similar behaviour close to a hyperbolic critical point)

Stable manifold theorem

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

WLOG, assume that $\mathbf{0} \subset \Omega$ is the hyperbolic critical point, *i.e.* $\mathbf{f}(\mathbf{0}; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{0})$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. In particular, our discussion of linear systems is applicable to the linear system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$ with $M = D\mathbf{f}(\mathbf{0})$.

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Then there exists (local results):

- a k-dimensional differentiable manifold M^s_{loc} tangent to the stable subspace E^s of the linear system at $\mathbf{0}$ such that for all $t \geq 0$, we have $\phi_t(M^s_{\mathrm{loc}}) \subset M^s_{\mathrm{loc}}$ and for all $\mathbf{x}_0 \in M^s_{\mathrm{loc}}$, we have
$$\lim_{t\to\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$
- an $(n-k)$-dimensional differentiable manifold M^u_{loc} tangent to the unstable subspace E^u of the linear system at $\mathbf{0}$ such that for all $t \leq 0$, we have $\phi_t(M^u_{\mathrm{loc}}) \subset M^u_{\mathrm{loc}}$ and for all $\mathbf{x}_0 \in M^u_{\mathrm{loc}}$, we have
$$\lim_{t\to-\infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$$

Stable manifold theorem: example

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - x_2^2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_2 + x_1^2$$

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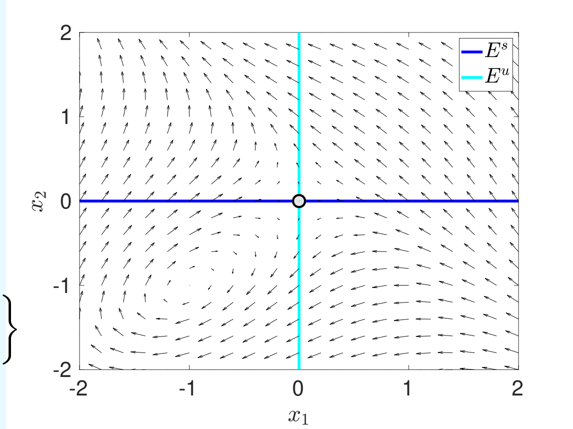
$\mathbf{0} = [0, 0]$ is a fixed point

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E^s = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E^u = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$



Stable manifold theorem: example

example: $\dfrac{dx_1}{dt} = -x_1 - x_2^2$

$$\frac{dx_2}{dt} = x_2 + x_1^2$$

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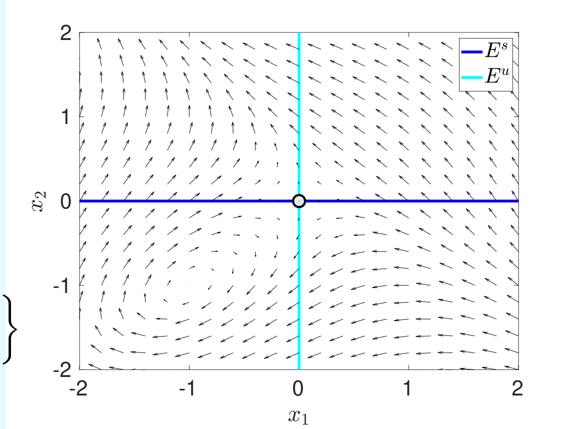
M^s_{loc} is of the form $x_2 = c_1 x_1^2 + \mathcal{O}(x_1^3)$

M^u_{loc} is of the form $x_1 = c_2 x_2^2 + \mathcal{O}(x_2^3)$



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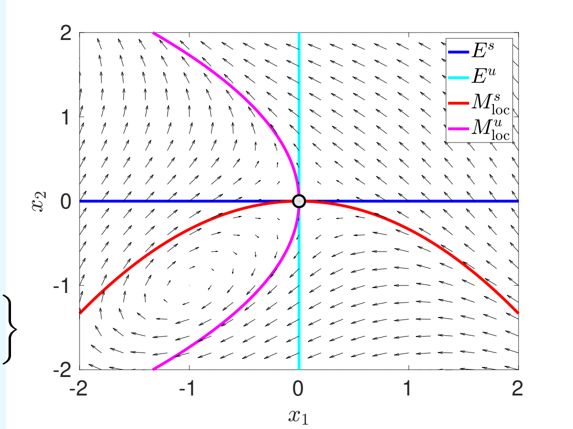

M^s_{loc} is of the form $x_2 = c_1 x_1^2 + \mathcal{O}(x_1^3)$
M^u_{loc} is of the form $x_1 = c_2 x_2^2 + \mathcal{O}(x_2^3)$

differentiating these approximations, we get $c_1 = c_2 = -\frac{1}{3}$, *i.e.*

M^s_{loc} is of the form $x_2 = -\frac{x_1^2}{3} + \mathcal{O}(x_1^3)$ and M^u_{loc} is of the form $x_1 = -\frac{x_2^2}{3} + \mathcal{O}(x_2^3)$

Global stable and unstable manifolds

global stable and unstable manifolds: $M^s = \bigcup_{t \leq 0} \phi_t(M^s_{\text{loc}})$ and $M^u = \bigcup_{t \geq 0} \phi_t(M^u_{\text{loc}})$

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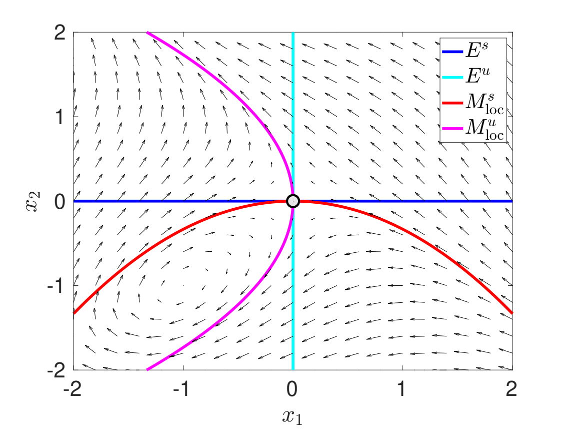

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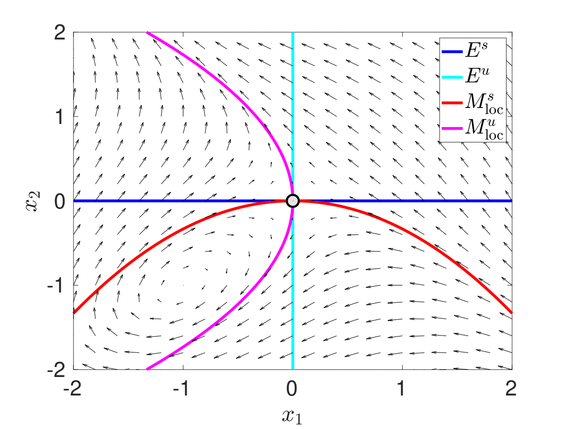
observe that $A = 3x_1x_2 + x_1^3 + x_2^3$ is time independent:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 3(x_2 + x_1^2)\frac{\mathrm{d}x_1}{\mathrm{d}t} + 3(x_1 + x_2^2)\frac{\mathrm{d}x_2}{\mathrm{d}t}$$
$$= 3(x_2 + x_1^2)(-x_1 - x_2^2)$$
$$+ 3(x_1 + x_2^2)(x_2 + x_1^2) = 0$$



consequently, both stable and unstable manifolds satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = 0$

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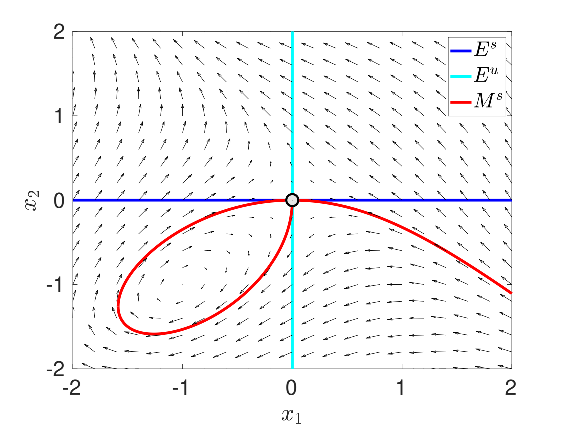

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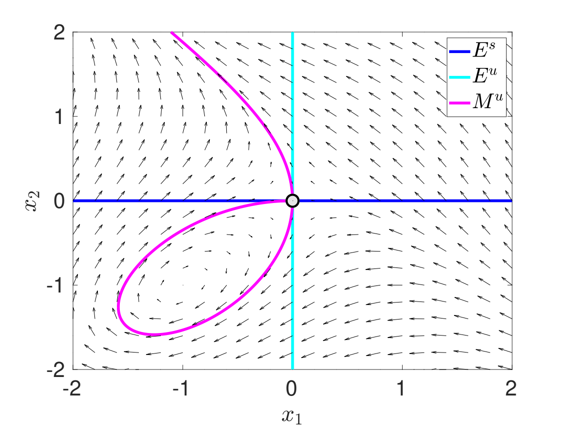

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$$= 3(x_2 + x_1^2)(-x_1 - x_2^2)$$
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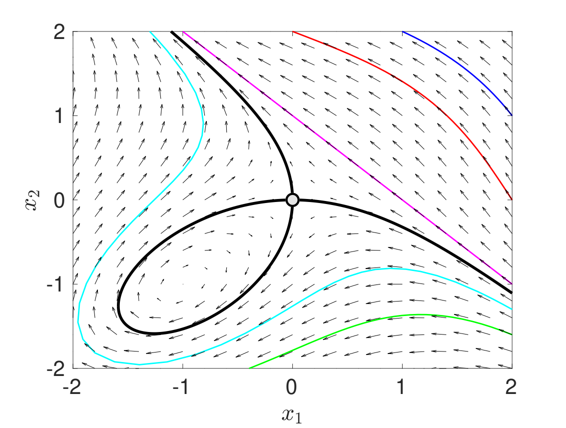


consequently, both stable and unstable manifolds satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = 0$

Periodic solutions (closed orbits)

[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

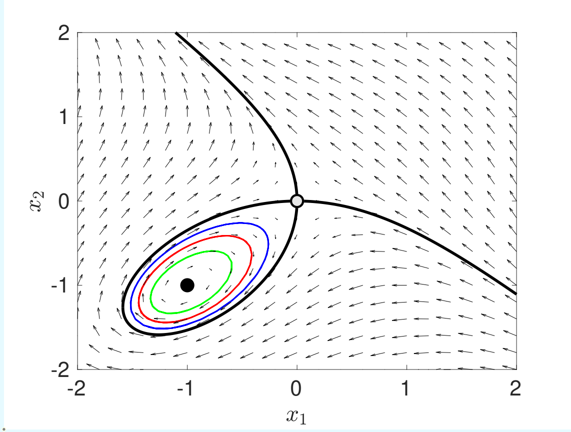
example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = -x_1 - x_2^2$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_2 + x_1^2$$

observe that $A = 3x_1x_2 + x_1^3 + x_2^3$

is time independent:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= 3(x_2 + x_1^2)\frac{\mathrm{d}x_1}{\mathrm{d}t} + 3(x_1 + x_2^2)\frac{\mathrm{d}x_2}{\mathrm{d}t} \\ &= 3(x_2 + x_1^2)(-x_1 - x_2^2) \\ &\quad + 3(x_1 + x_2^2)(x_2 + x_1^2) = 0\end{aligned}$$



periodic solutions around point $[-1, -1]$
satisfy $A = 3x_1x_2 + x_1^3 + x_2^3 = c$ for $c \in (0, 1)$

Periodic solutions (closed orbits)

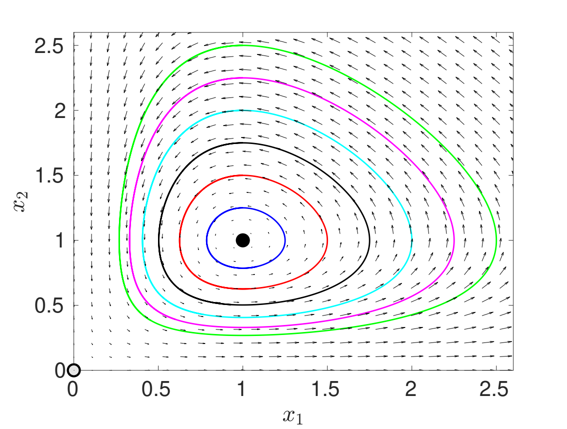
[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

example: $$\frac{dx_1}{dt} = x_1 - x_1 x_2$$
$$\frac{dx_2}{dt} = -x_2 + x_1 x_2$$

$$A = \log(x_1) - x_1 + \log(x_2) - x_2$$

is time independent

Lotka-Volterra predator-prey equations



Periodic solutions (closed orbits)

[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 - x_1 x_2$

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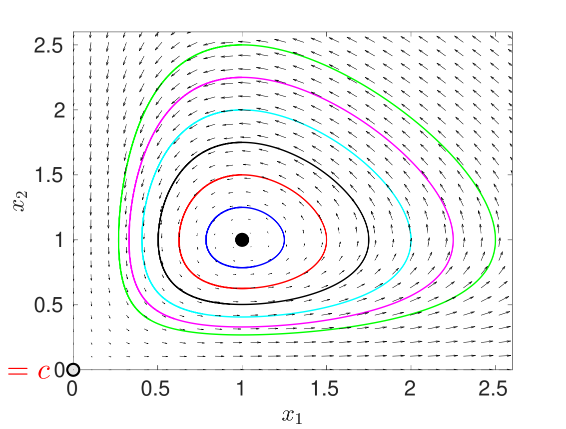
$A = \log(x_1) - x_1 + \log(x_2) - x_2$

is time independent

Lotka-Volterra predator-prey equations

periodic solutions around point $[1, 1]$

satisfy $A = \log(x_1) - x_1 + \log(x_2) - x_2 = c$



[Part A Differential Equations 1]:

see pages 39-41 of your lecture notes from last year

Periodic solutions (closed orbits)

[Part A Differential Equations 1]: The existence of suitable A is one possible approach to prove the existence of periodic solutions (closed orbits) in planar systems.

example:
$$\frac{dx_1}{dt} = x_1 - x_1 x_2$$
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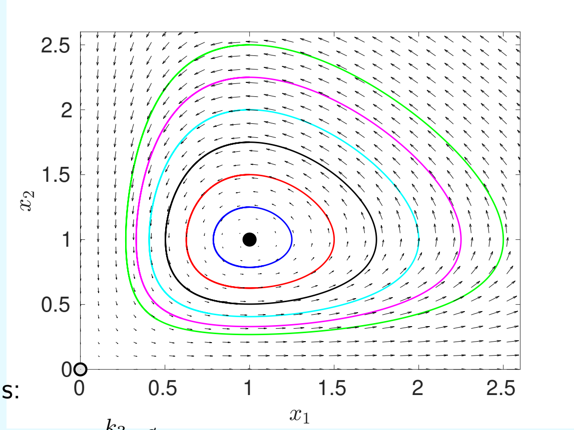

Note: Lotka-Volterra ODE system also describes a system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 3$ chemical reactions:

$$X_1 + X_2 \xrightarrow{k_1} 2X_2 \qquad X_1 \xrightarrow{k_2} 2X_1 \qquad X_2 \xrightarrow{k_3} \emptyset$$

where the values of the rate constants are: $k_1 = k_2 = k_3 = 1$

Poincaré-Bendixson theorem $(n = 2)$

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider the planar ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Suppose that $R \subset \Omega$ is compact (*i.e.* closed and bounded) and

- R does not contain any fixed points
- there exists $\mathbf{x}_0 \in R$ such that $\phi_t(\mathbf{x}_0) \in R$ for all $t \geq 0$, *i.e.* the trajectory is confined in R for $t \geq 0$

Poincaré-Bendixson theorem: Then either $\Gamma_{\mathbf{x}_0}$ is a closed orbit, or $\phi_t(\mathbf{x}_0)$ spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

Poincaré-Bendixson theorem ($n = 2$)

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application of the Poincaré-Bendixson theorem

- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary

Poincaré-Bendixson theorem ($n = 2$)

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider the planar ODE system

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application of the Poincaré-Bendixson theorem

- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary
- we need to show that any fixed point in the trapping region is unstable, and remove its small neighbourhood to construct R

Poincaré-Bendixson theorem ($n = 2$)

Given C^1 vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^2$, where $\Omega \subset \mathbb{R}^2$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider the planar ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Suppose that $R \subset \Omega$ is compact (*i.e.* closed and bounded) and

- R does not contain any fixed points
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Poincaré-Bendixson theorem: Then either $\Gamma_{\mathbf{x}_0}$ is a closed orbit, or $\phi_t(\mathbf{x}_0)$ spirals toward a closed orbit as $t \to \infty$. In either case, R contains a closed orbit.

application of the Poincaré-Bendixson theorem [Question 6 on Problem Sheet 1]

- we need to find a *trapping region*: compact connected subset of Ω such that the vector field $\mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$ points 'inward' everywhere on the boundary [Question 6(c)]
- we need to show that any fixed point in the trapping region is unstable, and remove its small neighbourhood to construct R [Question 6(d)]

Center manifold

Given C^r vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu})$$

Assume that $\mathbf{x}_c \subset \Omega$ is the critical point, *i.e.* $\mathbf{f}(\mathbf{x}_c; \boldsymbol{\mu}) = \mathbf{0}$ and matrix $D\mathbf{f}(\mathbf{x}_c)$ has $k > 0$ eigenvalues with zero real part and $n - k$ eigenvalues with non-zero real part.

Then there exists a k-dimensional C^r-manifold M^c_{loc} tangent to center subspace E^s of the linear system at $\mathbf{x}_c$ such that for all $t \geq 0$, we have $\phi_t(M^c_{\mathrm{loc}}) \subset M^c_{\mathrm{loc}}$.

Center manifold

Given C^r vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

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- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.

Center manifold

Given C^r vectory field $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$, $\Theta \subset \mathbb{R}^m$, and $\boldsymbol{\mu} \in \Theta$, we consider ODE system

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Then there exists a k-dimensional C^r-manifold M^c_{loc} tangent to center subspace E^s of the linear system at $\mathbf{x}_c$ such that for all $t \geq 0$, we have $\phi_t(M^c_{\mathrm{loc}}) \subset M^c_{\mathrm{loc}}$.

- If the unstable manifold is non-empty, then the fixed point $\mathbf{x}_c$ is unstable.
- Suppose the unstable manifold is empty and the system has both a non-empty stable and center manifold. Then the stability of the fixed point $\mathbf{x}_c$ is governed by the dynamics on the center manifold.

Reduction to the center manifold

example: $$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_1^2(x_2 - x_1^3)$$
$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1^2 - x_2$$

Reduction to the center manifold

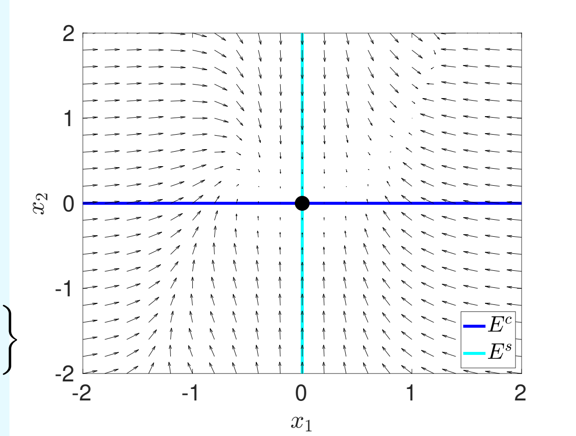
example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

consider fixed point $\mathbf{x}_c = \mathbf{0} = [0, 0]$

$$D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

$$E^c = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}, \quad E^s = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$



Reduction to the center manifold

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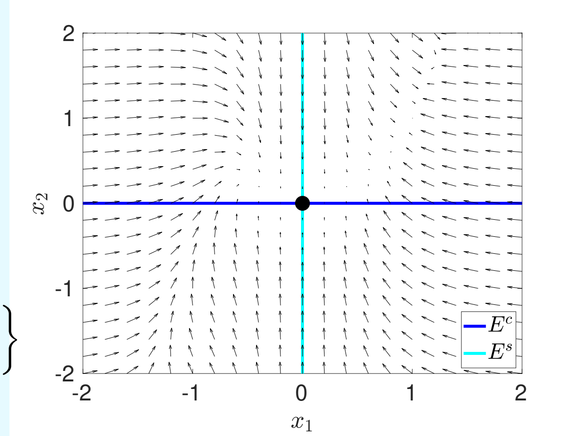


Warning: On the center linear subspace, we have $x_2 = 0$. Substituting $x_2 = 0$ into the first equation gives $\mathrm{d}x_1/\mathrm{d}t = -x_1^5$, but this does not mean that the origin is stable!

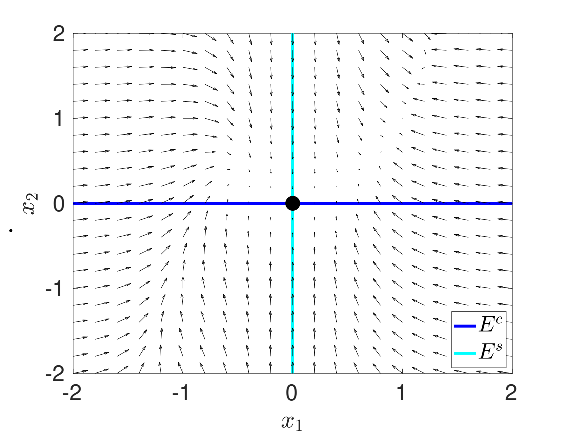
Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$

M^c_{loc} is of the form

$x_2 = h(x_1) = h_2\, x_1^2 + h_3\, x_1^3 + h_4\, x_1^4 + \ldots$



- differentiating, we get

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (2h_2x_1 + 3h_3x_1^2 + \ldots)\,\frac{\mathrm{d}x_1}{\mathrm{d}t}$$

Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$

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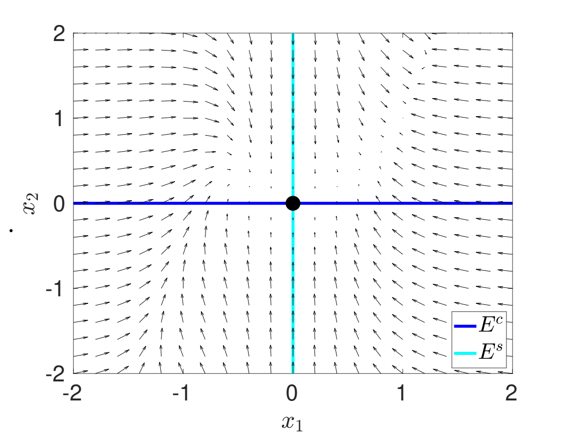

- differentiating, we get

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = (2h_2x_1 + 3h_3x_1^2 + \dots)\,\frac{\mathrm{d}x_1}{\mathrm{d}t}$$

- substituting

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3), \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - h(x_1)$$

we have $x_1^2 - h(x_1) = (2h_2x_1 + 3h_3x_1^2 + \dots)\,x_1^2(h(x_1) - x_1^3)$

Reduction to the center manifold

example: $\dfrac{dx_1}{dt} = x_1^2(x_2 - x_1^3)$

$\dfrac{dx_2}{dt} = x_1^2 - x_2$

M^c_{loc} is of the form

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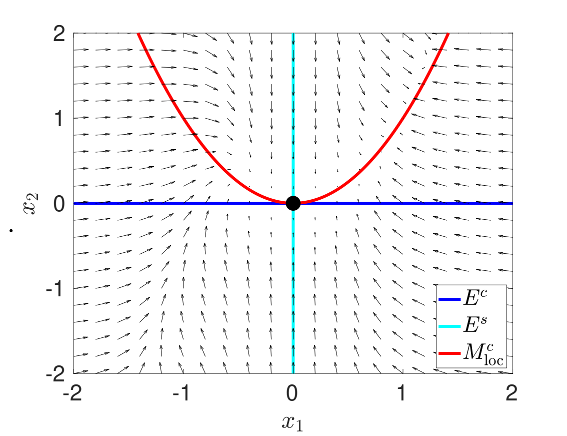


- differentiating, we get

  $\dfrac{dx_2}{dt} = (2h_2x_1 + 3h_3x_1^2 + \ldots)\,\dfrac{dx_1}{dt}$
- substituting

  $\dfrac{dx_1}{dt} = x_1^2(h(x_1) - x_1^3), \quad \dfrac{dx_2}{dt} = x_1^2 - h(x_1)$

we have $x_1^2 - h(x_1) = (2h_2x_1 + 3h_3x_1^2 + \ldots)\, x_1^2(h(x_1) - x_1^3)$

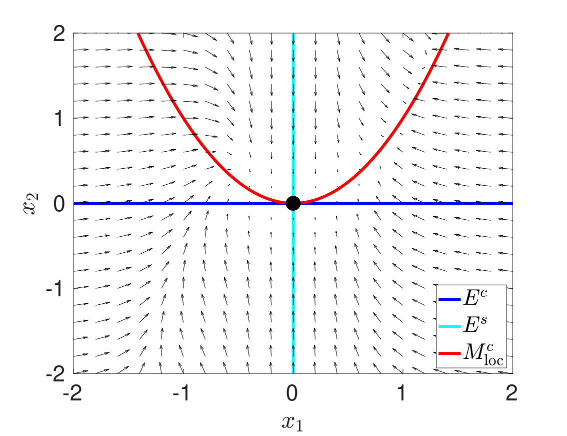
- equating coefficients of powers of x_1 gives $h_2 = 1$, $h_3 = 0$ and $h_4 = 0$

Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1^2 - x_2$$

M^c_{loc} is of the form

$$x_2 = h(x_1) = x_1^2 + \mathcal{O}(x_1^5)$$



Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
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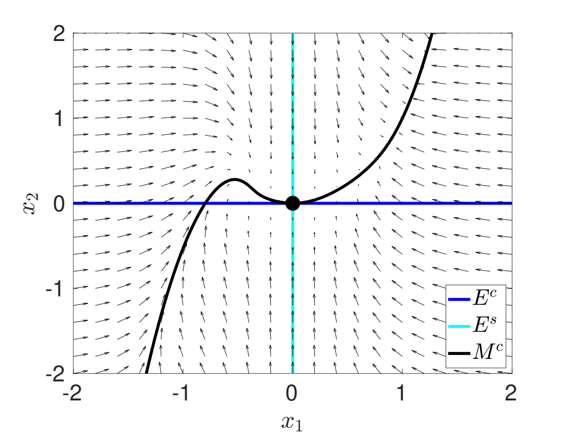
M^c_{loc} is of the form

$x_2 = h(x_1) = x_1^2 + \mathcal{O}(x_1^5)$

on the center manifold the dynamics is given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3)$$
$$= x_1^4 - x_1^5 + \mathcal{O}(x_1^7)$$



Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
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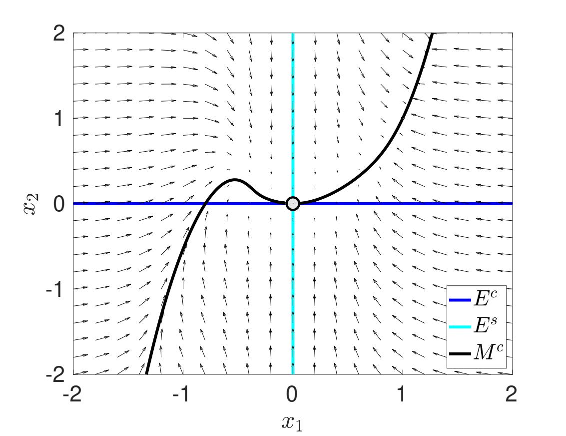
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3) = x_1^4 - x_1^5 + \mathcal{O}(x_1^7)$$

which implies that the origin is unstable



Reduction to the center manifold

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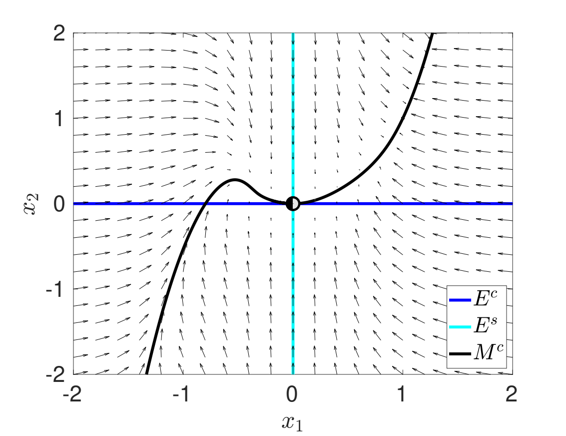

Reduction to the center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$$
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$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3)$$
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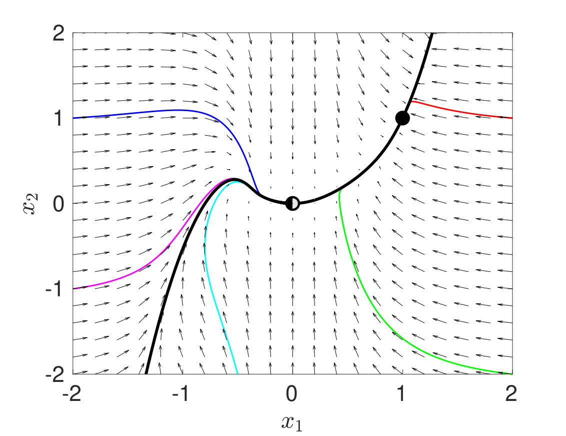

Reduction to the center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(x_2 - x_1^3)$

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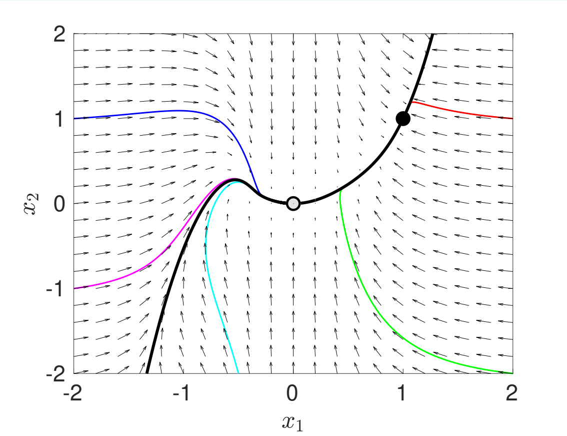
$x_2 = h(x_1) = x_1^2 + \mathcal{O}(x_1^5)$

on the center manifold the dynamics is given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2(h(x_1) - x_1^3) = x_1^4 - x_1^5 + \mathcal{O}(x_1^7)$$

which implies that the origin is unstable



another example: Question 3 on Problem Sheet 1

Bifurcations

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Bifurcations

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Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Some bifurcations can occur for $n = 1$, so we start with them.

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1; \mu) = \mu + x_1^2$$

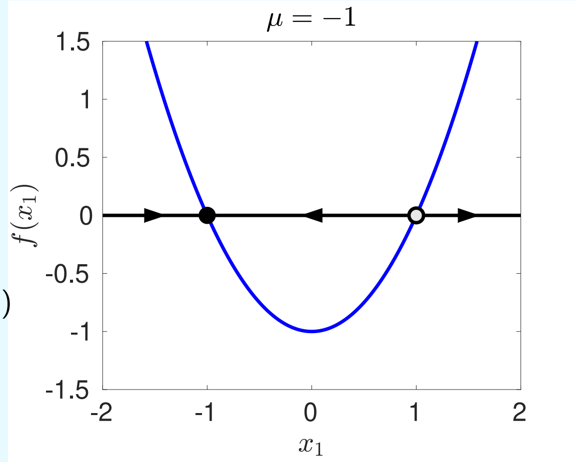
Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1; \mu) = \mu + x_1^2$$

$\mu < 0$

two fixed points at $x_1 = -\sqrt{-\mu}$ (stable) and $x_1 = \sqrt{-\mu}$ (unstable)



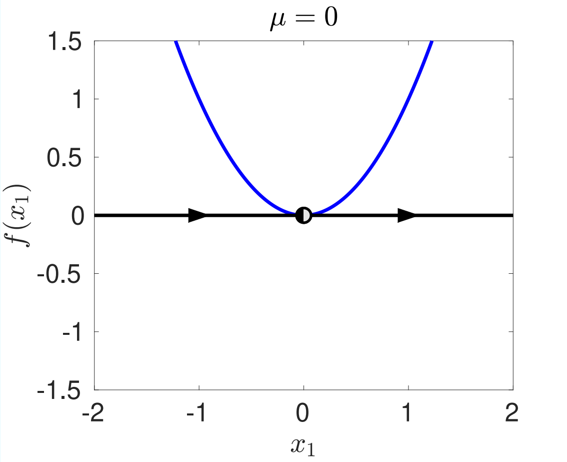
Saddle-node bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$$f(x_1;\mu) = \mu + x_1^2$$

as μ approaches zero from below, the two fixed points $-\sqrt{-\mu}$ and $\sqrt{-\mu}$ move toward each other

$\mu = 0$: the fixed points coalesce into a half-stable fixed point at $x_1 = 0$



Saddle-node bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

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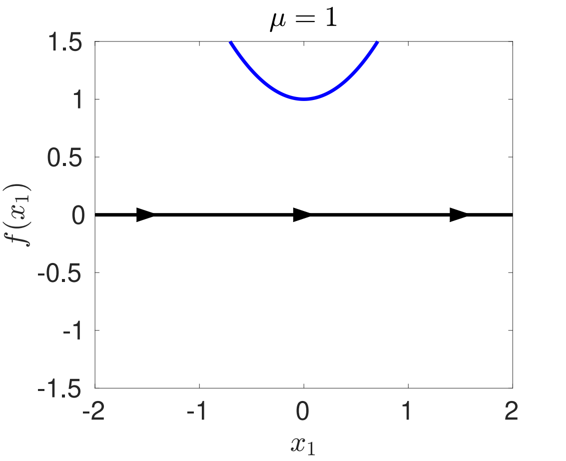
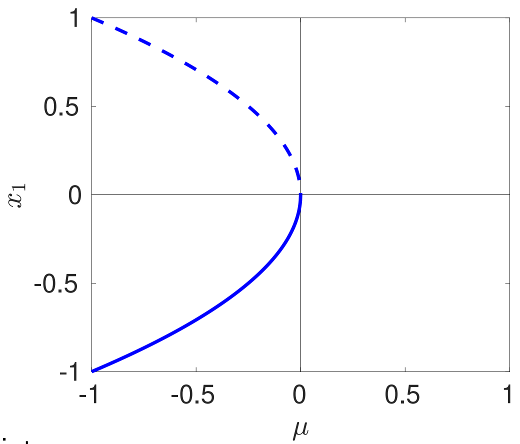
$\mu > 0$: no fixed points



Saddle-node bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$



bifurcation diagram

saddle node bifurcation is a simple mechanism by which critical points can be created or destroyed

terminology:

critical point: fixed point, equilibrium point

saddle-node bifurcation: fold bifurcation, turning point bifurcation

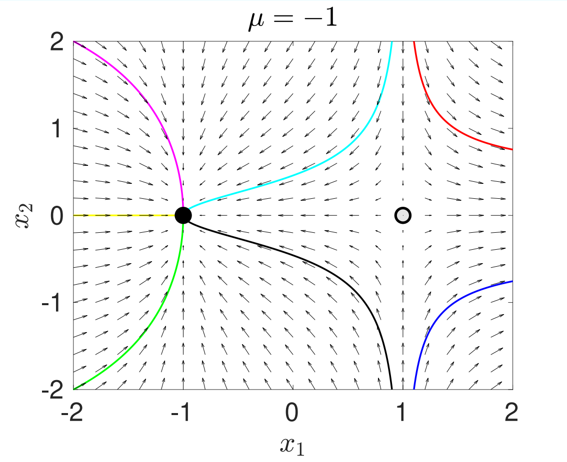
Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$

$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$

$\mu < 0$

two fixed points at
$\mathbf{x} = [-\sqrt{-\mu}, 0]$ stable node
and $\mathbf{x} = [\sqrt{-\mu}, 0]$ saddle



Saddle-node bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu + x_1^2$$
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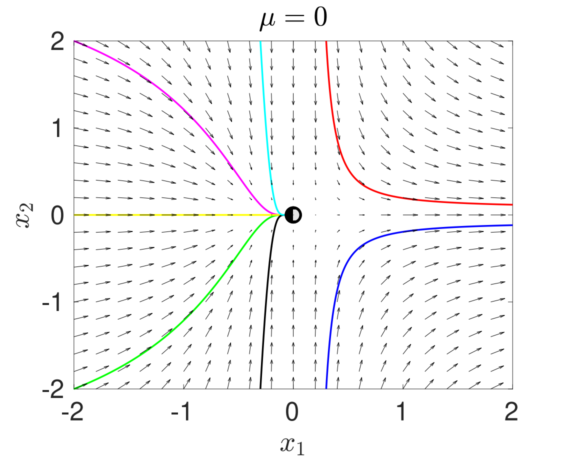
as μ approaches zero from below, the two fixed points $[-\sqrt{-\mu}, 0]$ and $[\sqrt{-\mu}, 0]$ move toward each other

$\mu = 0$: the fixed points coalesce into a (saddle-node) fixed point at $\mathbf{x} = [0, 0]$



Saddle-node bifurcation

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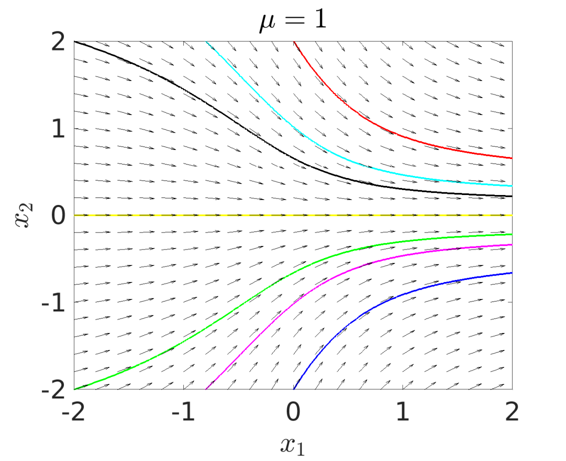
$\mu > 0$: no fixed points



Saddle-node bifurcation

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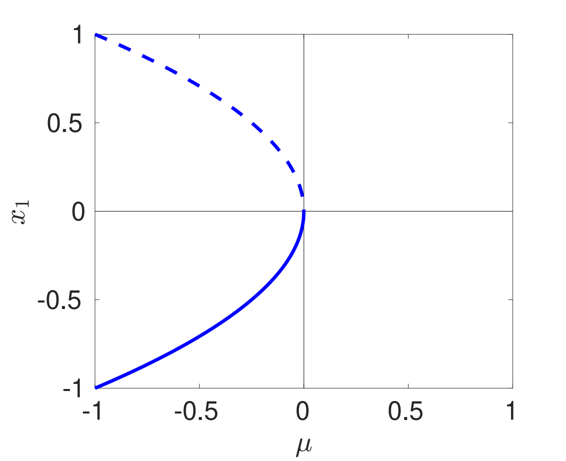
bifurcation diagram



Saddle-node bifurcation

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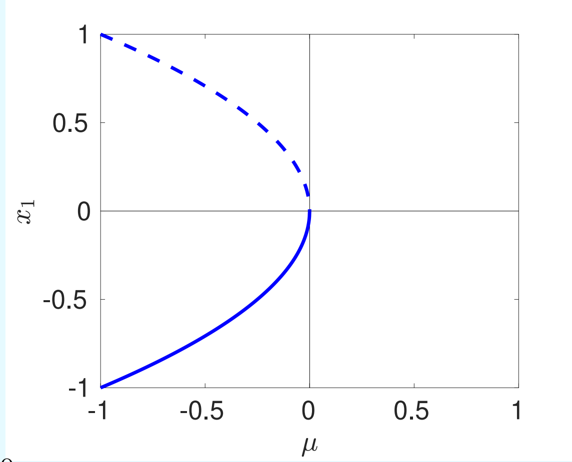
bifurcation diagram



saddle node bifurcation is a general mechanism by which critical points can be created or destroyed

if it occurs at $x_1 = x_c$ and $\mu = \mu_c$, we have

$$f(x_c; \mu_c) = 0 \text{ and } \frac{\partial f}{\partial x_1}(x_c; \mu_c) = 0$$

Taylor expansion:

$$f(x_1; \mu) = (\mu - \mu_c)\frac{\partial f}{\partial \mu}(x_c; \mu_c) + (x - x_c)^2\,\frac{1}{2}\,\frac{\partial^2 f}{\partial x_1^2}(x_c; \mu_c) + \ldots \qquad \text{(normal form)}$$

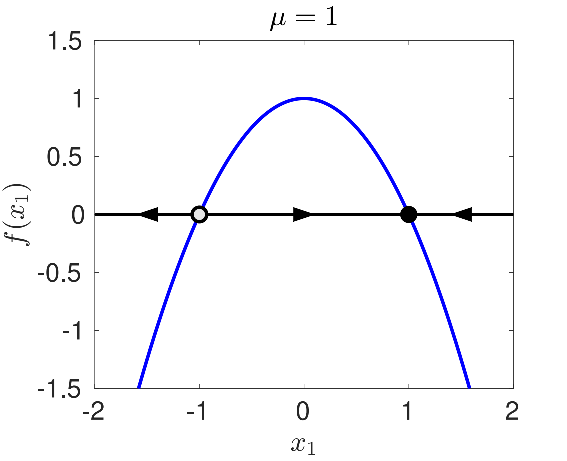
Saddle-node bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu - x_1^2$

$$f(x_1;\mu) = \mu - x_1^2$$

$\mu > 0$

two fixed points at $x_1 = \sqrt{\mu}$ (stable) and $x_1 = -\sqrt{\mu}$ (unstable)



Saddle-node bifurcation

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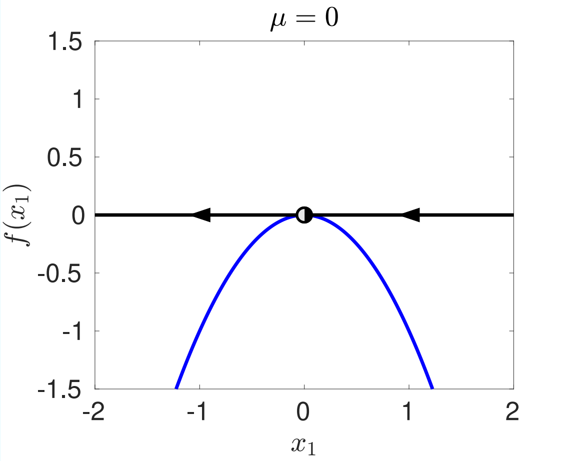

Saddle-node bifurcation

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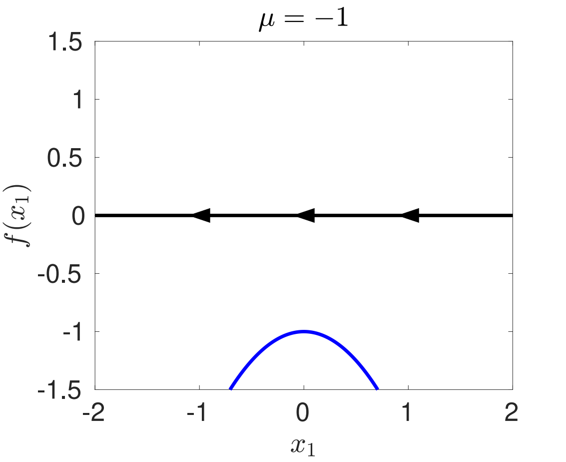


Saddle-node bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu - x_1^2$

$\dfrac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$

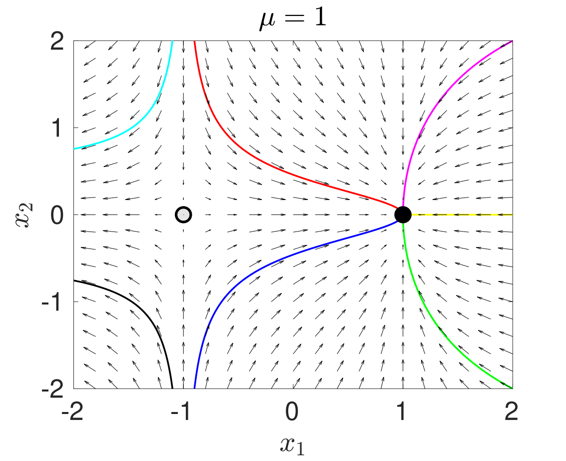
$\mu > 0$

two fixed points at

$\mathbf{x} = [-\sqrt{\mu}, 0]$ saddle

and

$\mathbf{x} = [\sqrt{\mu}, 0]$ stable node



Saddle-node bifurcation

example: $\dfrac{dx_1}{dt} = \mu - x_1^2$

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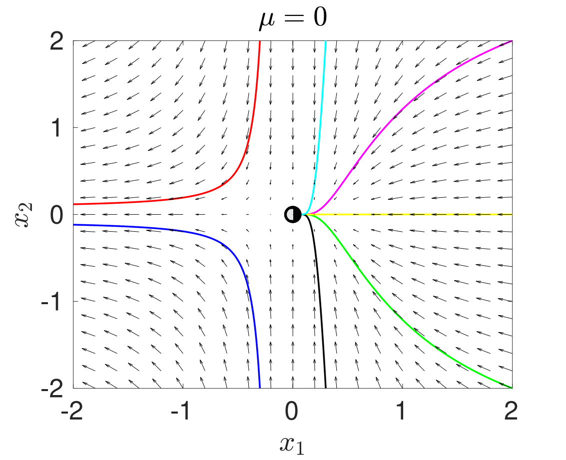

Saddle-node bifurcation

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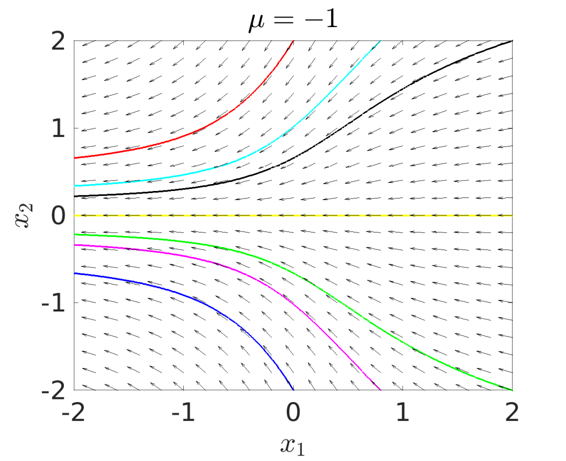

Saddle-node bifurcation

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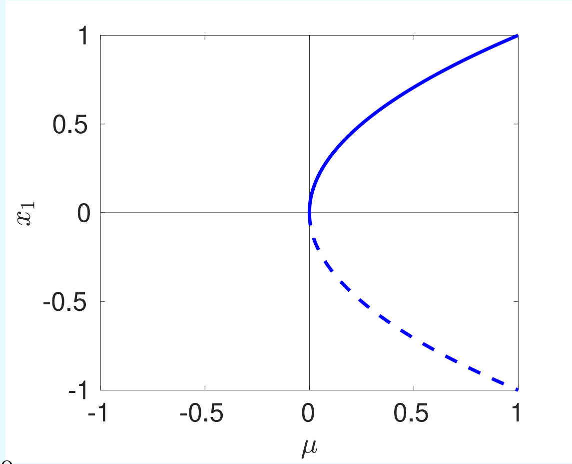
bifurcation diagram



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Transcritical bifurcation

example: $\dfrac{\mathsf{d}x_1}{\mathsf{d}t} = \mu x_1 - x_1^2$

$$f(x_1;\mu) = \mu x_1 - x_1^2$$

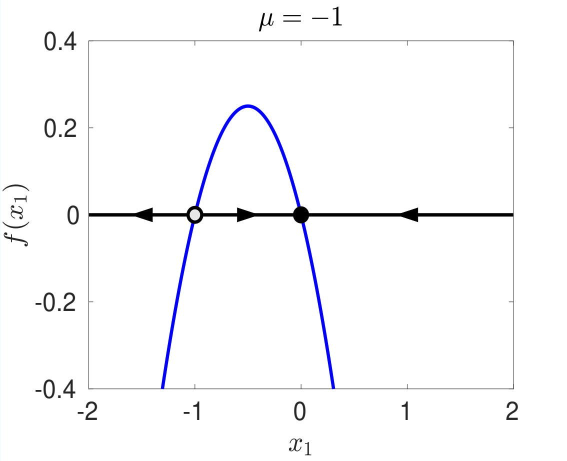
Transcritical bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^2$

$$f(x_1;\mu) = \mu x_1 - x_1^2$$

$\mu < 0$

two fixed points at $x_1 = \mu$ (unstable)
and $x_1 = 0$ (stable)



Transcritical bifurcation

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$$f(x_1;\mu) = \mu x_1 - x_1^2$$

as μ approaches zero,
the two fixed points μ and 0
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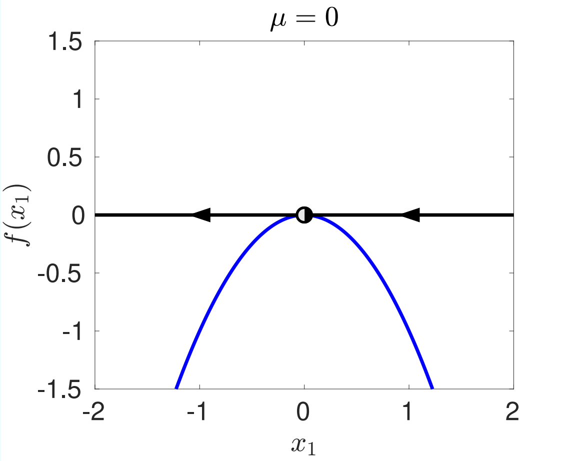

Transcritical bifurcation

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$\mu > 0$

two fixed points at $x_1 = \mu$ (stable)
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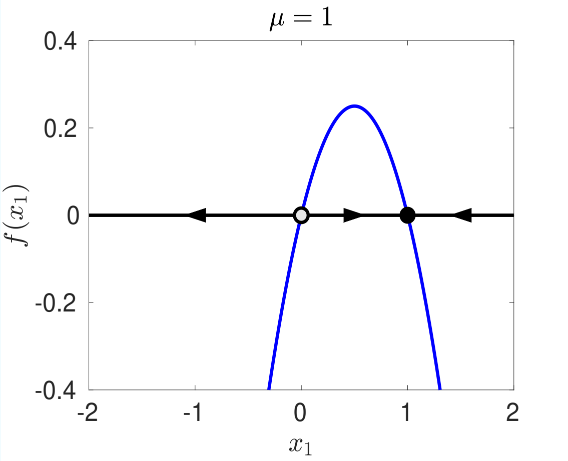
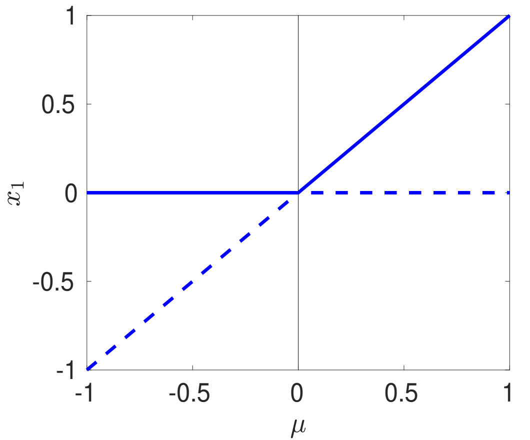


Transcritical bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^2$

bifurcation diagram



Transcritical bifurcation

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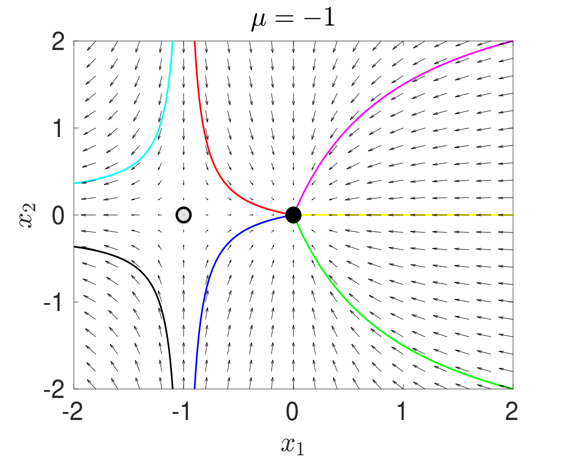
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

$\mu < 0$

two fixed points at

$\mathbf{x} = [\mu, 0]$ saddle (unstable)

and $\mathbf{x} = [0, 0]$ stable node



Transcritical bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^2$$
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as μ approaches zero,
the two fixed points $[\mu, 0]$ and $[0, 0]$ move toward each other

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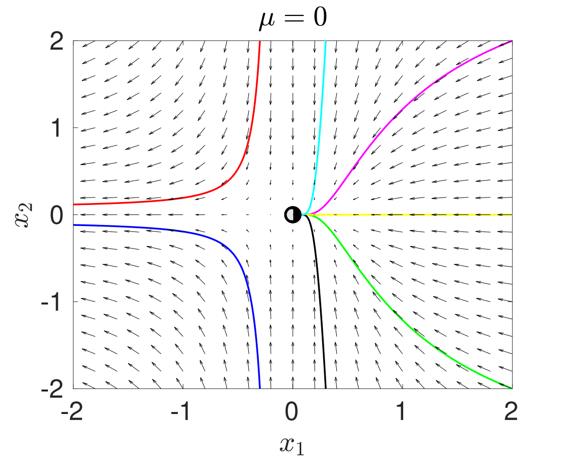

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two fixed points at

$\mathbf{x} = [\mu, 0]$ stable node

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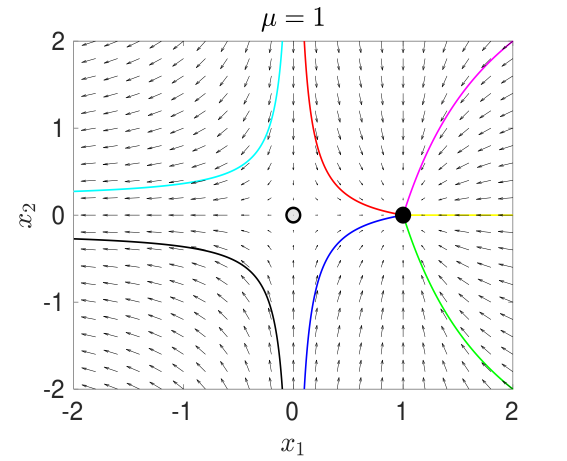

Transcritical bifurcation

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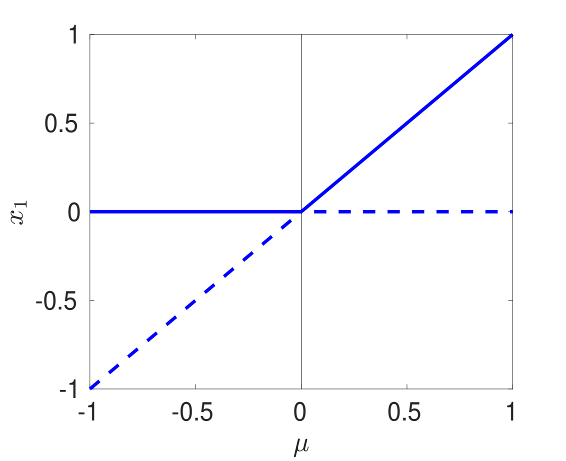
bifurcation diagram



Transcritical bifurcation

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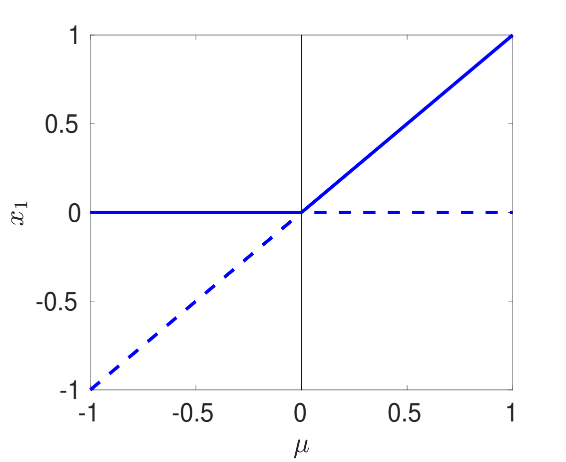
$$f(x_1;\mu) = \mu x_1 - x_1^2$$

bifurcation diagram



Other examples of ODE systems with bifurcations:

Questions 1, 2, 4 and 6 on Problem Sheet 2

Supercritical pitchfork bifurcation

example: $\dfrac{\mathsf{d}x_1}{\mathsf{d}t} = \mu x_1 - x_1^3$

$$f(x_1;\mu) = \mu x_1 - x_1^3$$

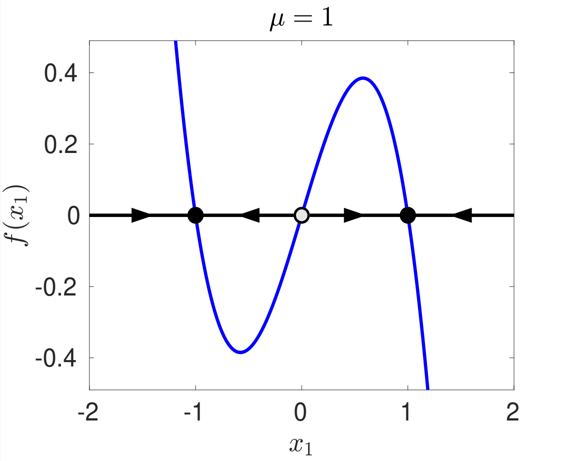
Supercritical pitchfork bifurcation

example: $\dfrac{dx_1}{dt} = \mu x_1 - x_1^3$

$$f(x_1; \mu) = \mu x_1 - x_1^3$$

$\mu > 0$

three fixed points at $x_1 = \pm\sqrt{\mu}$ (stable)
and $x_1 = 0$ (unstable)



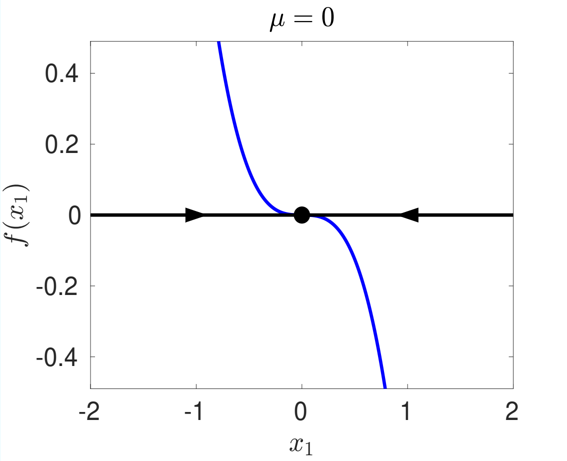
Supercritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 - x_1^3$

$$f(x_1; \mu) = \mu x_1 - x_1^3$$

as μ approaches zero from above, two fixed points $\sqrt{\mu}$ and $-\sqrt{\mu}$ move toward the third one

$\mu = 0$: the fixed points coalesce into a stable fixed point at $x_1 = 0$



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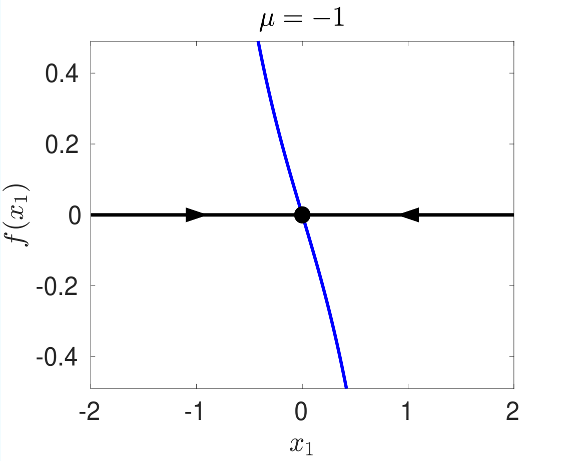
$$f(x_1;\mu) = \mu x_1 - x_1^3$$

$\mu < 0$: one stable fixed point at $x_1 = 0$



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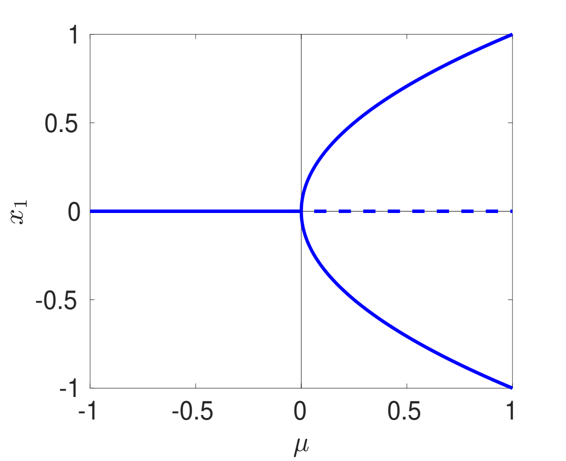

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$\mu > 0$

three fixed points at

$\mathbf{x} = [-\sqrt{\mu}, 0]$ (stable node)

$\mathbf{x} = [0, 0]$ (saddle)

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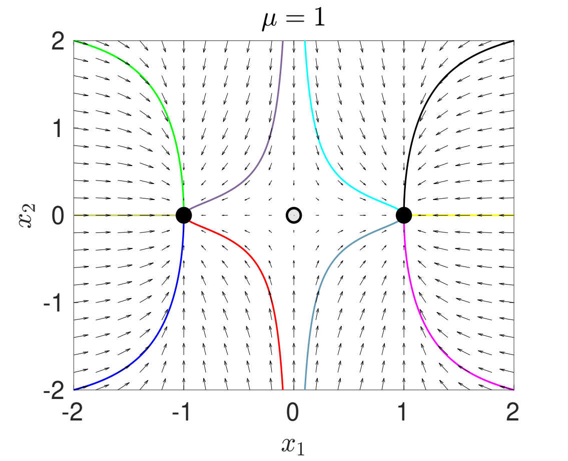

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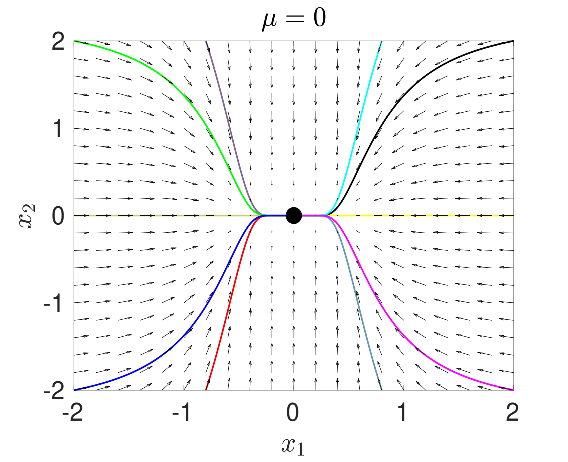

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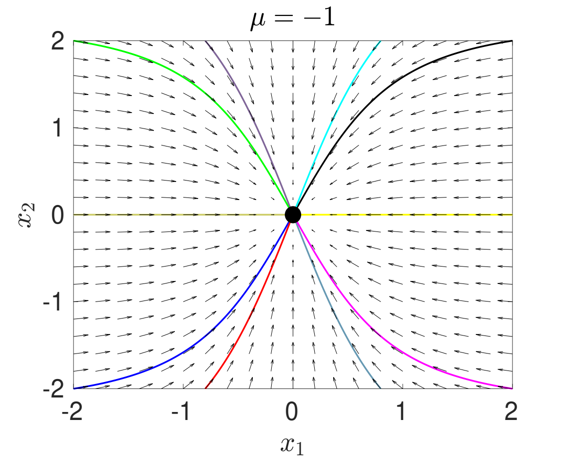
one stable fixed point at $\mathbf{x} = [0, 0]$



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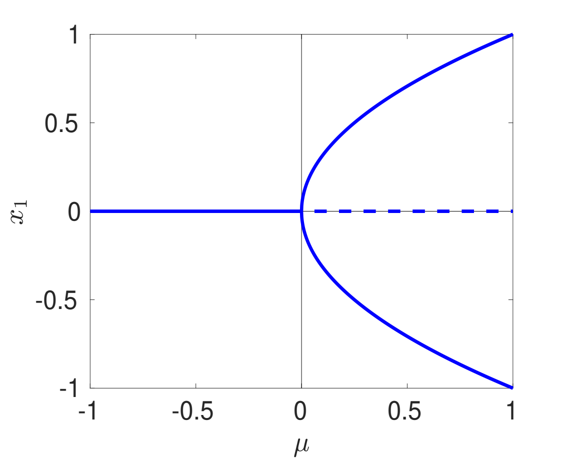
bifurcation diagram



Subcritical pitchfork bifurcation

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$$f(x_1;\mu) = \mu x_1 + x_1^3$$

Subcritical pitchfork bifurcation

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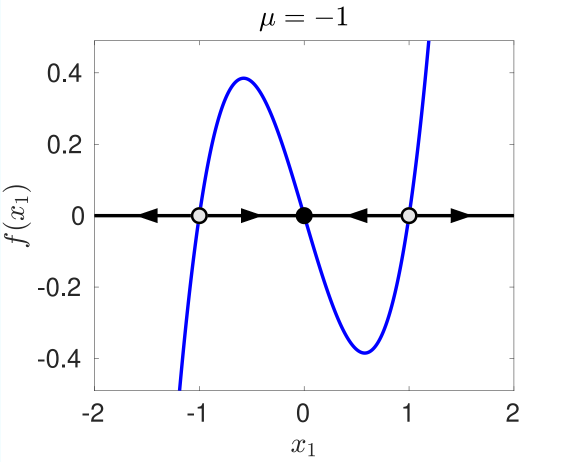

$\mu < 0$

three fixed points at
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Subcritical pitchfork bifurcation

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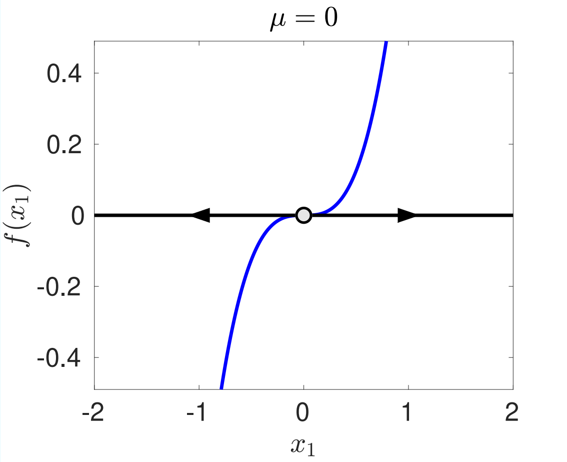

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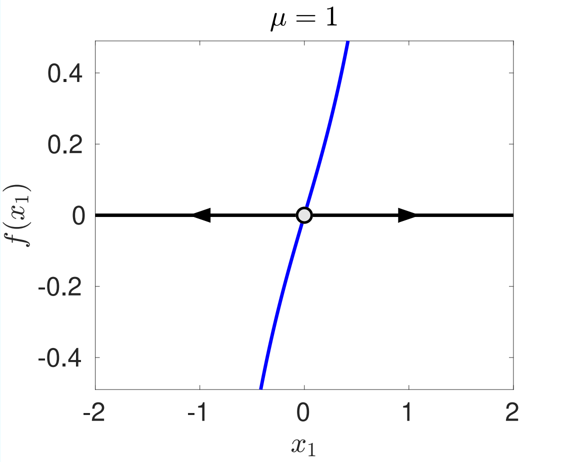
$\mu > 0$:
one unstable fixed point at $x_1 = 0$



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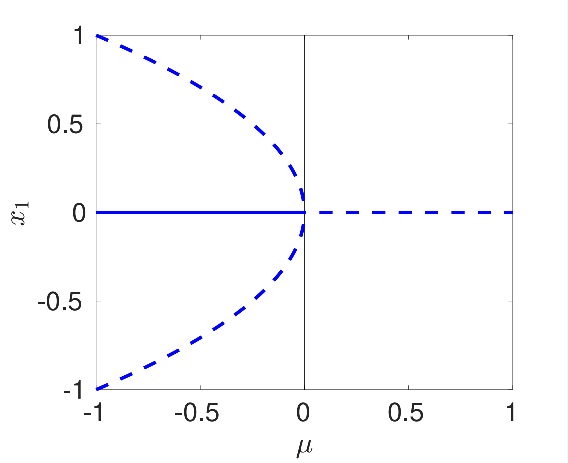

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$\mu < 0$

three fixed points at

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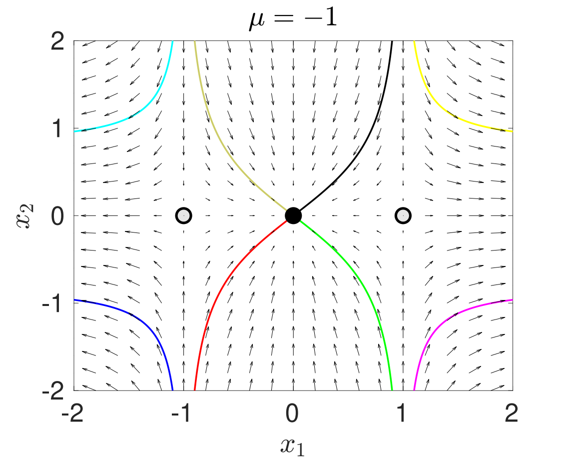
$\mathbf{x} = [0, 0]$ (stable node)

$\mathbf{x} = [\sqrt{-\mu}, 0]$ (saddle)



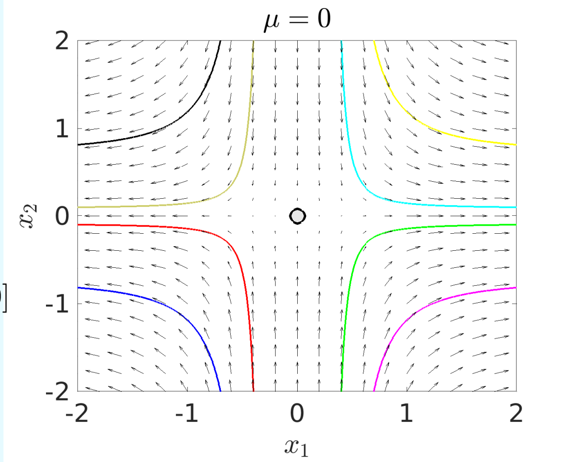
Subcritical pitchfork bifurcation

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$

$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$

as μ approaches zero from below, two fixed points $[-\sqrt{-\mu}, 0]$ and $\sqrt{-\mu}, 0]$ move toward the third one

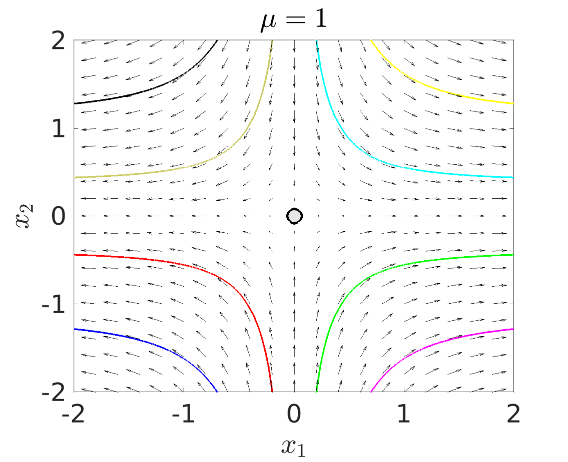
$\mu = 0$: the fixed points coalesce into an unstable fixed point at $\mathbf{x} = [0, 0]$



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

$\mu > 0$:
one unstable fixed point at $\mathbf{x} = [0, 0]$



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3$$
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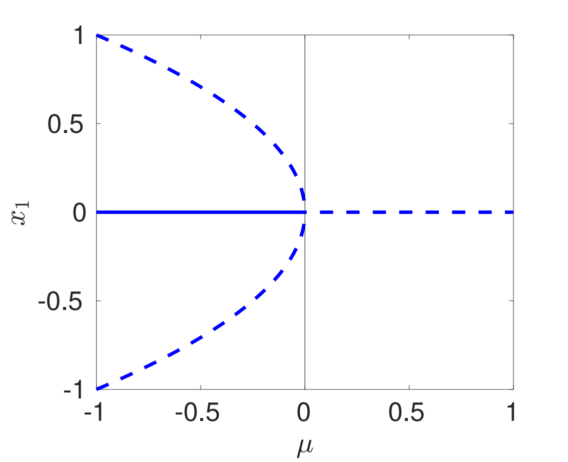
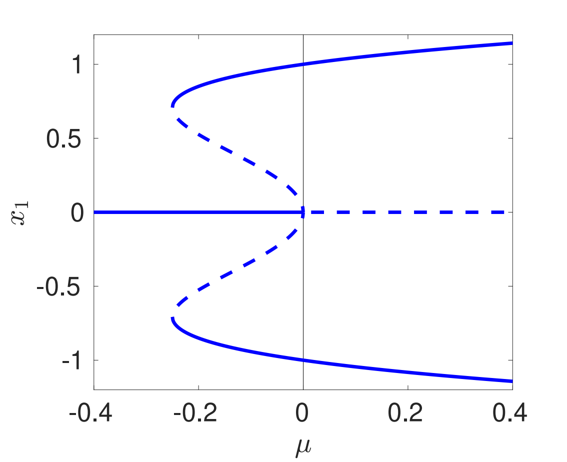
bifurcation diagram



Subcritical pitchfork bifurcation

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3 - x_1^5$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

bifurcation diagram



Subcritical pitchfork bifurcation

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = \mu x_1 + x_1^3 - x_1^5$

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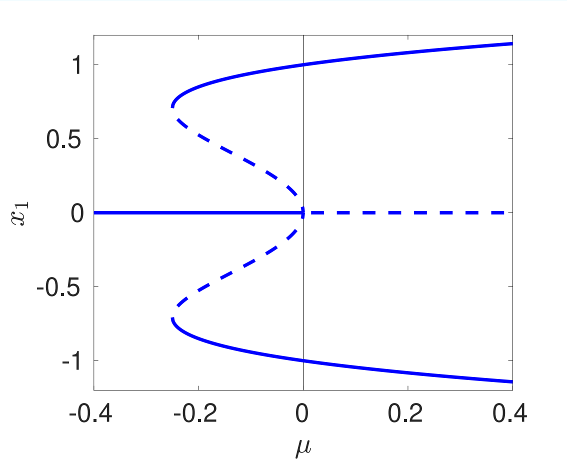
bifurcation diagram



Other examples:

Questions 1 and 6 on Problem Sheet 2

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

$[x_1, x_2] = [0, 0]$ is a critical point

linearized system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

for $M = \begin{pmatrix} -1 & 0 \\ 0 & \mu \end{pmatrix}$

Extended center manifold

example: $$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_2^2 - x_1$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = \mu x_2 - x_1 x_2$$

$$\frac{\mathsf{d}\mu}{\mathsf{d}t} = 0$$

$[x_1, x_2, \mu] = [0, 0, 0]$ is a critical point

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Extended center manifold

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the center manifold is given by

$$x_1 = h(x_2, \mu) = c_{20}\, x_2^2 + c_{11}\, \mu\, x_2 + c_{02}\, \mu^2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$
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$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, \frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial h}{\partial \mu}(x_2, \mu)\, \frac{\mathrm{d}\mu}{\mathrm{d}t}$$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$
$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

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the center manifold is given by

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$$x_2^2 - x_1 = \frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, \frac{\mathrm{d}x_2}{\mathrm{d}t} + \frac{\partial h}{\partial \mu}(x_2, \mu)\, \frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\partial h}{\partial x_2}(x_2, \mu)\, (\mu x_2 - x_1 x_2)$$

Extended center manifold

example: $\dfrac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

$[x_1, x_2, \mu] = [0, 0, 0]$ is a critical point

linearized system $\dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

for $M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

the center manifold is given by

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$$x_2^2 - c_{20}\, x_2^2 - c_{11}\, \mu\, x_2 - c_{02}\, \mu^2 = (2\, c_{20}\, x_2 + c_{11}\, \mu)\, \mu x_2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$$

Extended center manifold

example: $\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$

$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_1 x_2$

$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$

$[x_1, x_2, \mu] = [0, 0, 0]$ is a critical point

linearized system $\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = M\mathbf{x}$

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the center manifold is given by

$$x_1 = h(x_2, \mu) = c_{20}\, x_2^2 + c_{11}\, \mu\, x_2 + c_{02}\, \mu^2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$$

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$$c_{20} = 1, \qquad c_{11} = 0, \qquad c_{02} = 0$$

center manifold: $x_1 = x_2^2 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$

the dynamics on the center manifold: $\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_2^3 + \mathcal{O}(x_2^3, x_2^2\mu, x_2\mu^2, \mu^3)$

Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$
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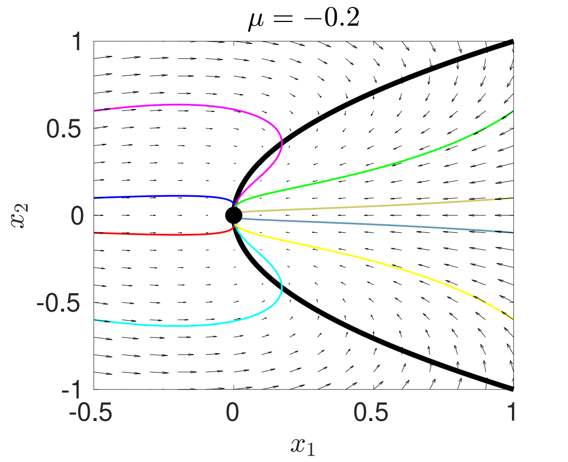

center manifold:
$$x_1 = x_2^2 + \ldots$$

dynamics on the center manifold:
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu x_2 - x_2^3 + \ldots$$

supercritical pitchfork bifurcation

Extended center manifold

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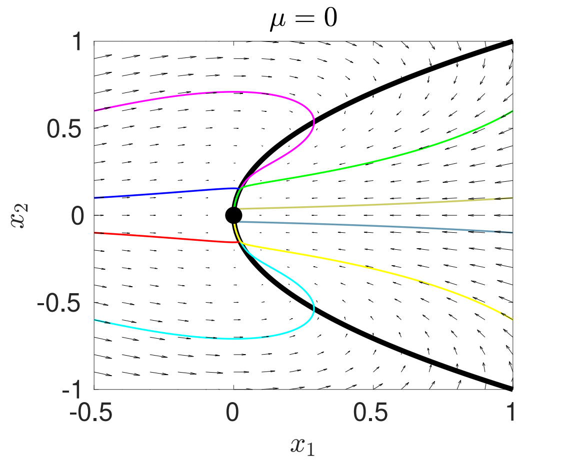

Extended center manifold

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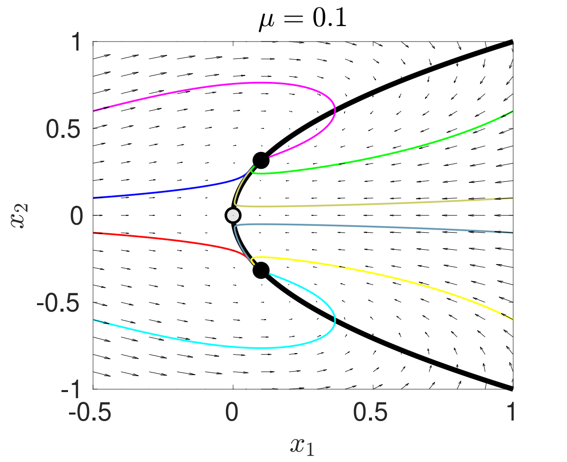

center manifold:
$$x_1 = x_2^2 + \dots$$

dynamics on the center manifold:
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Extended center manifold

example: $$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2^2 - x_1$$

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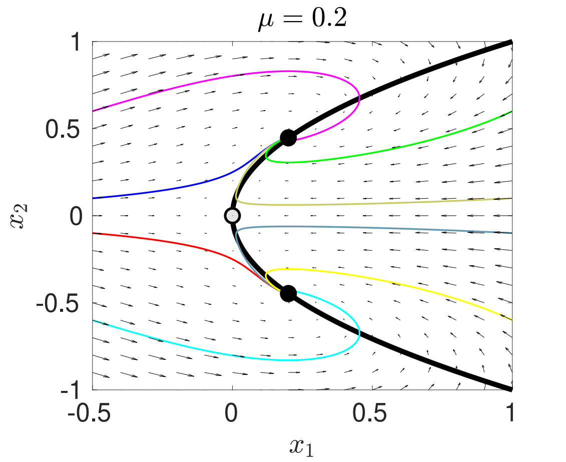

center manifold:

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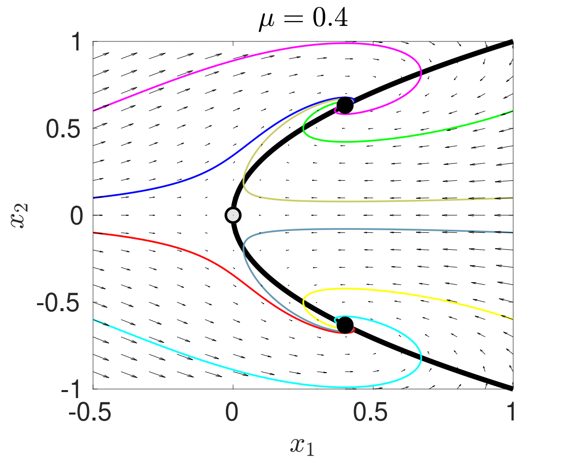

center manifold:

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dynamics on the center manifold:

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supercritical pitchfork bifurcation

Extended center manifold

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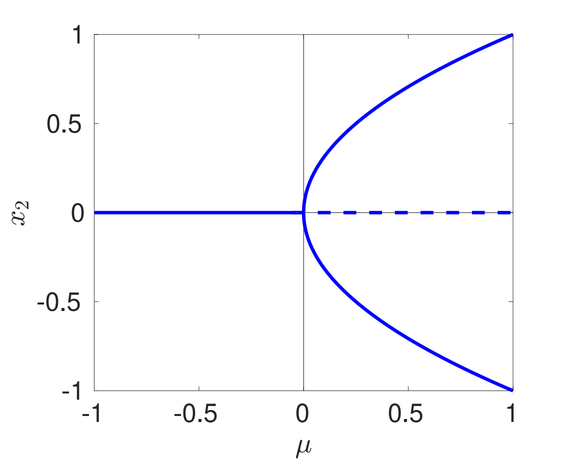

center manifold:

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dynamics on the center manifold:

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supercritical pitchfork bifurcation

Another example: Question 6 on Problem Sheet 2

Bifurcations of continuous-time dynamical systems – summary

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

We have discussed bifurcations of fixed points, which can occur for $n \geq 1$ and $m \geq 1$ (so, they can be explained on examples with $n = 1$ and $m = 1$):

- saddle-node bifurcation
- transcritical bifurcation
- pitchfork bifurcation (supercritical, subcritical)

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- saddle-node bifurcation
- transcritical bifurcation
- pitchfork bifurcation (supercritical, subcritical)

We will discuss later in the course:

- bifurcations of limit cycles $(n > 1)$
- bifurcations with more than one parameter $(m > 1)$

Next, we will discuss bifurcations of discrete-time dynamical systems.

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k; \boldsymbol{\mu})$$

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- $\boldsymbol{\alpha} \in \Omega$ is a *fixed point* if $\boldsymbol{\alpha} = \mathbf{F}(\boldsymbol{\alpha}; \boldsymbol{\mu})$
- fixed point $\boldsymbol{\alpha}$ is *stable* if

  $$\forall \varepsilon > 0 \, \exists \delta > 0 \text{ such that } \forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha}) \text{ and } k \in \mathbb{N}_0 \text{ we have } \mathbf{x}_k \in B_\varepsilon(\boldsymbol{\alpha})$$

  where the open ball of radius r is defined by $B_r(\boldsymbol{\alpha}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \| \mathbf{x} - \boldsymbol{\alpha} \| < r \right\}$

Fixed points

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- fixed point $\boldsymbol{\alpha}$ is *asymptotically stable* if (i) it is stable; and
  (ii) $\exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha})$ we have $\lim_{k \to \infty} \mathbf{x}_k = \boldsymbol{\alpha}$

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- Prelims Constructive Mathematics: we considered $n = 1$ where $x_{k+1} = F(x_k)$
  - $\alpha \in \mathbb{R}$ is a fixed point if $\alpha = F(\alpha)$
  - if $|F'(\alpha)| < 1$, then α is asymptotically stable

Example

$$x_{k+1} = 1 - 6\,x_k + 15\,x_k^2 - 10\,x_k^3$$

Example

$$x_{k+1} = 1 - 6\,x_k + 15\,x_k^2 - 10\,x_k^3$$

$$F(x) = 1 - 6\,x + 15\,x^2 - 10\,x^3$$

fixed points: solving $F(\alpha) = \alpha$

Example

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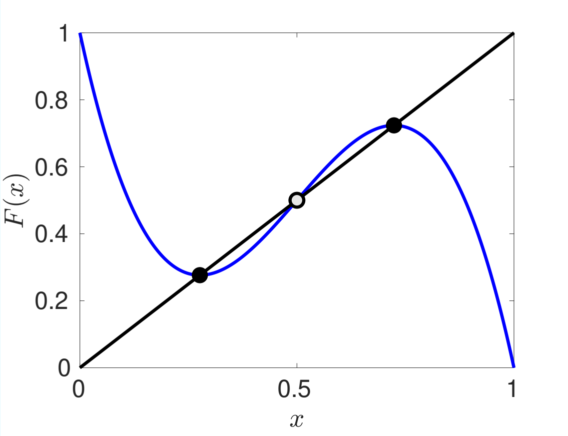
$$F(x) = 1 - 6\,x + 15\,x^2 - 10\,x^3$$

fixed points: solving $F(\alpha) = \alpha$, we get

$$\alpha_1 = \frac{1}{2} - \frac{\sqrt{5}}{10}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{1}{2} + \frac{\sqrt{5}}{10}$$



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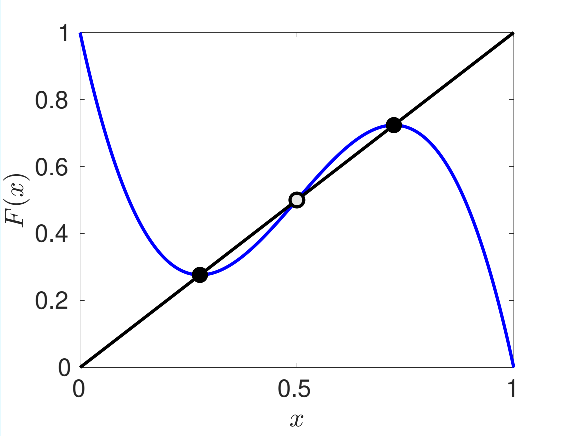
$$F'(x) = -6 + 30x - 30x^2$$

$$F'(\alpha_1) = F'(\alpha_3) = 0, \qquad F'(\alpha_2) = \frac{3}{2}$$

α_1 and α_3 are asymptotically stable

α_2 is unstable



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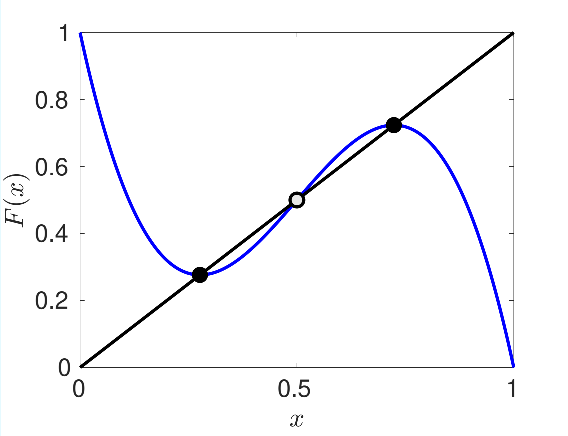


fixed point α with $|F'(\alpha)| < 1$ is asymptotically stable

fixed point α with $|F'(\alpha)| > 1$ is unstable

fixed point α with $|F'(\alpha)| = 0$ is called *super-attracting* because $|F'(\alpha)| = 0$ gives very fast convergence to the fixed point for nearby points

Example

$x_{k+1} = 1 - 6\,x_k + 15\,x_k^2 - 10\,x_k^3$

$F(x) = 1 - 6\,x + 15\,x^2 - 10\,x^3$

fixed points: solving $F(\alpha) = \alpha$, we get

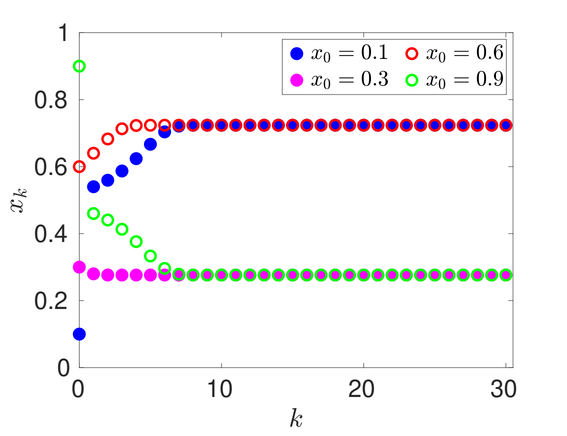
$\alpha_1 = \frac{1}{2} - \frac{\sqrt{5}}{10}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{1}{2} + \frac{\sqrt{5}}{10}$

$F'(x) = -6 + 30x - 30x^2$

$F'(\alpha_1) = F'(\alpha_3) = 0, \qquad F'(\alpha_2) = \frac{3}{2}$

α_1 and α_3 are asymptotically stable

α_2 is unstable



fixed point α with $|F'(\alpha)| < 1$ is asymptotically stable

fixed point α with $|F'(\alpha)| > 1$ is unstable

fixed point α with $|F'(\alpha)| = 0$ is called *super-attracting* because $|F'(\alpha)| = 0$ gives very fast convergence to the fixed point for nearby points

Fixed points

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}_k \in \Omega$ be defined iteratively by

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  $\forall \varepsilon > 0 \, \exists \delta > 0$ such that $\forall \mathbf{x}_0 \in B_\delta(\boldsymbol{\alpha})$ and $k \in \mathbb{N}_0$ we have $\mathbf{x}_k \in B_\varepsilon(\boldsymbol{\alpha})$

  where the open ball of radius r is defined by $B_r(\boldsymbol{\alpha}) = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \boldsymbol{\alpha}\| < r \right\}$

Fixed points

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  $$\mathbf{x}_k = \mathbf{F}_{\boldsymbol{\mu}}^{(k)}(\mathbf{x}_0)$$

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Fixed points, periodic points and N-cycles

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  and the set $\left\{\boldsymbol{\alpha}, \mathbf{F}_{\boldsymbol{\mu}}(\boldsymbol{\alpha}), \mathbf{F}_{\boldsymbol{\mu}}^{(2)}(\boldsymbol{\alpha}), \ldots, \mathbf{F}_{\boldsymbol{\mu}}^{(N-1)}(\boldsymbol{\alpha})\right\}$ is called an N-*cycle*

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- periodic point $\boldsymbol{\alpha} \in \Omega$ is *stable* if it is a *stable* fixed point of $\mathbf{F}_{\boldsymbol{\mu}}^{(N)}$ (resp. *asymptotically stable*, *unstable*)
- to find periodic points and the corresponding N-cycles, we need to solve $\boldsymbol{\alpha} = \mathbf{F}_{\boldsymbol{\mu}}^{(N)}(\boldsymbol{\alpha})$ and we also need to exclude solutions with some lesser period

  Question 5 on Problem Sheet 2

Fixed points, periodic points, N-cycles, orbits and bifurcations

Discrete-time dynamical system: Let $\mathbf{F} : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
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Fixed points, periodic points, N-cycles, orbits and bifurcations

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- if $\mathbf{x}_0$ is a periodic point with period N, then its orbit is a finite set (N-cycle)
- if orbit is a finite set, then it is (eventually) periodic,
  i.e. there exists $j \in \mathbb{N}_0$ such that $\mathbf{F}_{\boldsymbol{\mu}}^{(j)}(\mathbf{x}_0)$ is a *periodic point* with *period* $N \in \mathbb{N}$

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- if orbit is an infinite set, then it can approach a fixed point or an N-cycle, or it can be chaotic - we will illustrate this on examples with $n = 1$ and $m = 1$
- bifurcations: the qualitative behaviour of orbits can change as parameters $\boldsymbol{\mu}$ are varied (for example, fixed points or N-cycles can be created or destroyed, or their stability changes); the parameter values at which these qualitative changes in the dynamics occur are called bifurcation points

Example

$$x_{k+1} = (1 - x_k)(1 - 5\,x_k + \mu\,x_k^2)$$

Example

$$x_{k+1} = (1 - x_k)(1 - 5\,x_k + \mu\,x_k^2)$$

$$F(x;\mu) = (1 - x)(1 - 5\,x + \mu\,x^2)$$

fixed points: $F(x;\mu) = x$

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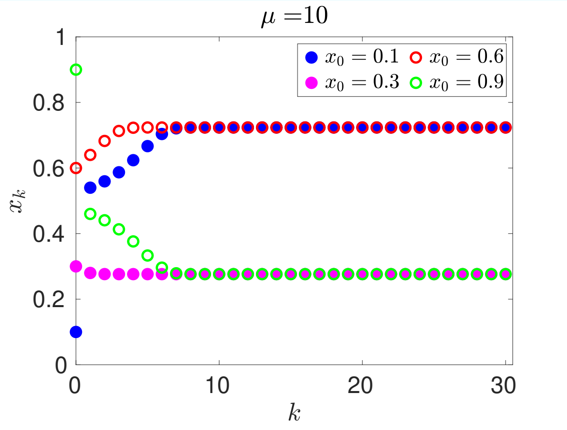
our previous example: $\mu = 10$

$$x_{k+1} = 1 \;-\; 6\,x_k \;+\; 15\,x_k^2 - 10\,x_k^3$$

$$F(x;10) = 1 \;-\; 6\,x \;+\; 15\,x^2 - 10\,x^3$$



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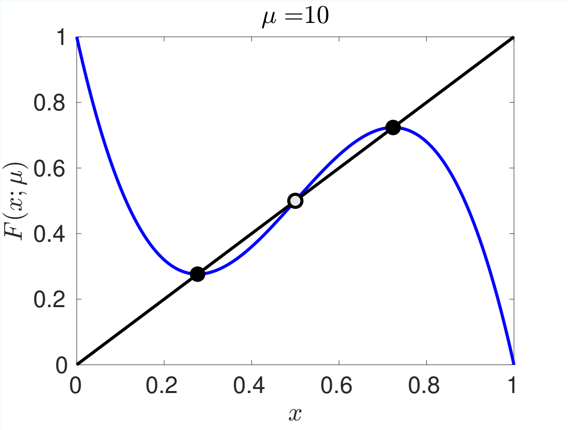

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If $\mu \in \Theta = [6.3, 11.8]$,
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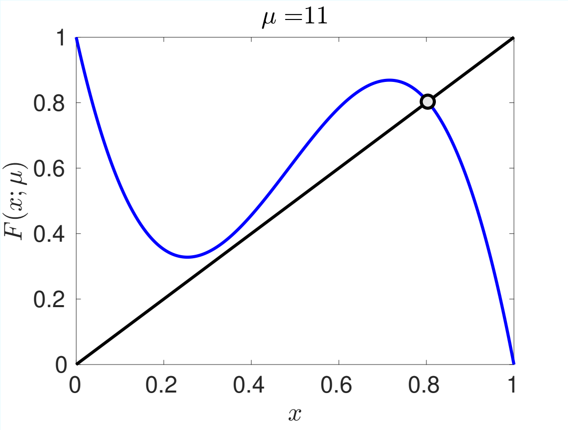
We study dynamics and bifurcations of $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$.



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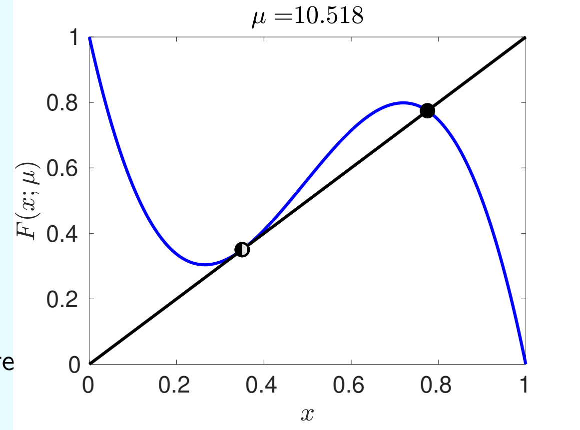
three fixed points for $\mu \in (\mu_1, \mu_2)$ where $\mu_1 = 9.7066\ldots$ and $\mu_2 = 10.518\ldots$

one fixed point for $\mu < \mu_1$ and $\mu > \mu_2$



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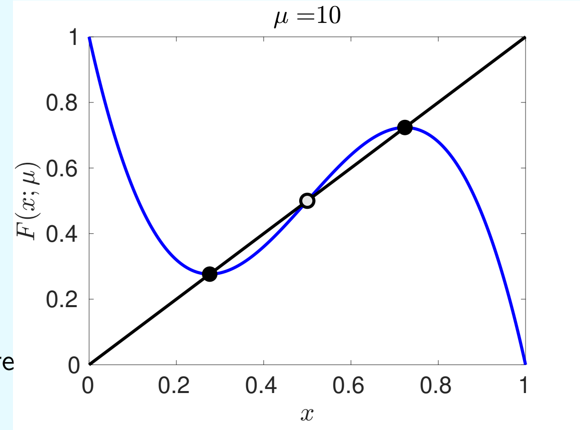

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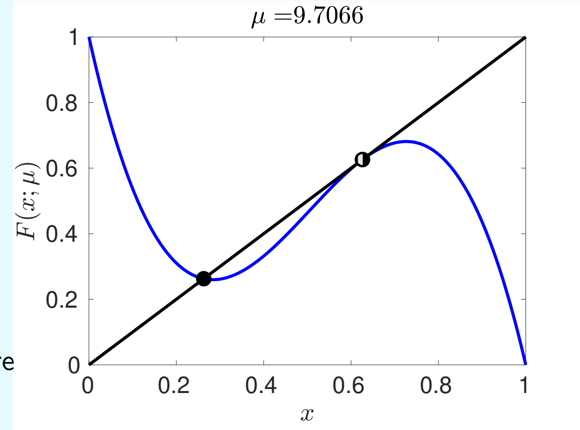

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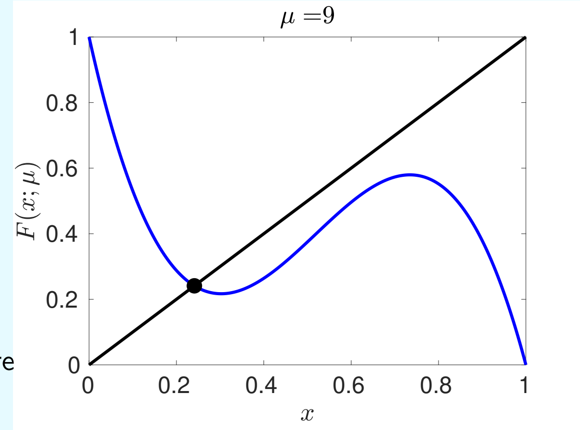

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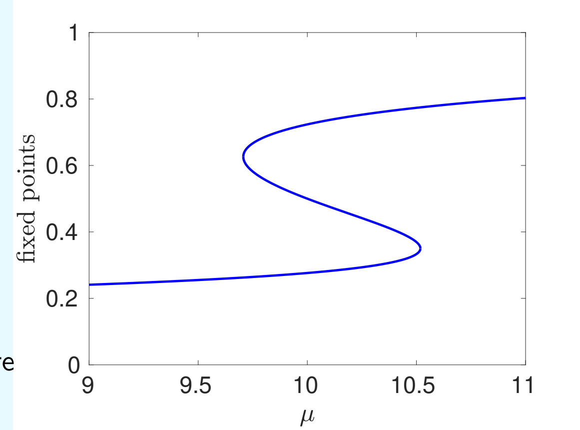

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$$\mu(x) = \frac{7x - 5x^2 - 1}{(1-x)\,x^2}$$

μ_1 and μ_2 can be found by solving $0 = \mu'(x) = \dfrac{2 - 10x + 14x^2 - 5x^3}{(1-x)^2\,x^3} = 0$

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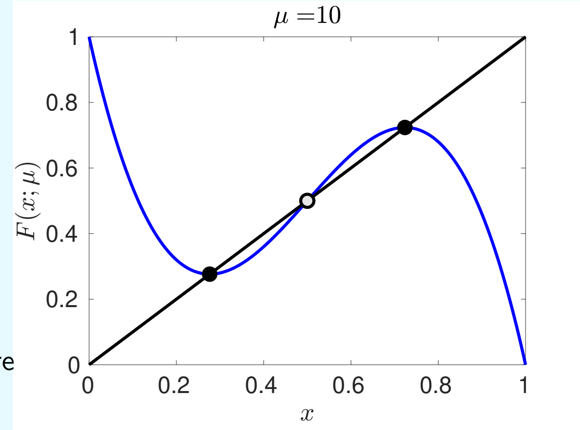

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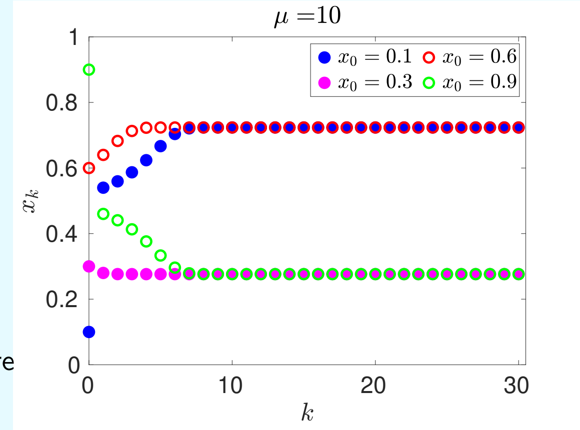

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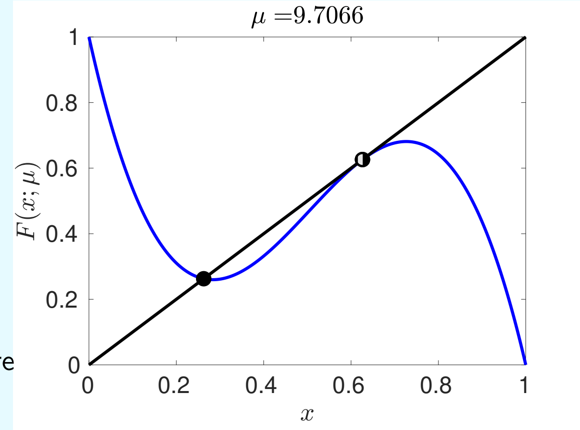

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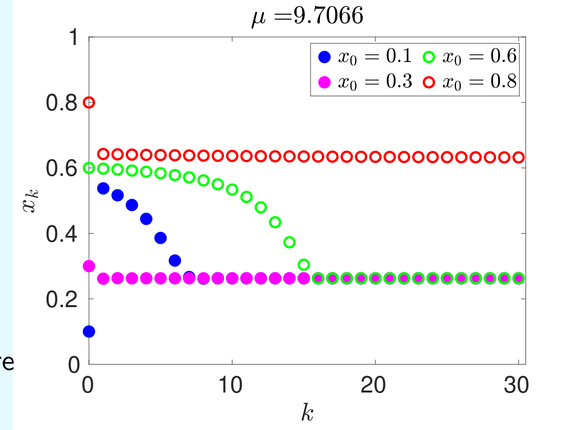

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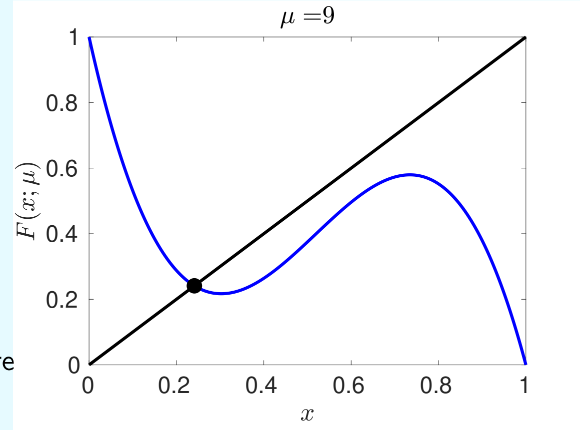

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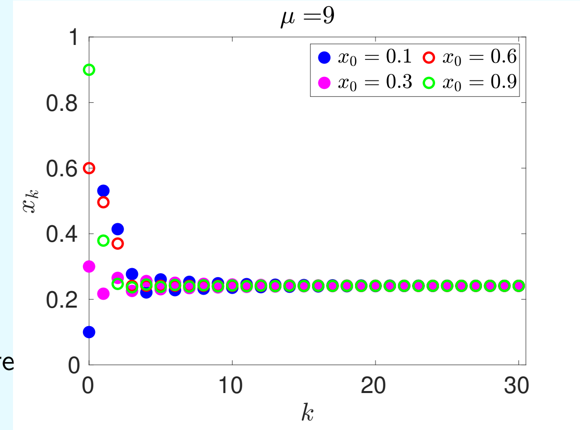

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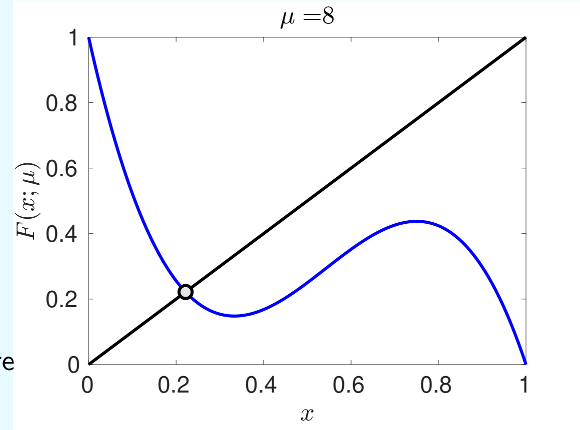

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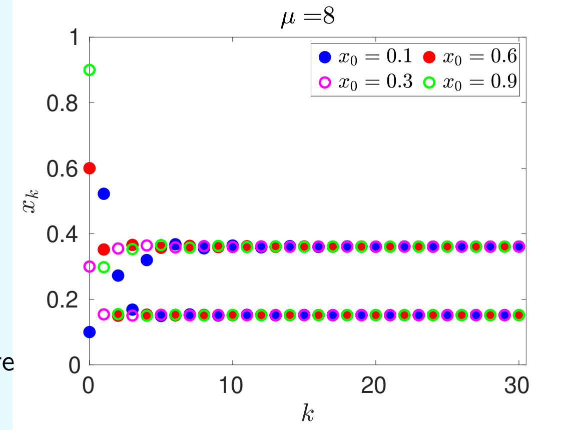

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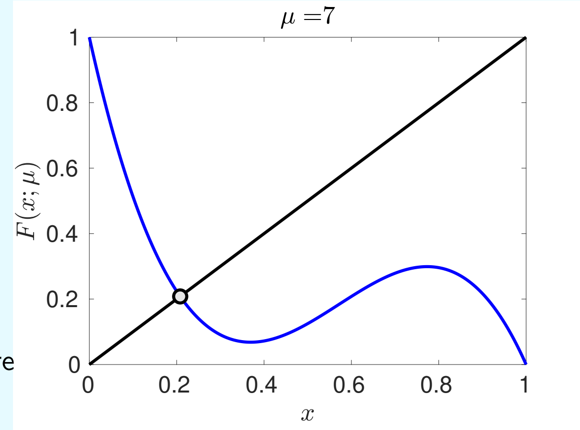

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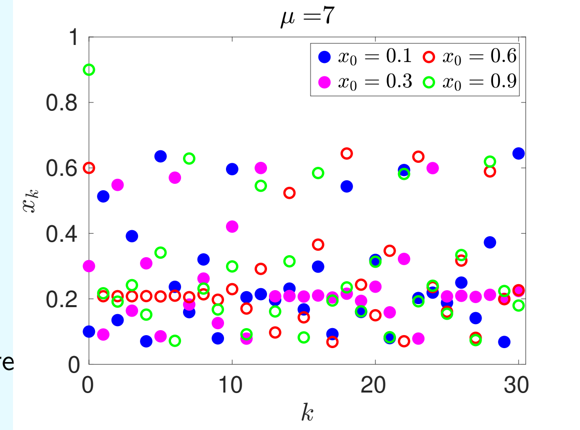

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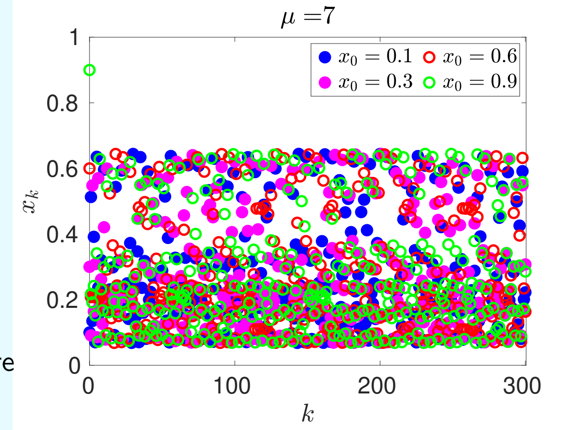

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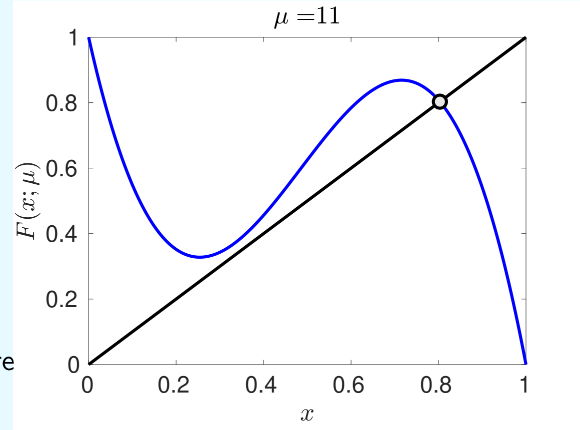

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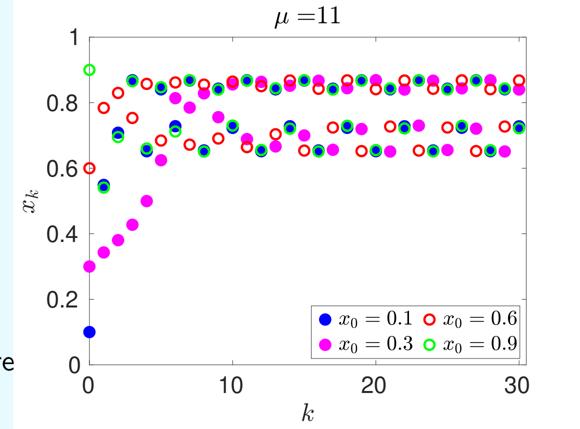

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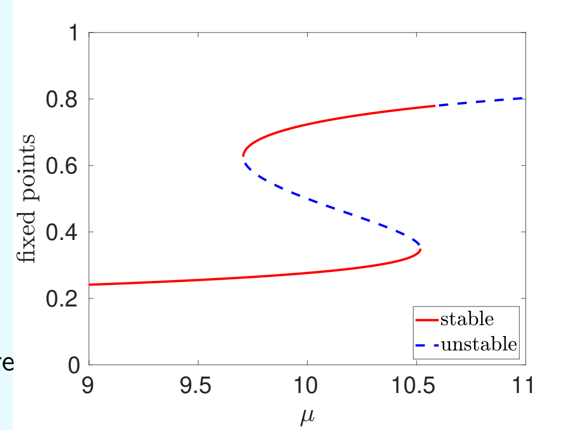

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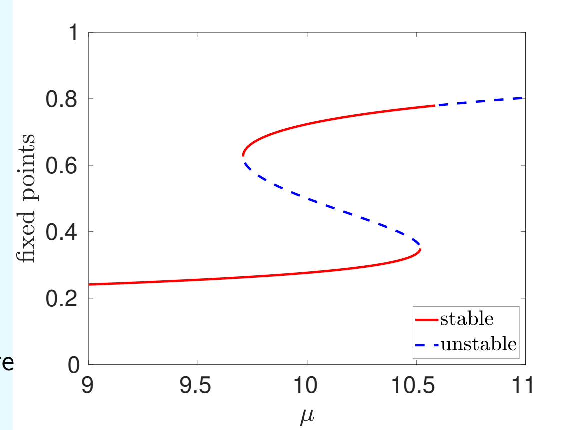

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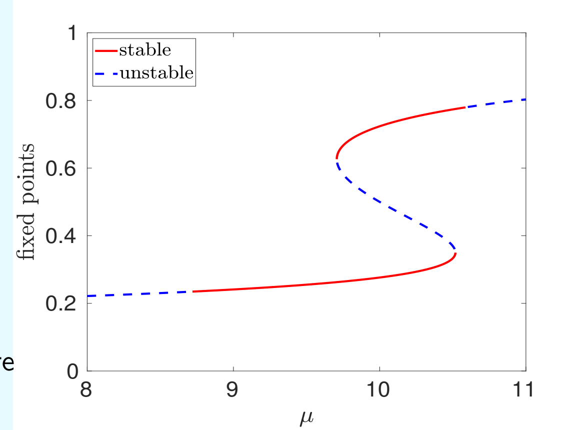

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Stability of N-cycles

Discrete-time dynamical system ($n = 1$, $m = 1$):
Let $F : \Omega \times \Theta \to \Omega$, where $\Omega \subset \mathbb{R}$ and $\Theta \subset \mathbb{R}$.
Let $x_0 \in \Omega$, $\mu \in \Theta$ and $x_k \in \Omega$ be defined iteratively by

$$x_{k+1} = F(x_k; \mu) = F_\mu(x_k)$$

- $\alpha \in \Omega$ is a *periodic point* with *period* $N \in \mathbb{N}$ if

  $$\alpha = F_\mu^{(N)}(\alpha) \text{ and } \alpha \neq F_\mu^{(k)}(\alpha) \text{ for } k = 1, 2, \ldots, N-1$$

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- N-cycle is *unstable* if $\left|F'_\mu(x_0)\, F'_\mu(x_1)\, \ldots\, F'_\mu(x_{N-1})\right| > 1$

Example: logistic map $x_{k+1} = \mu\, x_k\,(1 - x_k)$

$F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$, $\Theta = [0,4]$ and $F(x;\mu) = F_\mu(x) = \mu\, x\,(1-x)$

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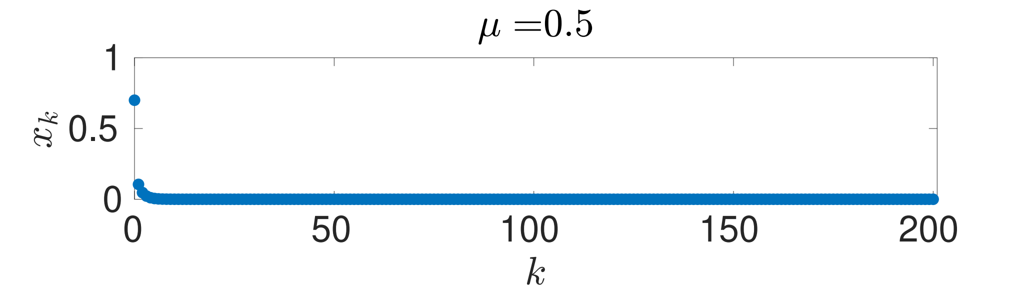
Problem Sheet 0 Question 3: Starting with $x_0 = 0.7 \in \Omega$, we calculate x_k, for $k = 0, 1, 2, \ldots, 200$, by $x_{k+1} = F(x_k; \mu) = F_\mu(x_k)$ as follows:



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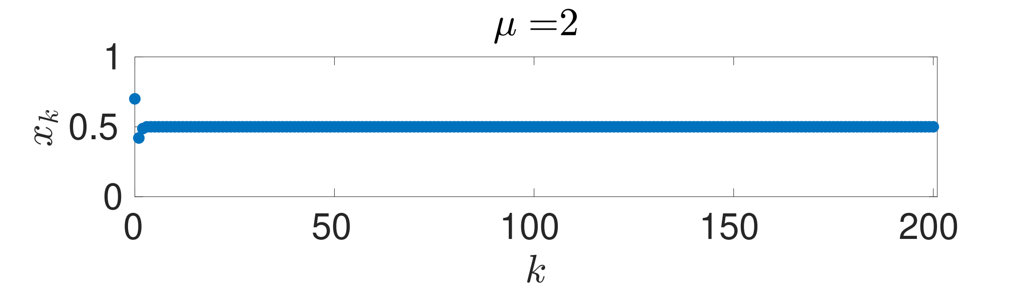

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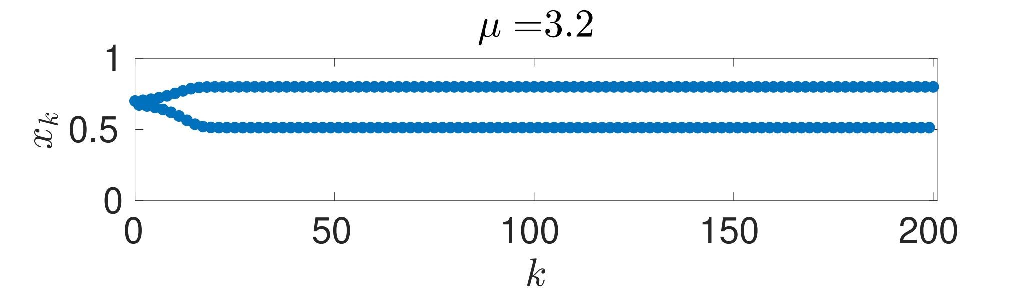

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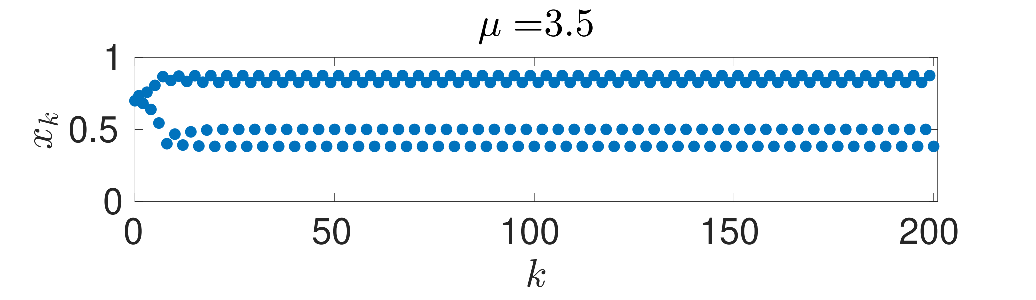

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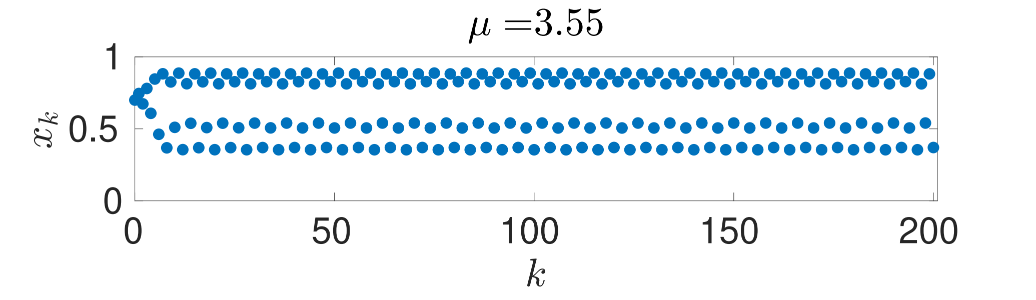

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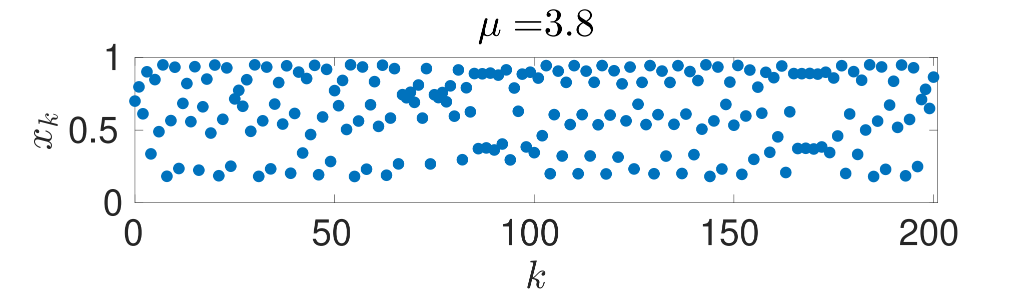

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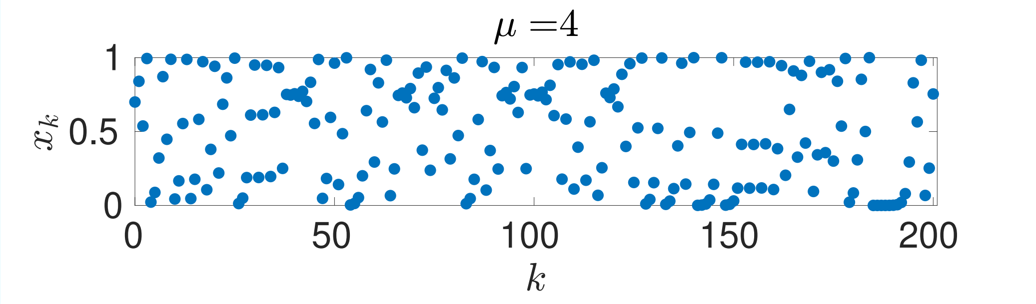

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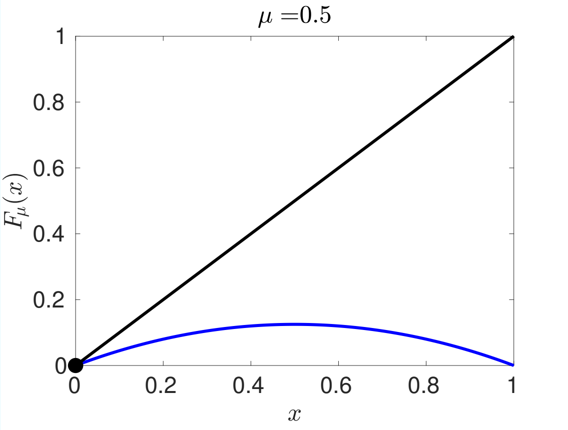
$\alpha_1 = 0, \quad F'_\mu(\alpha_1) = \mu$

α_1 is asymptotically stable for $\mu \in [0, 1]$



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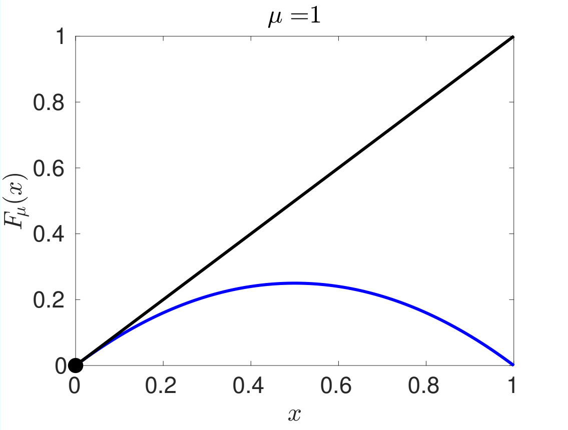

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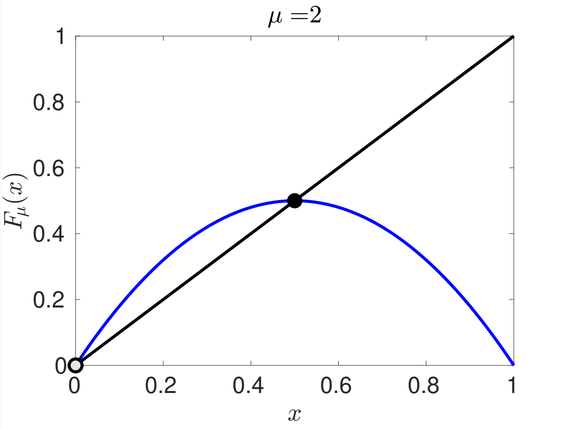
$\alpha_1 = 0$ is unstable for $\mu > 1$

$\alpha_2 = 1 - \dfrac{1}{\mu}, \quad F'_\mu(\alpha_2) = 2 - \mu$

α_2 is asymptotically stable for $\mu \in [1, 3]$

α_2 is unstable for $\mu > 3$

α_2 is super-attracting for $\mu = 2$



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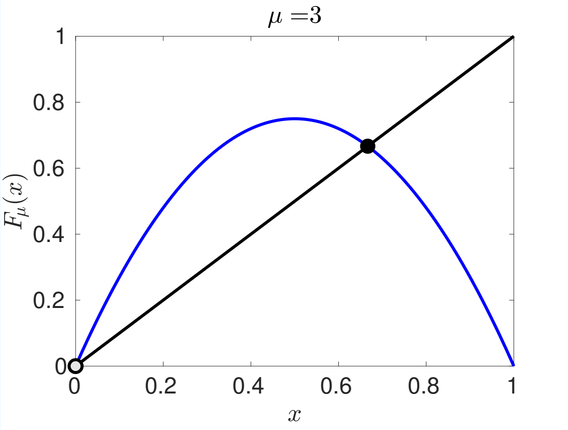

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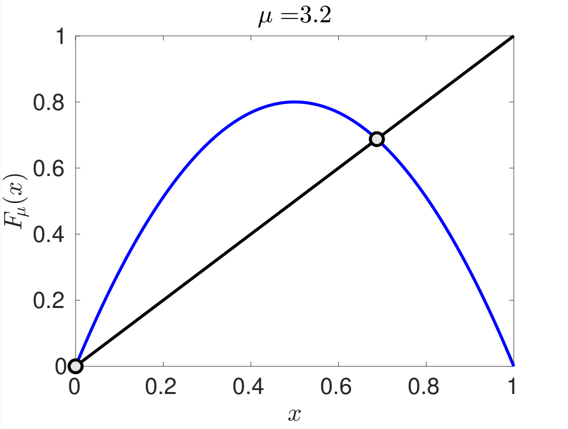

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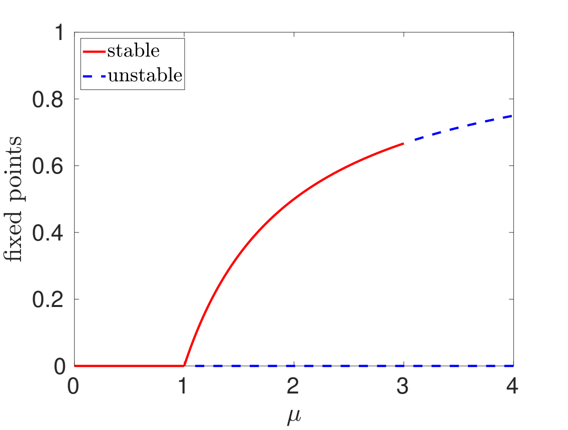

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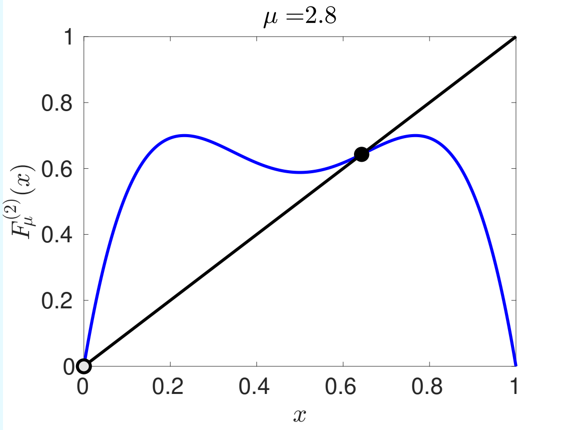

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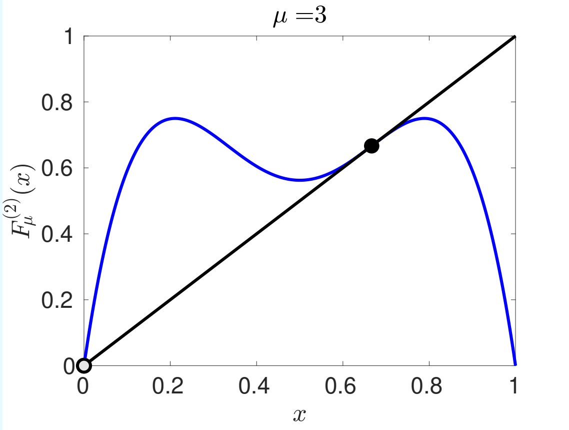
$\{c_-, c_+\}$ for

$$c_\pm = \frac{1 + \mu \pm \sqrt{(\mu-3)(\mu+1)}}{2\,\mu}$$



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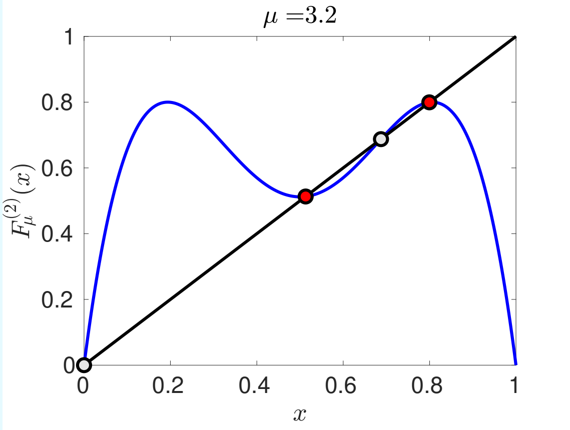

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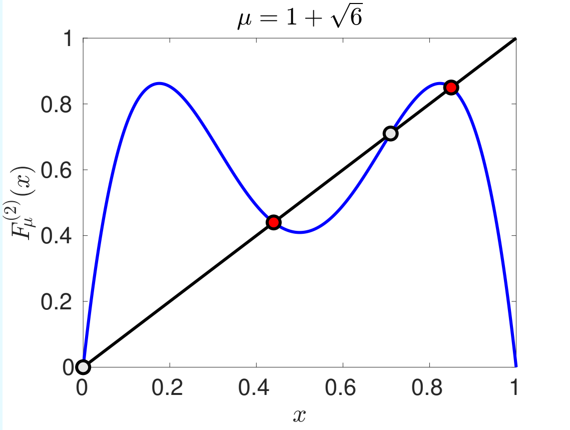

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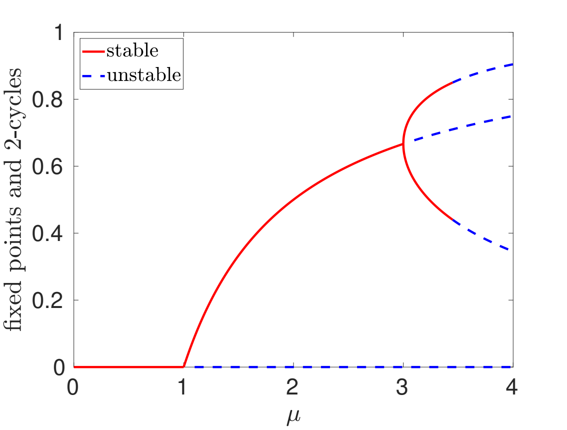

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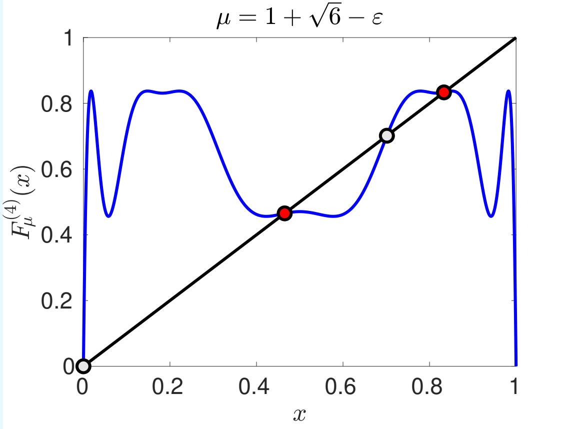

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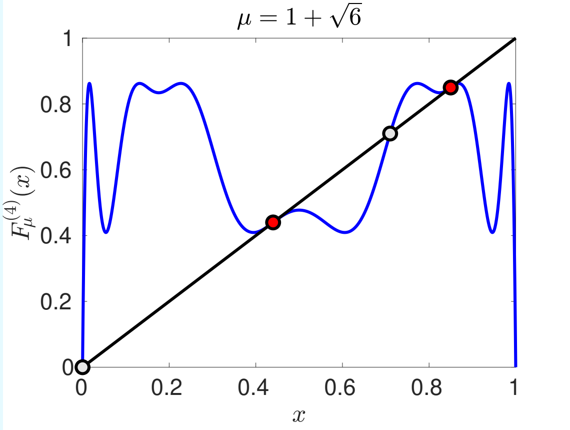

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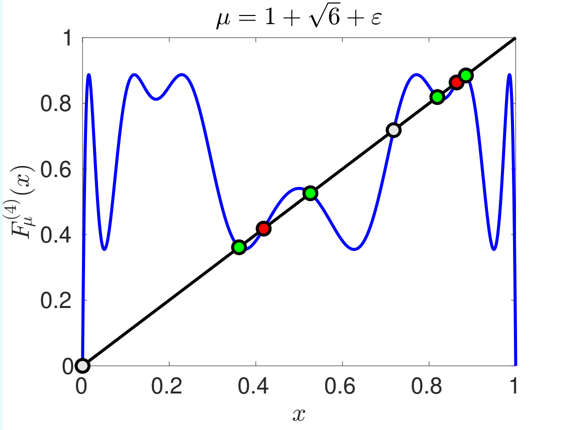
4-cycle exists and is asymptotically stable for $\mu \in \left(1 + \sqrt{6},\, 3.544090\ldots\right]$



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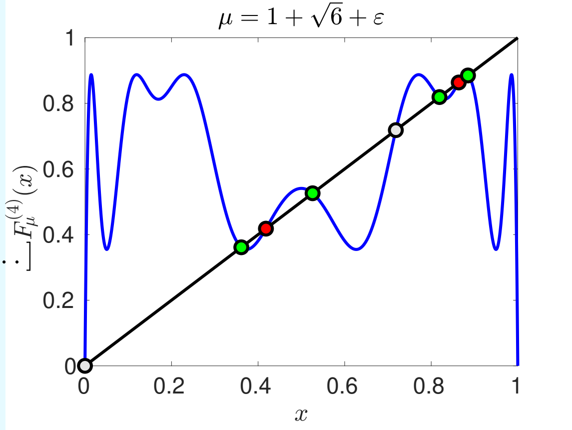
8-cycle exists and is asymptotically stable for $\mu \in (3.544090\ldots,\, 3.564407\ldots]$

this is called the period doubling route to chaos

additional example:
Question 3 on Problem Sheet 2



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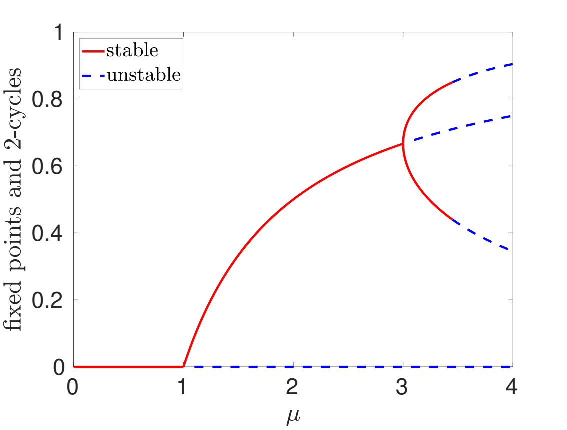
bifurcation diagram



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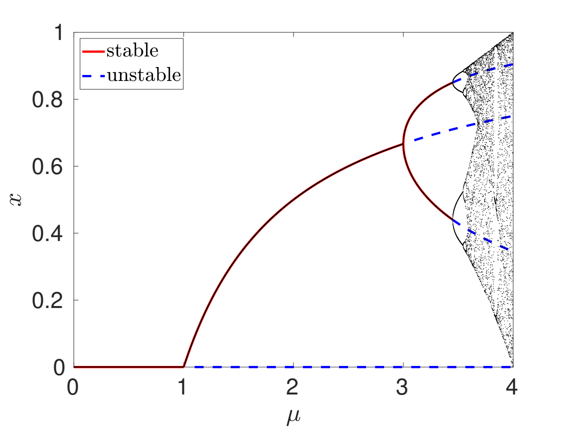

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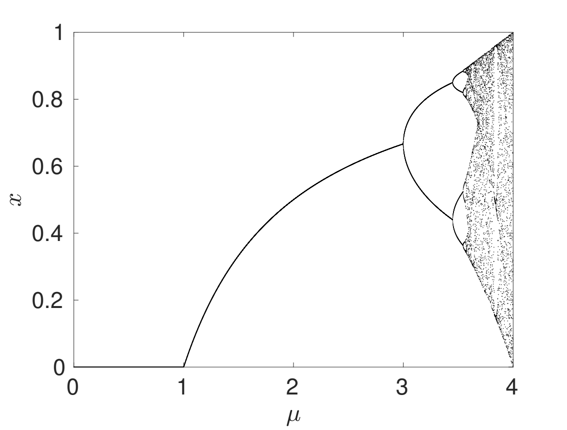
bifurcation diagram

α_2 is asymptotically stable for $\mu \in (1,3]$

asymptotically stable 2-cycle exists for $\mu \in \left(3, 1+\sqrt{6}\right]$

asymptotically stable 4-cycle exists for $\mu \in \left(1+\sqrt{6},\, 3.544090\ldots\right]$

asymptotically stable 8-cycle exists for $\mu \in (3.544090\ldots,\, 3.564407\ldots]$



16-cycle, 32-cycle, 64-cycle, . . .

period doubling route to chaos

Example: logistic map $x_{k+1} = \mu\, x_k\, (1 - x_k)$

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bifurcation diagram

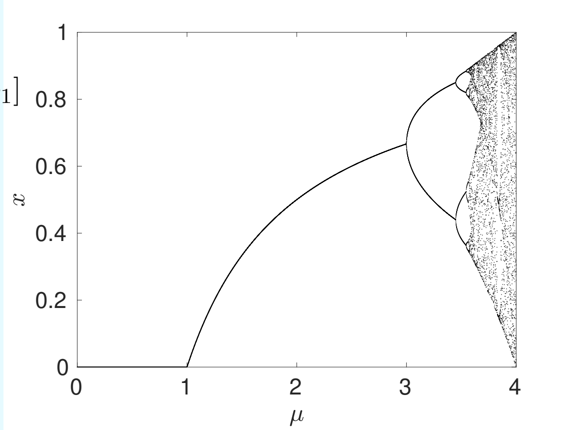
α_2 is asymptotically stable for $\mu \in (1, b_1]$

asymptotically stable 2-cycle exists for $\mu \in (b_1, b_2]$

asymptotically stable 2^k-cycle exists for $\mu \in (b_k, b_{k+1}]$

Feigenbaum's constant:

$$\lim_{k\to\infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k} = 4.6692016\ldots$$



additional example: Question 3 on Problem Sheet 2

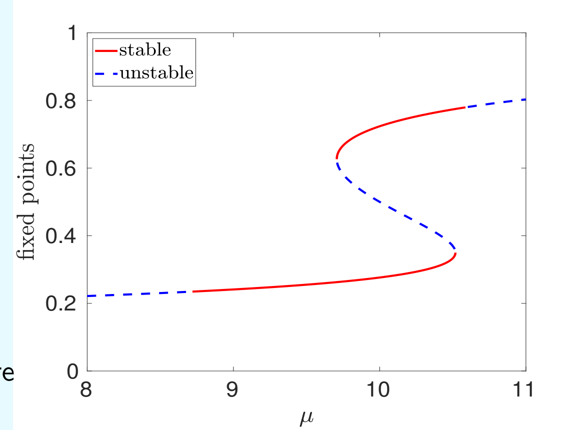
Example (from Lecture 5)

$$x_{k+1} = (1 - x_k)(1 - 5\,x_k + \mu\,x_k^2)$$

$$F(x;\mu) = (1 - x)(1 - 5\,x + \mu\,x^2)$$

If $\mu \in \Theta = [6.3, 11.8]$,
then $F(x;\mu) \in [0,1]$ for all $x \in [0,1]$.

We have studied dynamics
of $F : \Omega \times \Theta \to \Omega$, where $\Omega = [0,1]$.



three fixed points for $\mu \in (\mu_1, \mu_2)$ where
$\mu_1 = 9.7066\ldots$ and $\mu_2 = 10.518\ldots$

one fixed point for $\mu < \mu_1$ and $\mu > \mu_2$

we have saddle-node bifurcations at $\mu = \mu_1$ and $\mu = \mu_2$

we also have period doubling bifurcations at $\mu \approx 8.71988\ldots$ and $\mu \approx 10.5877\ldots$

Example (from Lecture 5) – bifurcation diagram

$$x_{k+1} = (1 - x_k)(1 - 5\,x_k + \mu\,x_k^2)$$

$$F(x;\mu) = (1 - x)(1 - 5\,x + \mu\,x^2)$$

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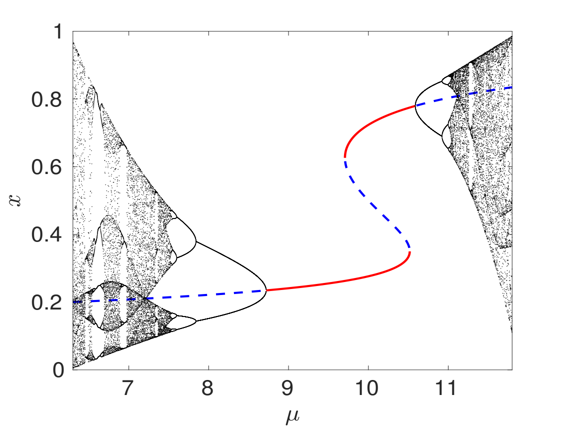

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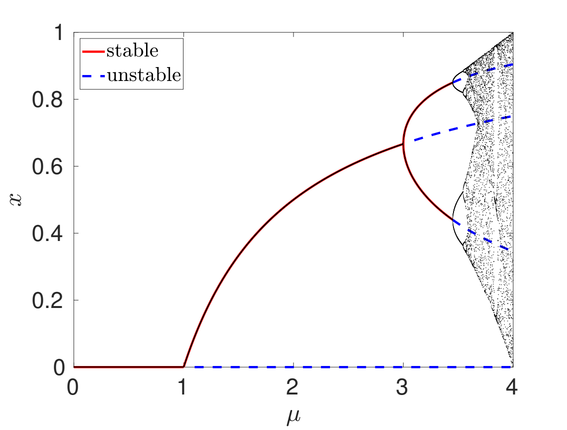
we also have period doubling bifurcations at $\mu \approx 8.71988\ldots$ and $\mu \approx 10.5877\ldots$

3-cycles: logistic map $x_{k+1} = \mu\, x_k\,(1 - x_k)$

asymptotically stable 2^k-cycle exists
for $\mu \in (b_k, b_{k+1}]$ where

$$b_1 = 3$$
$$b_2 = 1 + \sqrt{6}$$
$$b_3 = 3.544090\ldots$$
$$b_4 = 3.564407\ldots$$

$$\lim_{k\to\infty} b_k = 3.56994567\ldots$$



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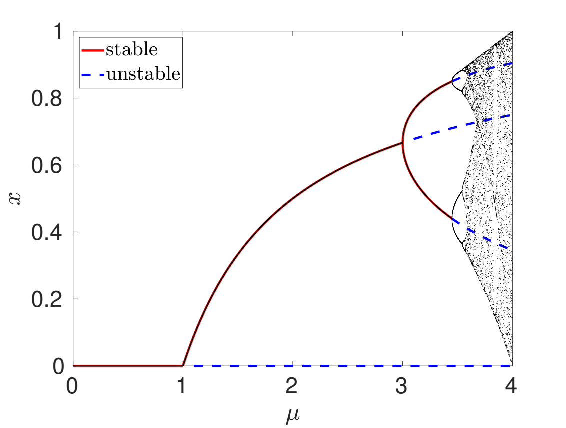

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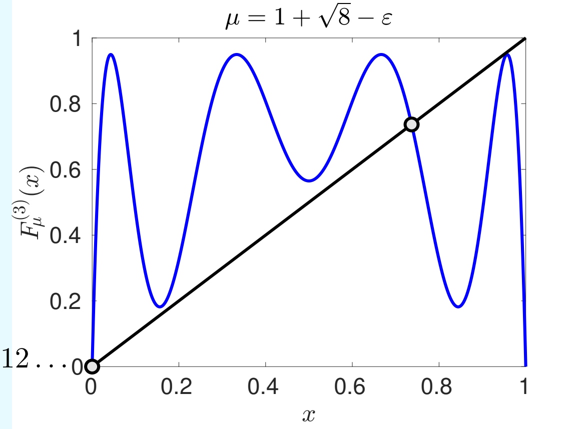
no 3-cycles for $\mu < 1 + \sqrt{8} = 3.82842712\ldots$



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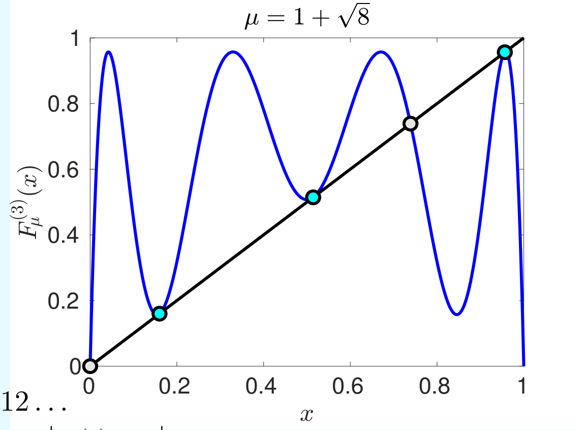

$$\{c_1, c_2, c_3\} = \left\{c_1, F_\mu(c_1), F_\mu^{(2)}(c_1)\right\} \text{ where } \left|F_\mu^{(3)}(c_1)\right| = \left|F_\mu'(c_1)\, F_\mu'(c_2) F_\mu'(c_3)\right| = 1$$

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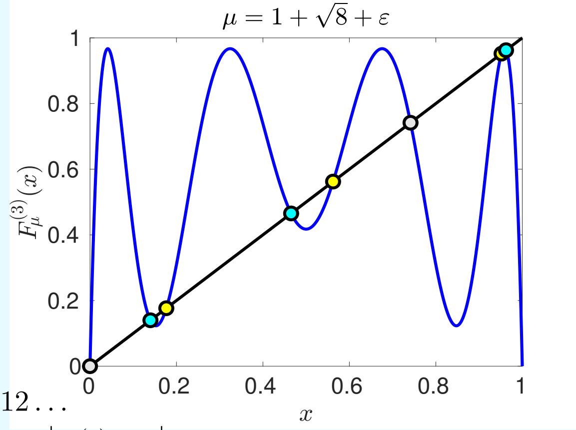

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two 3-cycles for $\mu \in \left(1 + \sqrt{8},\, 4\right]$ $\ldots$ 'cyan 3-cycle' and 'yellow 3-cycle'

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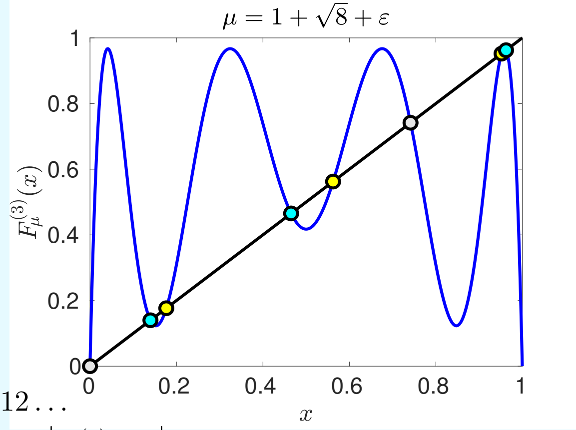

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two 3-cycles for $\mu \in \left(1 + \sqrt{8},\ 4\right]$ ... 'cyan 3-cycle' and 'yellow 3-cycle'

'cyan 3-cycle' is stable for $\mu = 1 + \sqrt{8} + \varepsilon$ for sufficiently small ε

'cyan 3-cycle' is super-attracting for $\varepsilon = 0.00344693\ldots$, i.e. for $\mu = 3.831874\ldots$

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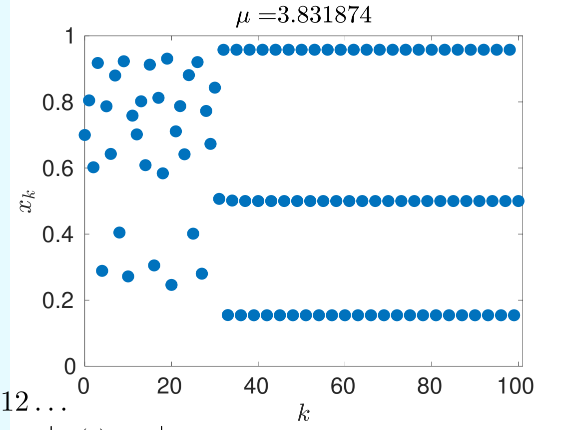

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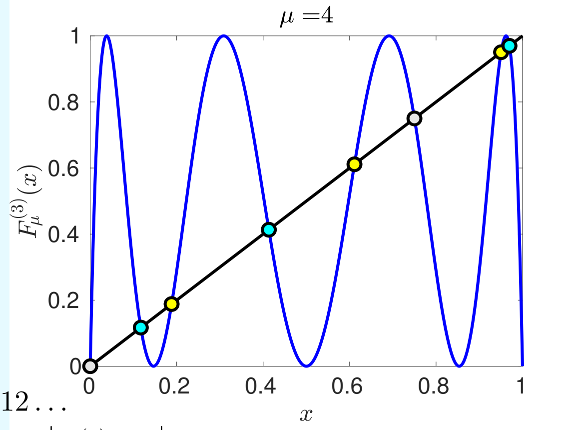

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two 3-cycles for $\mu \in \big(1 + \sqrt{8},\, 4\big]$ ... 'cyan 3-cycle' and 'yellow 3-cycle'

Question 5 on Problem Sheet 2: closed formulas for both 3-cycles derived for $\mu = 4$

both 3-cycles are unstable because $F_\mu^{(3)}(c_1) = F'_\mu(c_1)\,F'_\mu(c_2)F'_\mu(c_3) = \pm 2^3 = \pm 8$

Sharkovsky's Theorem

Sharkovsky's ordering:

$$3 \triangleright 5 \triangleright 7 \triangleright \ldots \triangleright 2 \times 3 \triangleright 2 \times 5 \triangleright \ldots \triangleright 2^2 \times 3 \triangleright 2^2 \times 5 \triangleright \ldots \triangleright 2^3 \times 3 \triangleright 2^3 \times 5 \triangleright \ldots$$

$$\ldots \triangleright 2^n \times 3 \triangleright 2^n \times 5 \triangleright \ldots \triangleright 2^n \triangleright 2^{n-1} \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1$$

Sharkovsky's Theorem (1964):
Let $\Omega = [a, b] \subset \mathbb{R}$ be an interval and $F : \Omega \to \Omega$ be continuous.
If F has a point of period n, then it has points of period k for all $k \in \mathbb{N}$ with $n \triangleright k$.

Sharkovsky's Theorem

Sharkovsky's ordering:

$$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2 \times 3 \rhd 2 \times 5 \rhd \ldots \rhd 2^2 \times 3 \rhd 2^2 \times 5 \rhd \ldots \rhd 2^3 \times 3 \rhd 2^3 \times 5 \rhd \ldots$$

$$\ldots \rhd 2^n \times 3 \rhd 2^n \times 5 \rhd \ldots \rhd 2^n \rhd 2^{n-1} \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

Sharkovsky's Theorem (1964):
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If F has a point of period n, then it has points of period k for all $k \in \mathbb{N}$ with $n \rhd k$.

We have shown that the logistic map $x_{k+1} = \mu\, x_k\, (1 - x_k)$ has 3-cycles (points of period 3) for any $\mu \in \left[1 + \sqrt{8},\, 4\right]$.

Sharkovsky's theorem implies that the logistic map has points of period k (i.e. k-cycles) for all $k \in \mathbb{N}$ for $\mu \in \left[1 + \sqrt{8},\, 4\right]$.

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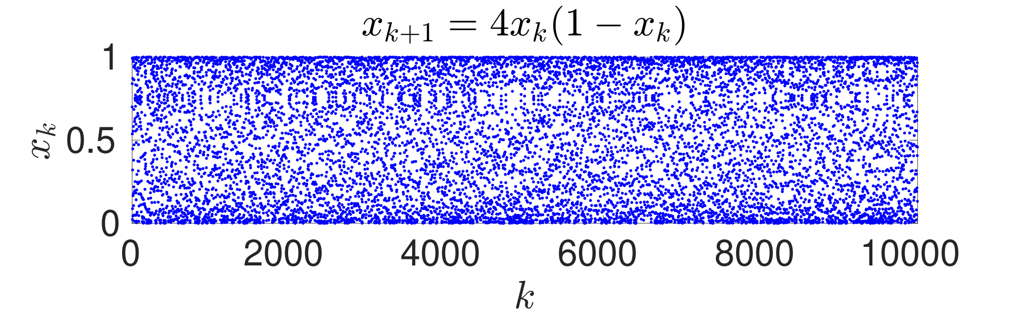
Sharkovsky's theorem implies that the logistic map has points of period k (i.e. k-cycles) for all $k \in \mathbb{N}$ for $\mu \in \left[1 + \sqrt{8},\, 4\right]$.

Question 5 on Problem Sheet 2: closed formulas for k-cycles can be derived for $\mu = 4$, we can also show that k-cycles are unstable by calculating the corresponding derivatives

Invariant distribution (Question 7 on Problem Sheet 2)

Questions 3 and 4 on Problem Sheet 0: Starting with $x_0 = 0.7$, we obtain x_k as:



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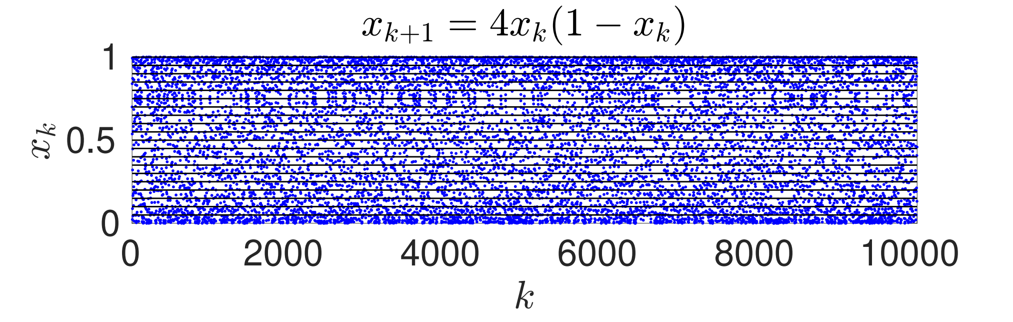
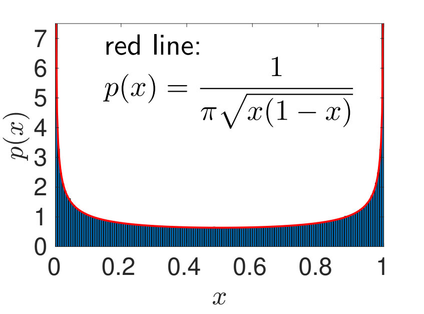


Invariant distribution (Question 7 on Problem Sheet 2)

Histogram of values x_k, for $k = 0, 1, 2, \ldots, 10^6$ (blue bars): $x_{k+1} = 4\,x_k\,(1 - x_k)$



Question 4 on Problem Sheet 0:

Let X_k be a continuous random variable on interval $[0, 1]$ with the probability density function $p(x)$. Then the random variable $X_{k+1} = F(X_k) = 4\,X_k\,(1 - X_k)$ has the same probability density function $p(x)$.

[Prelims Probability and Calculus]

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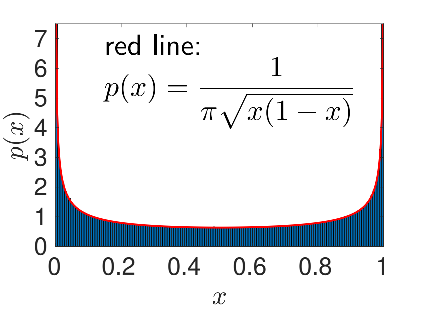

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[Prelims Probability and Calculus]

invariant distribution $p(x)$: if the random variable X is distributed according to $p(x)$, then the random variable $F(X)$ is also distributed according to $p(x)$

Question 7 on Problem Sheet 2: calculate invariant distributions for some other chaotic maps and compare them with the histograms of orbits

theoretical justification is given by ergodic theory (Birkhoff ergodic theorem)

Problem Sheet 2: bifurcations of continuous-time dynamical systems

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x};\boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Questions 1, 2, 4 and 6 on Problem Sheet 2 cover bifurcations of fixed points, which can occur for $n \geq 1$ and $m \geq 1$:

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

We have explained them in our lectures on examples with $n = 1, 2$ and $m = 1$.

Next, we will discuss some additional examples to help you solve Problem Sheet 2, including examples with $m \geq 2$ and $n = 3$.

Example: $n = 1$, $m = 2$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

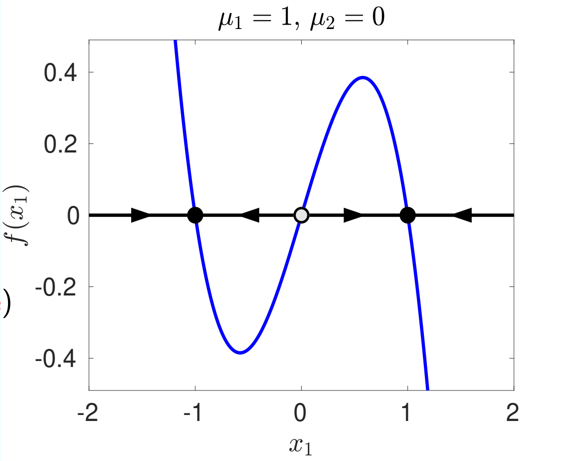
$\mu_2 = 0$: supercritical pitchfork bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

$\mu_1 > 0$, $\mu_2 = 0$

three fixed points at $x_1 = \pm\sqrt{\mu_1}$ (stable)
and $x_1 = 0$ (unstable)



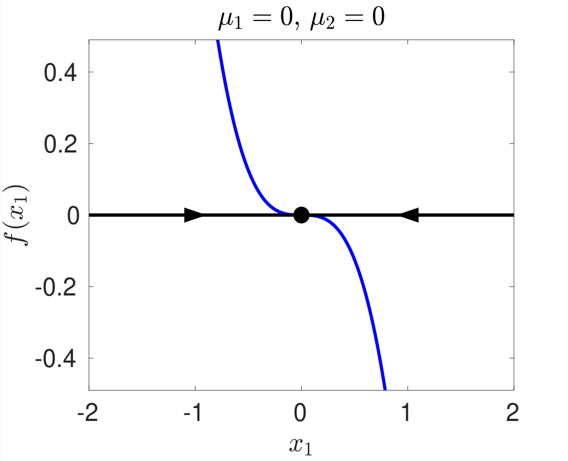
$\mu_2 = 0$: supercritical pitchfork bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 \, x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

as μ_1 approaches zero from above, two fixed points $\sqrt{\mu_1}$ and $-\sqrt{\mu_1}$ move toward the third one

$\mu_1 = 0$: the fixed points coalesce into a stable fixed point at $x_1 = 0$



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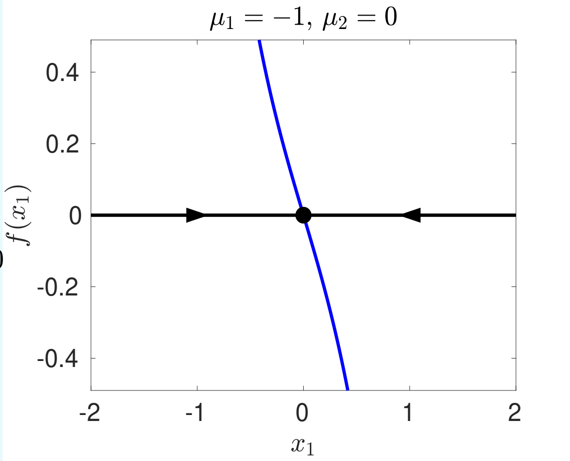


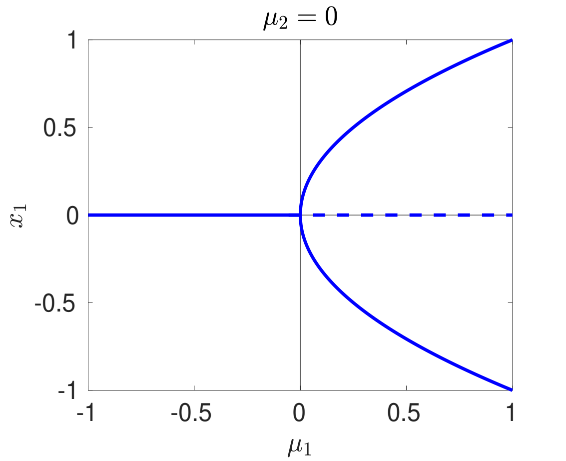
$\mu_1 < 0$: one stable fixed point at $x_1 = 0$

$\mu_2 = 0$: supercritical pitchfork bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 \, x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

bifurcation diagram



$\mu_2 = 0$: supercritical pitchfork bifurcation $(n = 2)$

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1\, x_1 - x_1^3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_2$$

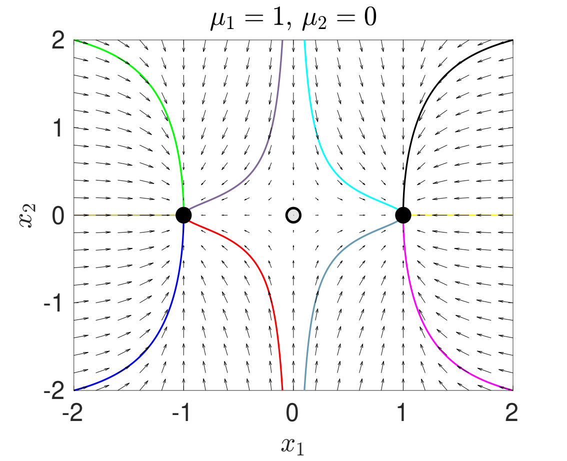
$\mu_1 > 0,\ \mu_2 = 0$

three fixed points at

$\mathbf{x} = [-\sqrt{\mu_1}, 0]$ (stable node)

$\mathbf{x} = [0, 0]$ (saddle)

$\mathbf{x} = [\sqrt{\mu_1}, 0]$ (stable node)



$\mu_2 = 0$: supercritical pitchfork bifurcation ($n = 2$)

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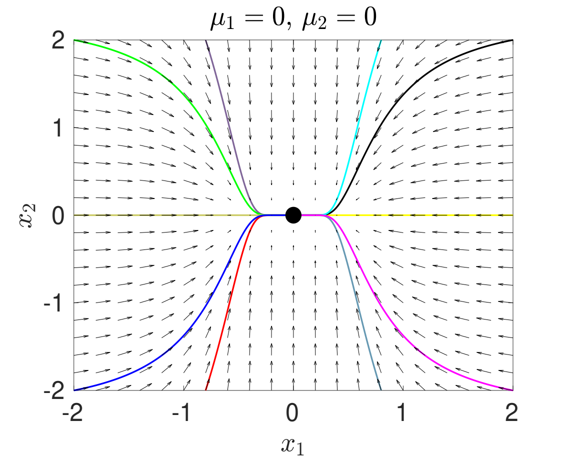
as μ_1 approaches zero from above, two fixed points $[-\sqrt{\mu_1}, 0]$ and $\sqrt{\mu_1}, 0]$ move toward the third one

$\mu_1 = 0$: the fixed points coalesce into a stable fixed point at $\mathbf{x} = [0, 0]$



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$\mu < 0$:
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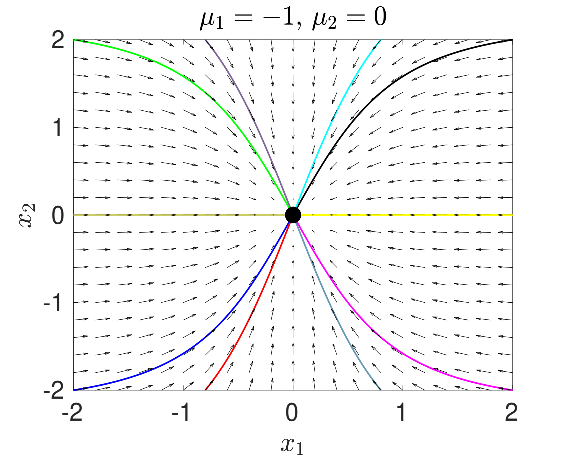

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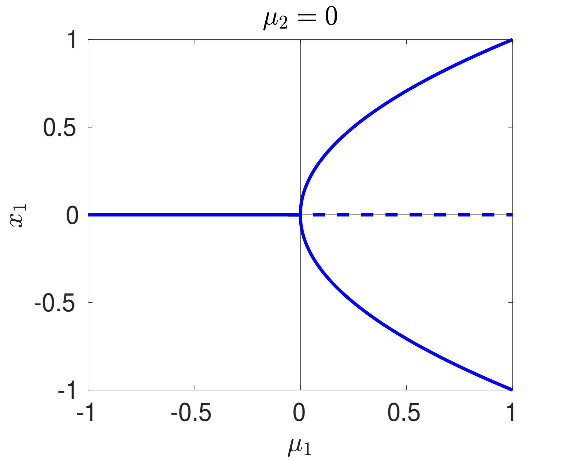
bifurcation diagram



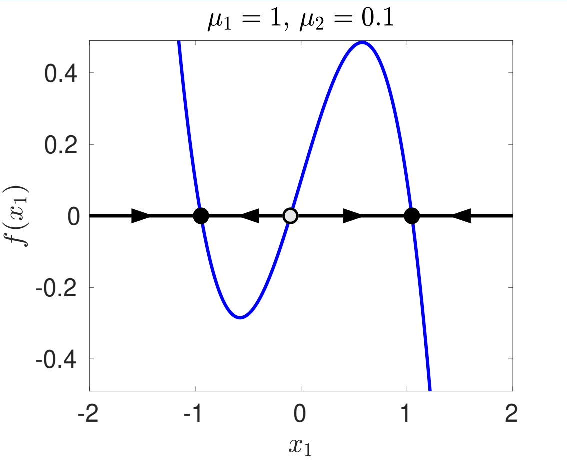
Cusp catastrophe ($\mu_2 = 0.1$)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

$\mu_1 = 1, \ \mu_2 = 0.1$

three fixed points given as solutions of $\mu_2 + \mu_1 x_1 - x_1^3 = 0$



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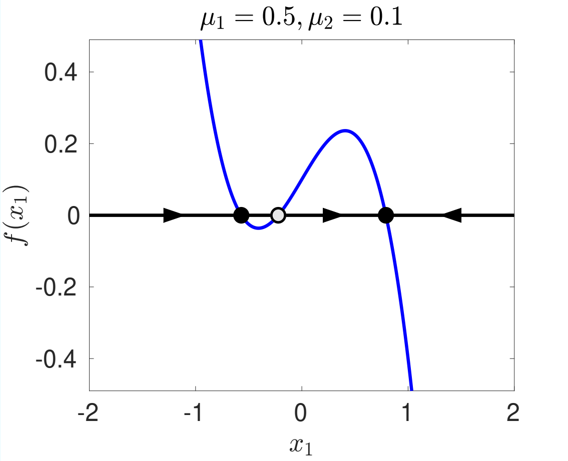
$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

$\mu_1 = 0.5,\ \mu_2 = 0.1$

three fixed points given as solutions of $\mu_2 + \mu_1 x_1 - x_1^3 = 0$



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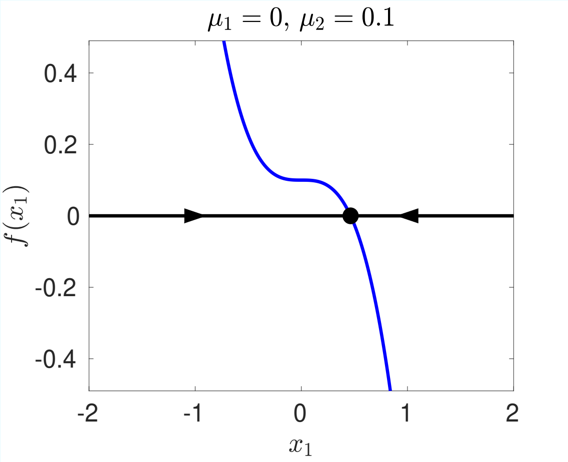


as μ_1 approaches the bifurcation value

$$\mu_c = \left(\frac{27\mu_2^2}{4}\right)^{1/3}$$

from above, two (smaller) fixed points move toward each other (saddle-node bifurcation)

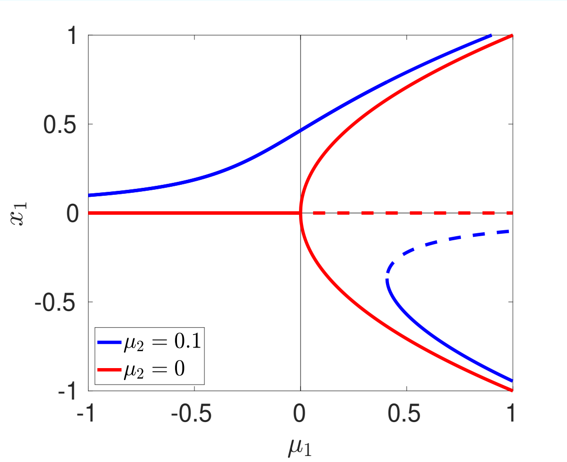
$\mu_1 < \mu_c$: one stable fixed point

Cusp catastrophe ($\mu_2 = 0.1$)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 + \mu_1 x_1 - x_1^3$$

$$f(x_1; \boldsymbol{\mu}) = \mu_2 + \mu_1 x_1 - x_1^3$$

bifurcation diagram



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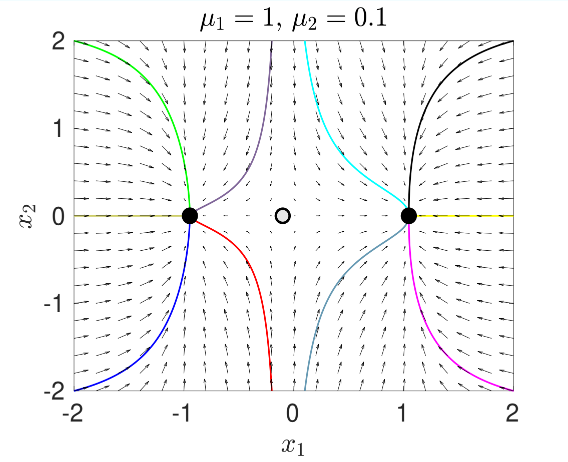
$\mu_1 > \mu_c$: three fixed points given as solutions of $\mu_2 + \mu_1 x_1 - x_1^3 = 0$



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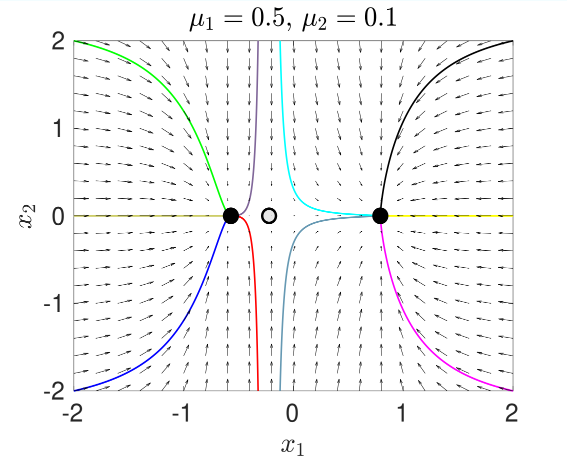

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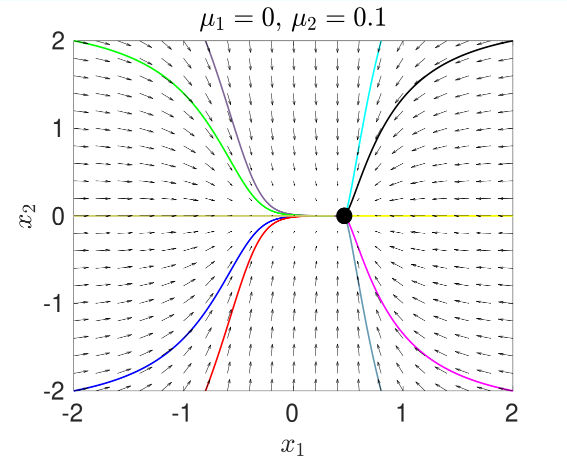

as μ_1 approaches the bifurcation value

$$\mu_c = \left(\frac{27\mu_2^2}{4}\right)^{1/3}$$

from above, two (smaller) fixed points move toward each other (saddle-node bifurcation)

$\mu_1 < \mu_c$: one stable fixed point

Cusp catastrophe

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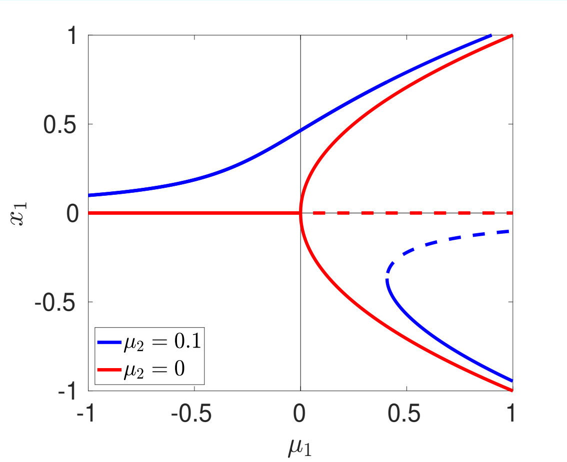

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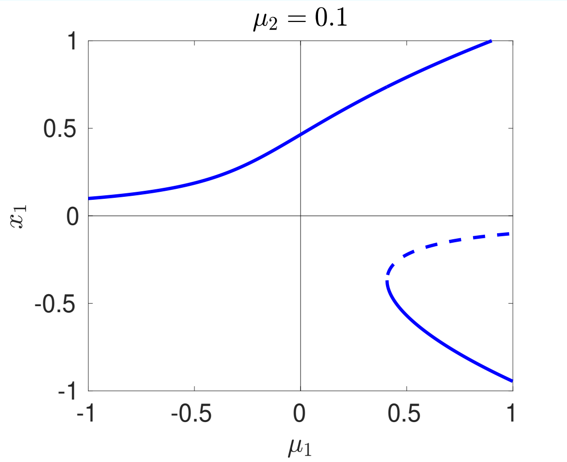

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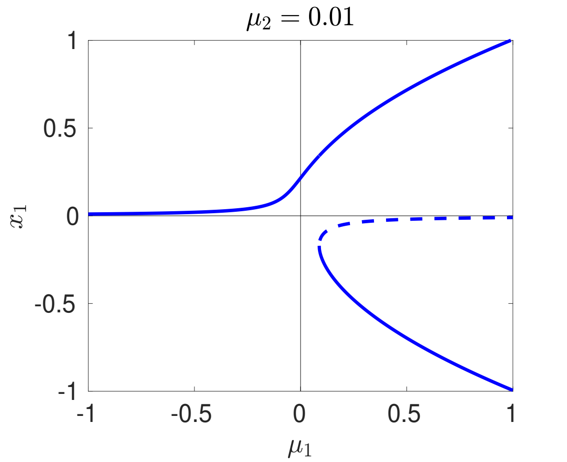

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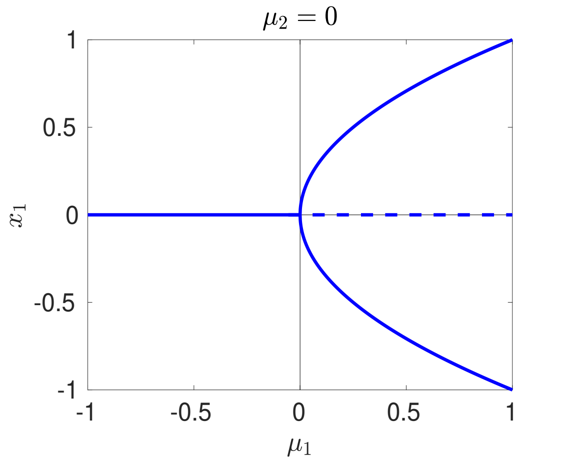

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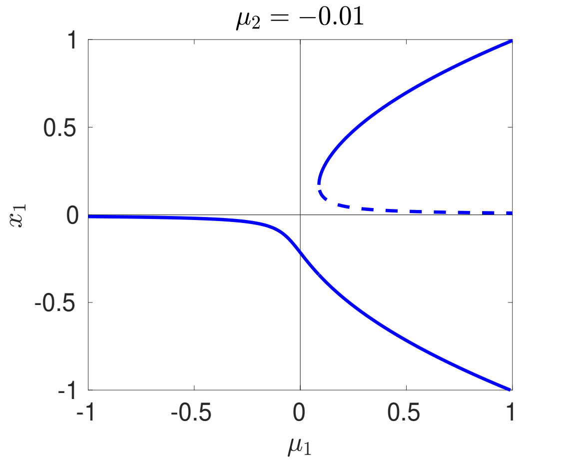

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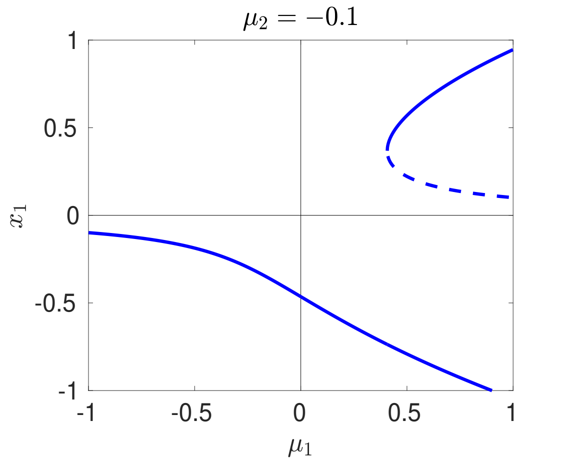

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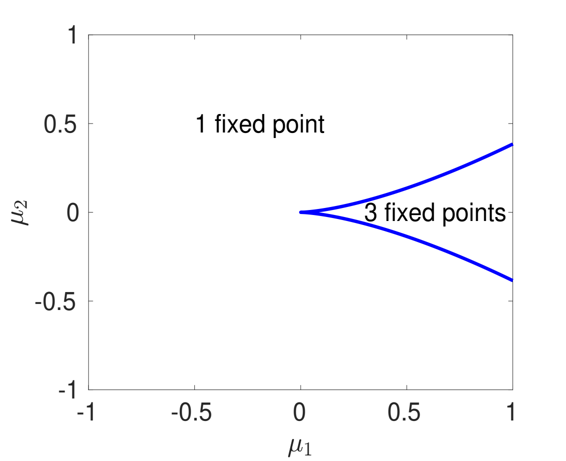

Example: $n = 2$, $m = 2$

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu_2 + \mu_1 x_1 - x_1^2$$

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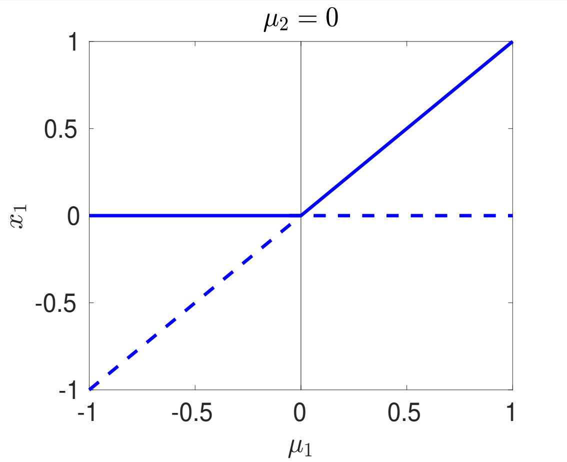
$\mu_2 = 0$: transcritical bifurcation



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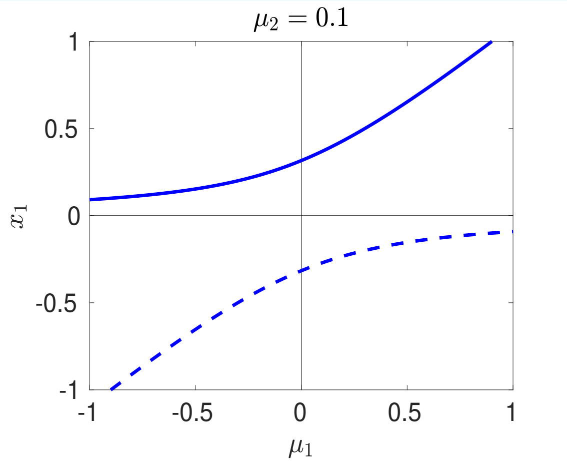
The saddle-node bifurcation is robust under small changes of parameters, but transcritical and pitchfork bifurcations change under small perturbations.



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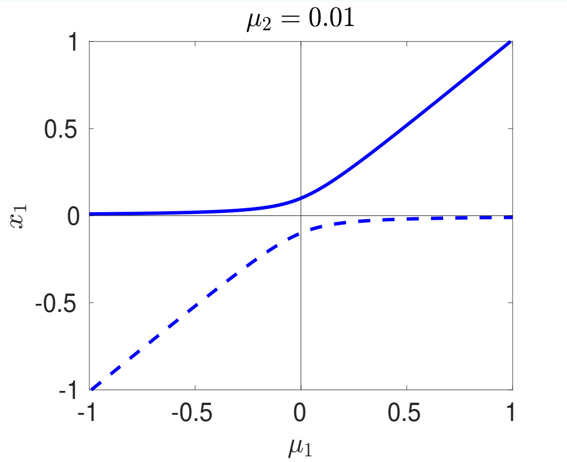

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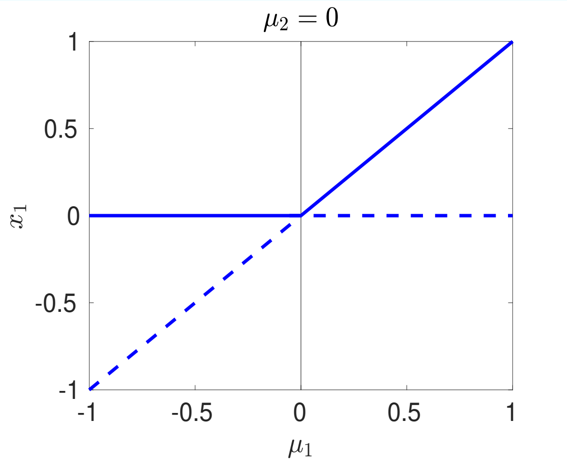

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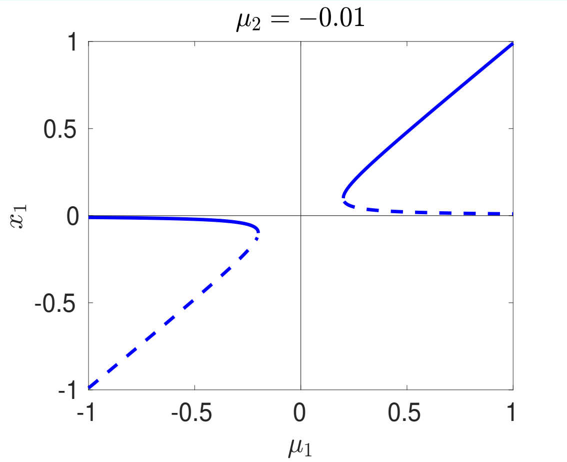

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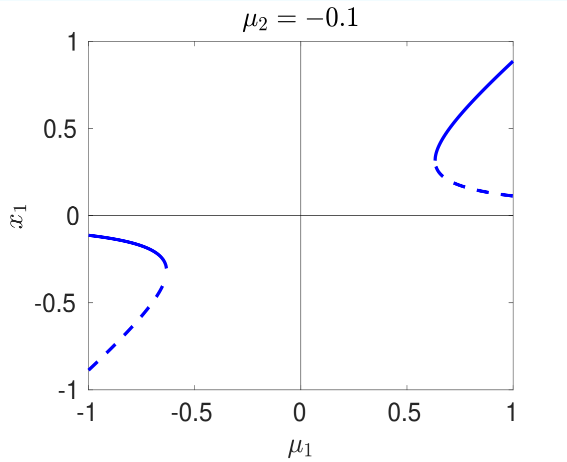

Example with $n = 3$, $m = 3$: Lorenz equations (Part 1)

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \mu_3\,x_3$$

- Lecture 8: we started with a 3D demonstration viewing trajectories in the phase space for different values of parameters μ_1, μ_2, μ_3 and illustrating the convergence to fixed points, limit cycles, chaos, strange attractor and transient chaos
- derivation on whiteboard (no slides): we used the Lorenz system in Lecture 8 to further practice techniques on Problem Sheets 1 and 2 including:
  - finding the Lyapunov function to prove the global stability of the fixed point at origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$
  - using the extended center manifold theory to analyze the supercritical pitchfork bifurcation at $\mu_1 = 1$, calculating the center manifold and the dynamics on it
- Part 2: we will consider the Lorenz system again when we discuss chaos in ODEs

Bifurcations

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Bifurcations of fixed points

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$. Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

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Problem Sheet 2: bifurcations of fixed points

they can occur for $n \geq 1$, we studied examples with $n = 1$, $n = 2$ and $n = 3$

- saddle-node bifurcation
- transcritical bifurcation
- supercritical pitchfork bifurcation
- subcritical pitchfork bifurcation

Bifurcations of limit cycles

Continuous-time dynamical system: Let $\mathbf{f} : \Omega \times \Theta \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ and $\Theta \subset \mathbb{R}^m$.
Let $\mathbf{x}_0 \in \Omega$, $\boldsymbol{\mu} \in \Theta$ and $\mathbf{x}(t) \in \Omega$ be a solution of the ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}; \boldsymbol{\mu}) \quad \text{with the initial condition} \quad \mathbf{x}(0) = \mathbf{x}_0 \in \Omega$$

Bifurcations: The qualitative structure of the flow can change as parameters $\boldsymbol{\mu}$ are varied. For example, critical points (fixed points) can be created or destroyed, or limit cycles can be created or destroyed. The parameter values at which these qualitative changes in the dynamics occur are called bifurcation points.

Problem Sheet 3: bifurcations of limit cycles

they can occur for $n \geq 2$, we will first explain them on the case $n = 2$

- supercritical Hopf bifurcation
- subcritical Hopf bifurcation
- saddle-node bifurcation of cycles
- infinite-period bifurcation (SNIC, SNIPER)
- homoclinic bifurcation (saddle-loop bifurcation)

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \; - \; x_2 \; - \; x_1(x_1^2 + x_2^2)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = x_1 \; + \; \mu\, x_2 \; - \; x_2(x_1^2 + x_2^2)$$

Supercritical Hopf bifurcation

example:

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 \,-\, x_2 \,-\, x_1(x_1^2 + x_2^2)$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = x_1 \,+\, \mu\, x_2 \,-\, x_2(x_1^2 + x_2^2)$$

fixed point at $\mathbf{0} = [0, 0]$

linearization $D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$

eigenvalues $\lambda_{\pm} = \mu \pm i$

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \; - \; x_2 \; - \; x_1(x_1^2 + x_2^2)$$

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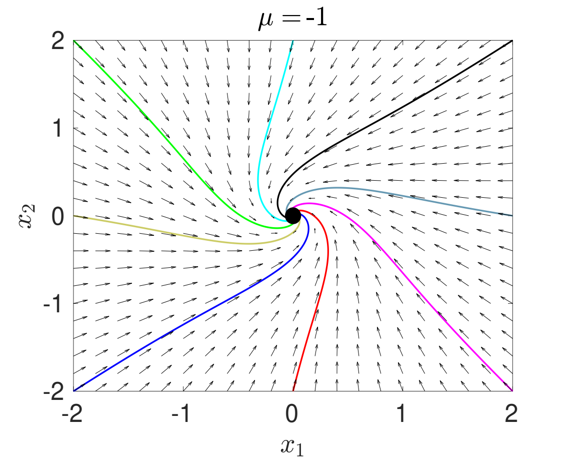

$\mu < 0$: fixed point $\mathbf{0} = [0, 0]$ is a stable spiral

Supercritical Hopf bifurcation

example:

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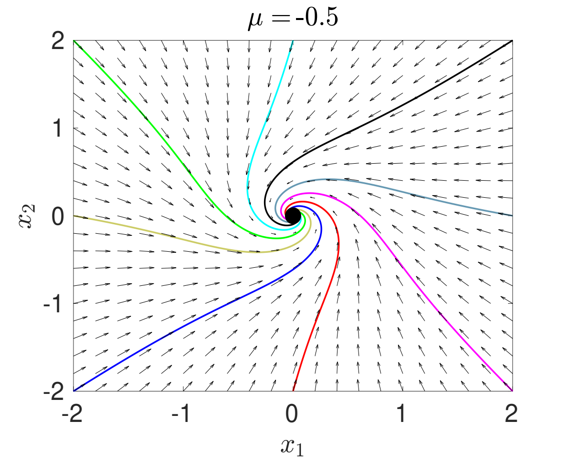

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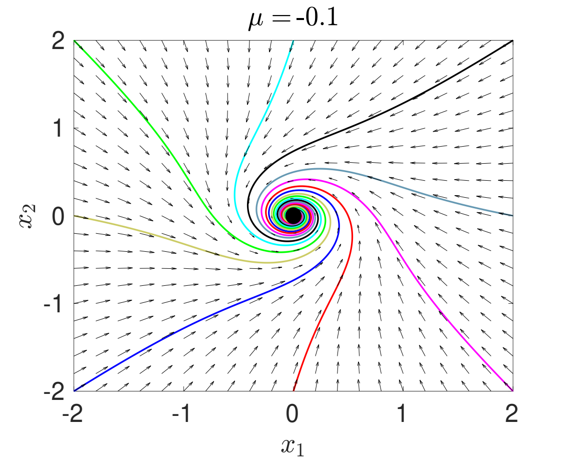

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Supercritical Hopf bifurcation

example:

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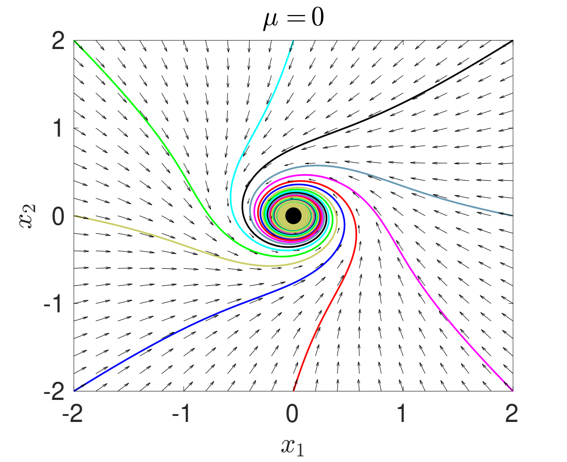

as μ increases from negative to positive values, eigenvalues cross the imaginary axis from left to right

$\mu = 0$: fixed point $\mathbf{0} = [0, 0]$ is a still stable spiral, though a very weak one

Supercritical Hopf bifurcation

example:

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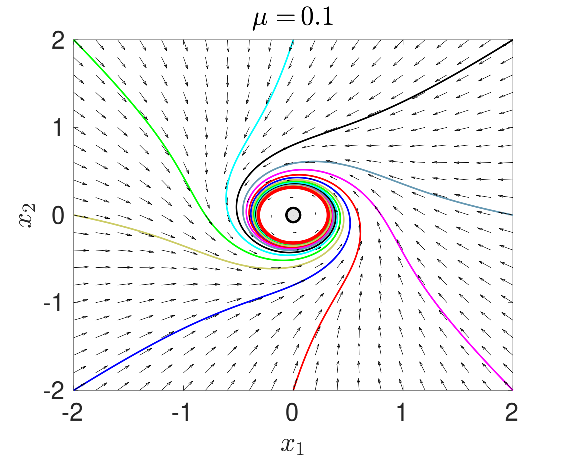

$\mu > 0$: fixed point $\mathbf{0} = [0,0]$ is an unstable spiral

stable circular limit cycle of radius $r = \sqrt{\mu}$

Supercritical Hopf bifurcation

example:

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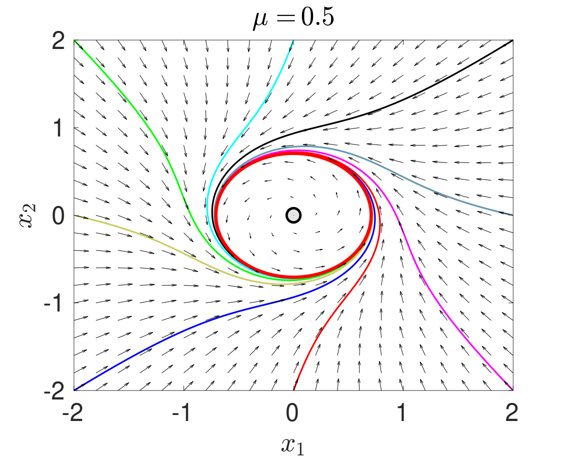

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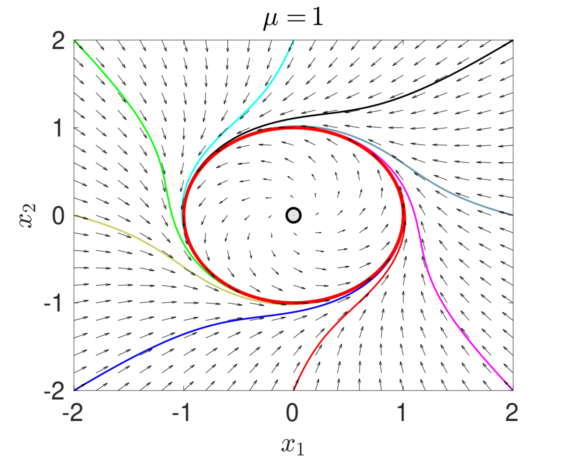

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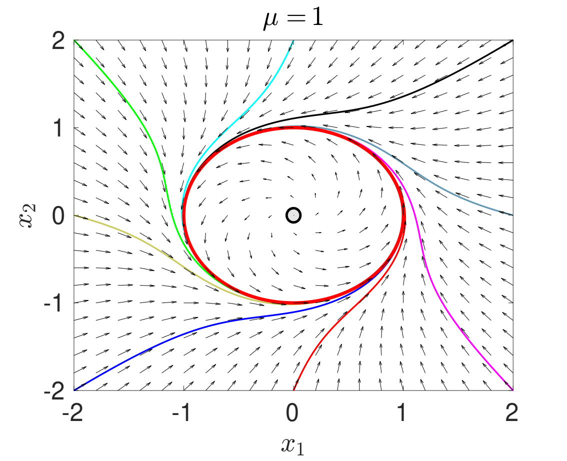

We transform the ODEs to polar coordinates by using variables $r(t)$ and $\theta(t)$, where $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. We obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r(\mu - r^2) \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

Supercritical Hopf bifurcation

example:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 \;-\; x_2 \;-\; x_1(x_1^2 + x_2^2)$$

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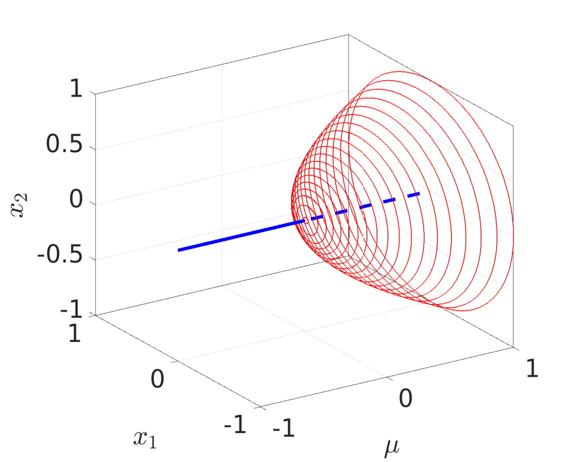

bifurcation diagram

[show 3D animation]

Hopf bifurcation - general case

general case: eigenvalues $\lambda(\mu) = \alpha(\mu) \pm i\,\omega(\mu)$ with $\alpha(0) = 0$ and $\omega(0) \neq 0$

behaviour close to the fixed point: normal form (in polar coordinates)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha(\mu)\,r + a(\mu)\,r^3 + \mathcal{O}(r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(\mu) + b(\mu)\,r^2 + \mathcal{O}(r^4)$$

Taylor expanding:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r + a(0)\,r^3 + \mathcal{O}(\mu^2 r, \mu r^3, r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) + \omega'(0)\,\mu + b(0)\,r^2 + \mathcal{O}(\mu^2, \mu r^2, r^4)$$

Hopf bifurcation - general case

general case: eigenvalues $\lambda(\mu) = \alpha(\mu) \pm i\,\omega(\mu)$ with $\alpha(0) = 0$ and $\omega(0) \neq 0$

behaviour close to the fixed point: normal form (in polar coordinates)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha(\mu)\,r + a(\mu)\,r^3 + \mathcal{O}(r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(\mu) + b(\mu)\,r^2 + \mathcal{O}(r^4)$$

Taylor expanding:

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our previous example: $\alpha'(0) = 1$, $a(0) = -1$, $\omega(0) = 1$, $\omega'(0) = b(0) = 0$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Supercritical Hopf bifurcation

general case: $a(0) < 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\, r \;+\; a(0)\, r^3$$

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eigenvalues $\lambda_{\pm} = \alpha'(0)\,\mu \,\pm\, i\,\omega(0)$

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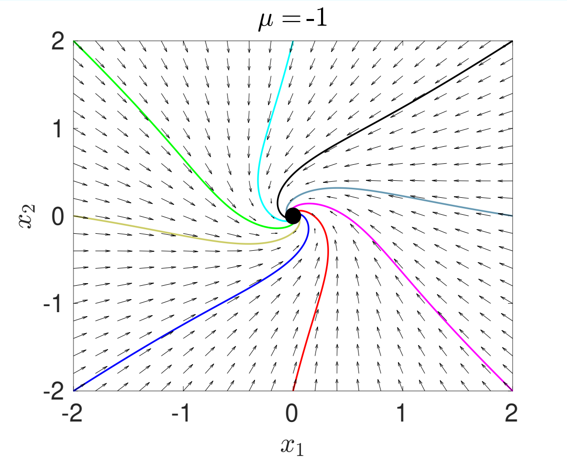
example: $\alpha'(0) = 1,\ a(0) = -1,$
$\omega(0) = 1,\ \omega'(0) = b(0) = 0$

$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

$\mu > 0$: $\mathbf{0} = [0, 0]$ is an unstable spiral

stable circular limit cycle of radius $r = \sqrt{\mu}$



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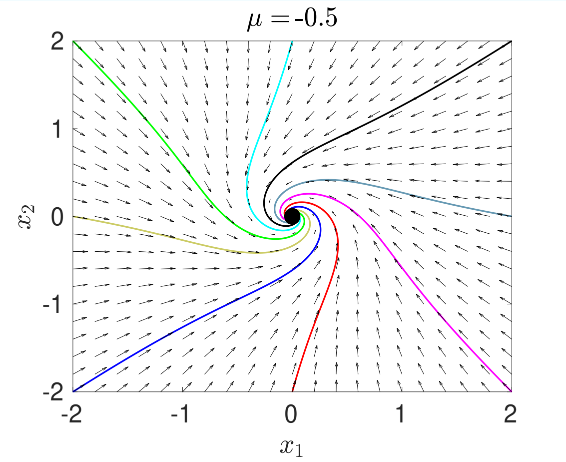

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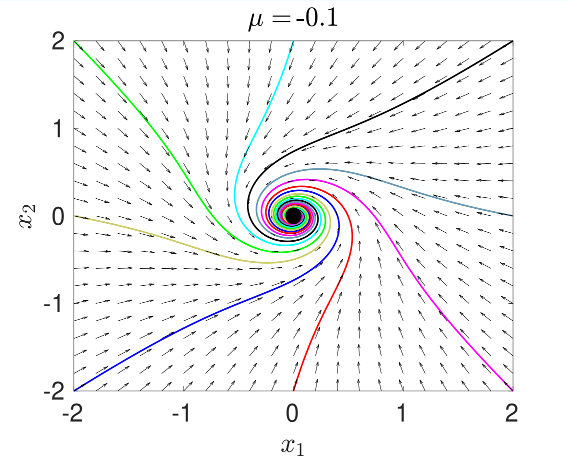

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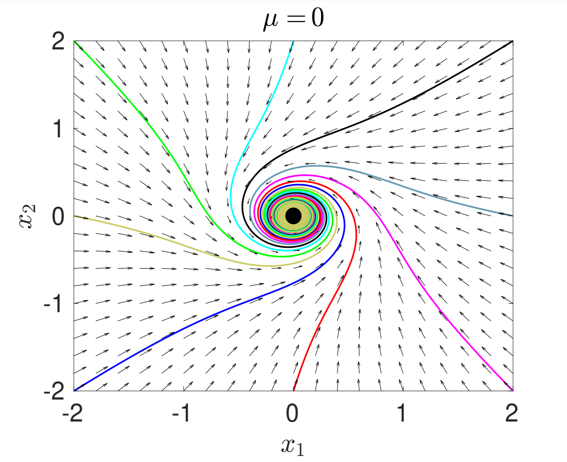

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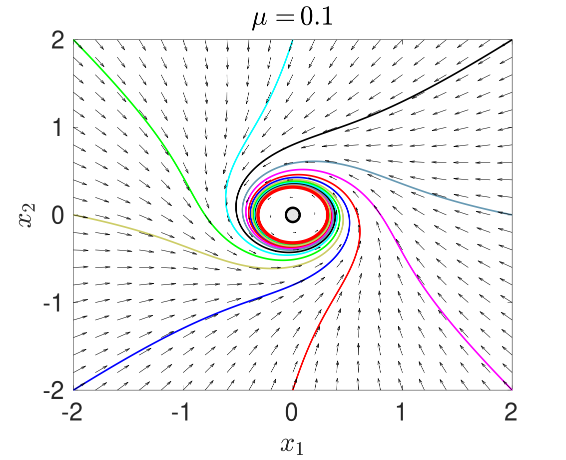

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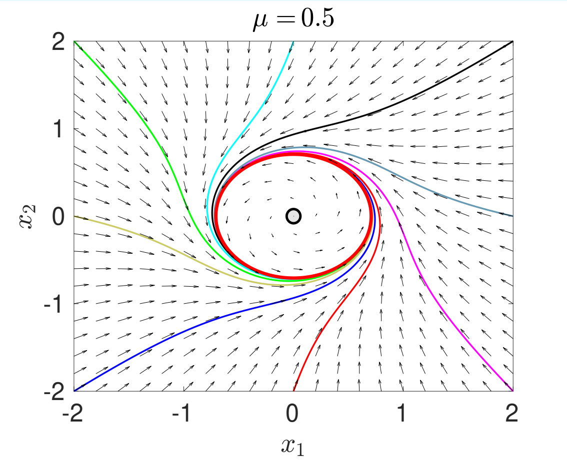

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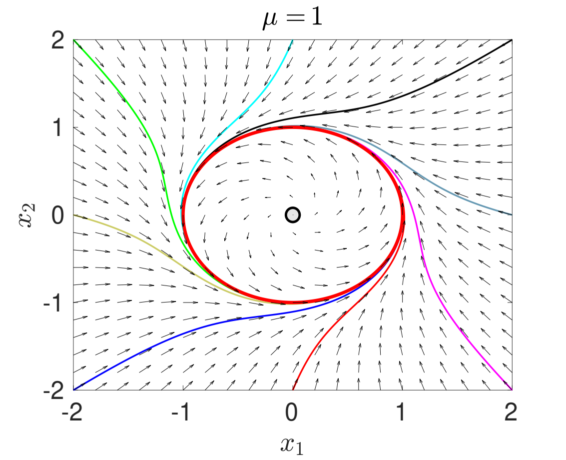

Supercritical Hopf bifurcation

general case: $a(0) < 0$

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$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) \;+\; \omega'(0)\,\mu \;+\; b(0)\,r^2$$

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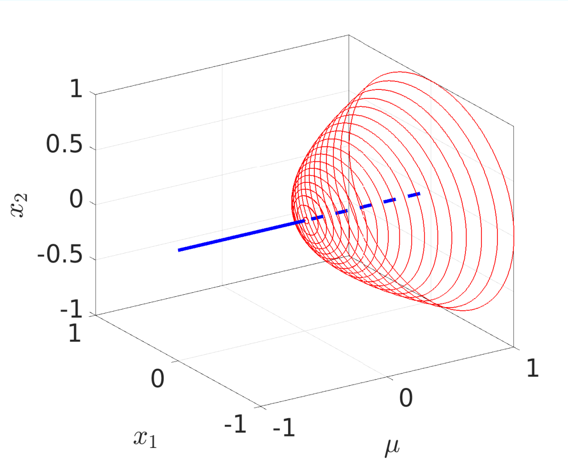

Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \,+\, a(0)\,r^3$$

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eigenvalues $\lambda_\pm = \alpha'(0)\,\mu \pm i\,\omega(0)$

Subcritical Hopf bifurcation

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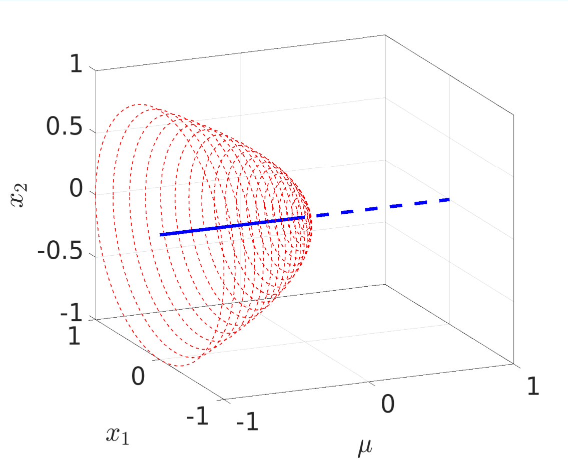

$\mu < 0$: $\mathbf{0} = [0,0]$ is a stable spiral
unstable circular limit cycle of radius $r = \sqrt{\mu}$

$\mu > 0$: $\mathbf{0} = [0,0]$ is an unstable spiral

Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3$$

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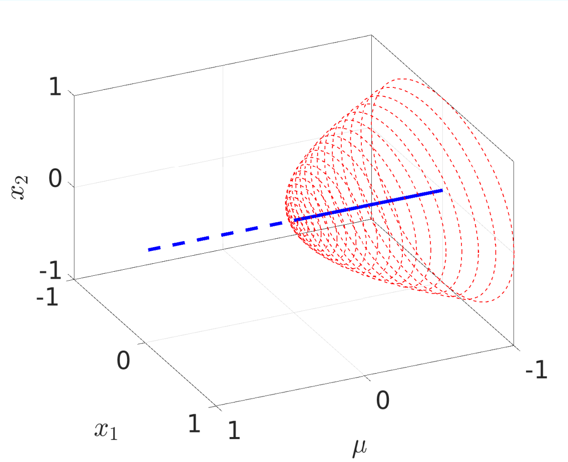

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unstable circular limit cycle of radius $r = \sqrt{\mu}$

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Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

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subcritical Hopf bifurcation at $\mu = 0$

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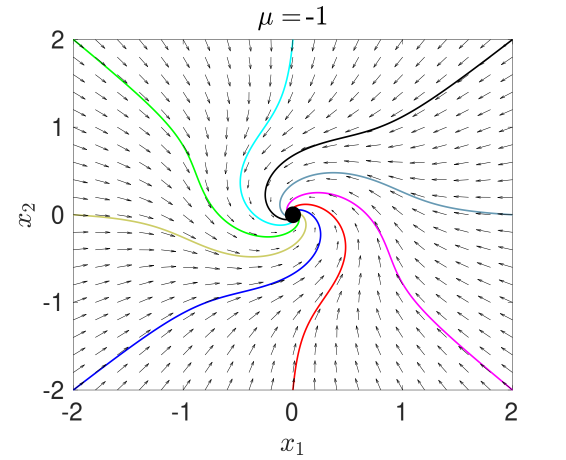

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general case: $a(0) > 0$

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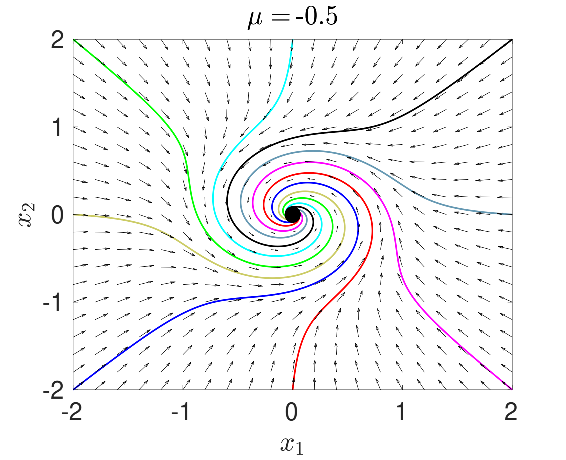
subcritical Hopf bifurcation at $\mu = 0$



Saddle-node bifurcation of cycles

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) \;+\; \omega'(0)\,\mu \;+\; b(0)\,r^2$$

eigenvalues $\lambda_{\pm} = \alpha'(0)\,\mu \pm i\,\omega(0)$

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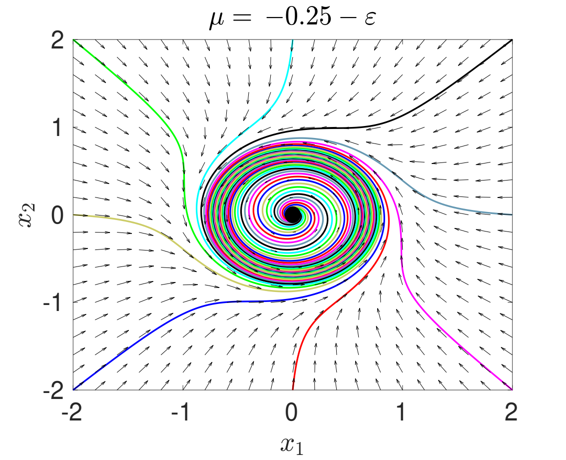
subcritical Hopf bifurcation at $\mu = 0$



saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

Saddle-node bifurcation of cycles

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$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

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example: $\alpha'(0) = 1,\ a(0) = 1,$
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$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

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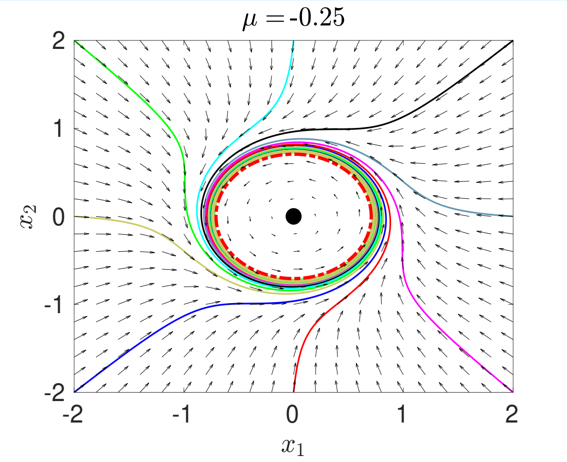
subcritical Hopf bifurcation at $\mu = 0$



saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

Saddle-node bifurcation of cycles

general case: $a(0) > 0$

$$\frac{dr}{dt} = \alpha'(0)\,\mu\,r + a(0)\,r^3 - r^5$$

$$\frac{d\theta}{dt} = \omega(0) + \omega'(0)\,\mu + b(0)\,r^2$$

eigenvalues $\lambda_\pm = \alpha'(0)\,\mu \pm i\,\omega(0)$

example: $\alpha'(0) = 1,\ a(0) = 1,$
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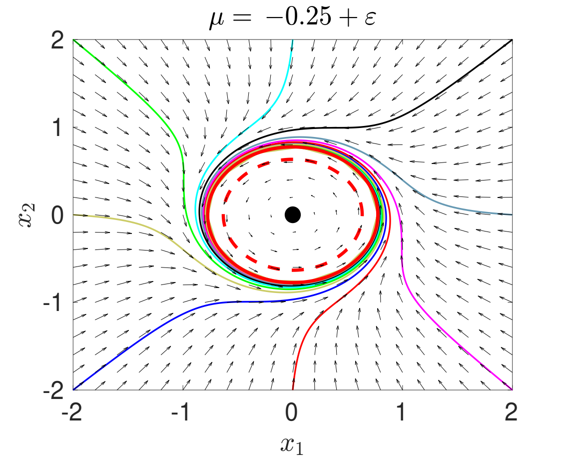
subcritical Hopf bifurcation at $\mu = 0$

saddle-node bifurcation of cycles at $\mu = -1/4$: viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through $\mu = -1/4$



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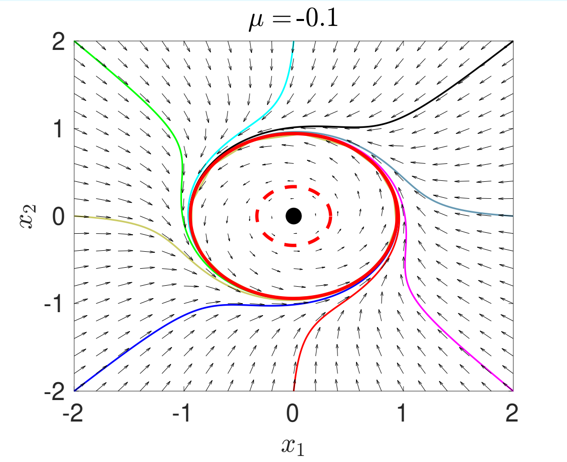

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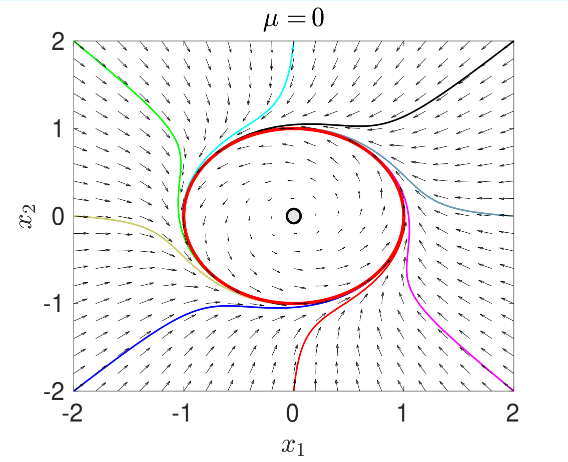

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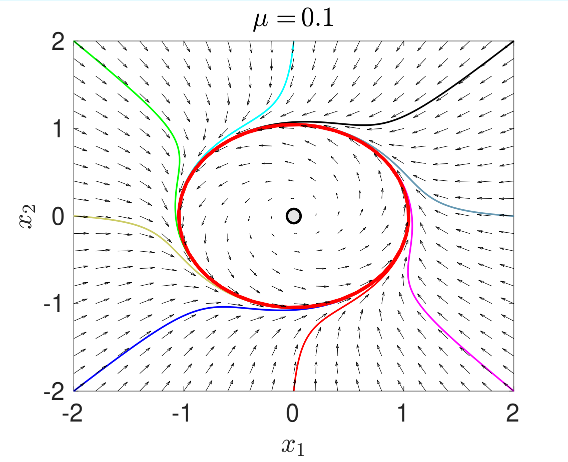

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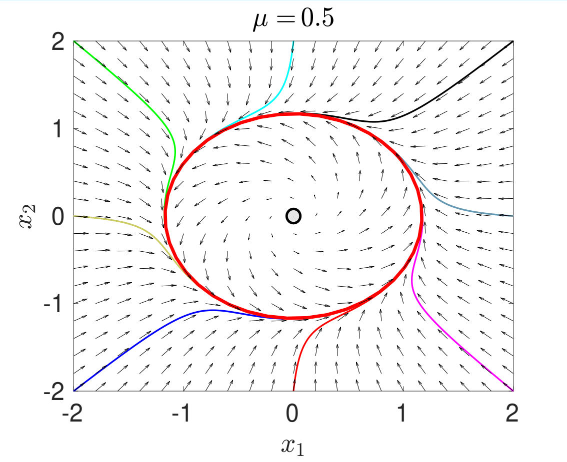

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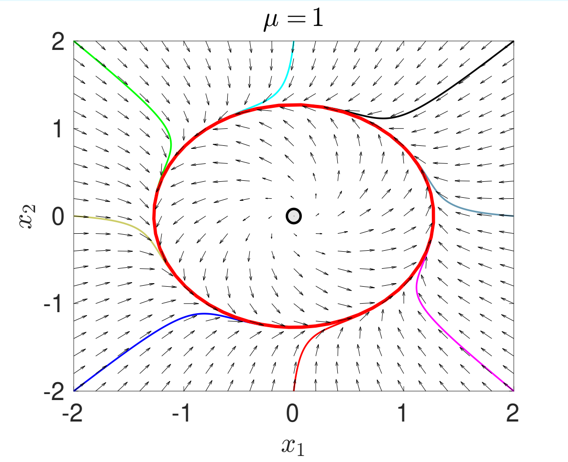
subcritical Hopf bifurcation at $\mu = 0$



Subcritical Hopf bifurcation and saddle-node bifurcation of cycles

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) \;+\; \omega'(0)\,\mu \;+\; b(0)\,r^2$$

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$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

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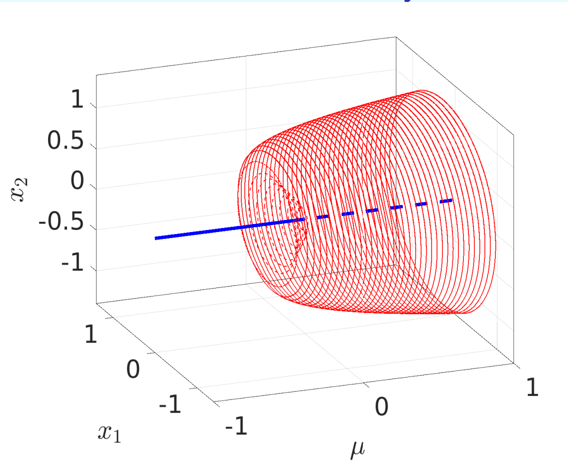
subcritical Hopf bifurcation at $\mu = 0$

saddle-node bifurcation of cycles at $\mu = -1/4$



Subcritical Hopf bifurcation

general case: $a(0) > 0$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\mu\,r \;+\; a(0)\,r^3 \;-\; r^5$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) \;+\; \omega'(0)\,\mu \;+\; b(0)\,r^2$$

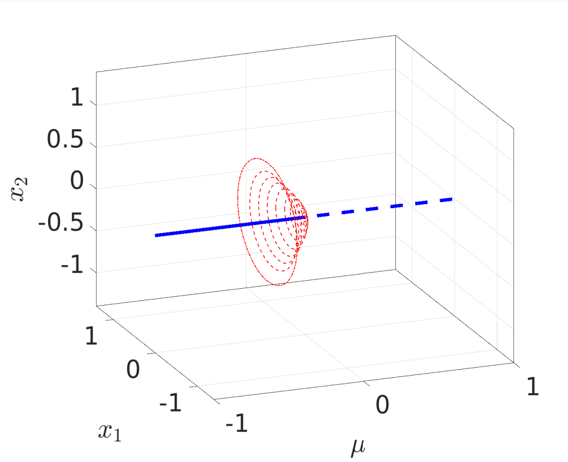
eigenvalues $\lambda_\pm = \alpha'(0)\,\mu \pm i\,\omega(0)$

example: $\alpha'(0) = 1,\ a(0) = 1,$
$\omega(0) = 1,\ \omega'(0) = b(0) = 0$

$\mu < 0$: $\mathbf{0} = [0, 0]$ is a stable spiral

$\mu > 0$: $\mathbf{0} = [0, 0]$ is an unstable spiral

subcritical Hopf bifurcation at $\mu = 0$



Question 6 on Problem Sheet 1

System of $n = 2$ chemical species X_1 and X_2 which are subject to $\ell = 4$ reactions:

$$2X_1 + X_2 \xrightarrow{k_1} 3X_1 \qquad \emptyset \xrightarrow{k_2} X_1 \qquad X_1 \xrightarrow{k_3} \emptyset \qquad \emptyset \xrightarrow{k_4} X_2$$

Assuming mass action kinetics, concentrations $x_1(t)$ and $x_2(t)$ evolve by

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Using $k_1 = k_2 = 1$, $k_3 = \mu$ and $k_4 = 2$, we get:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\, x_2 + 1 - \mu\, x_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\, x_1^2\, x_2 + 2$$

Question 6 on Problem Sheet 1: We considered $\mu = 9$. We showed that the fixed point $[1/3, 18]$ is unstable and we found a trapping region (closed bounded connected set such that the vector field points inward everywhere on its boundary). We applied the Poincaré-Bendixson theorem to prove that there exists a periodic solution.

Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\,x_2 + 1 - \mu\,x_1$$

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Question 6 on Problem Sheet 1

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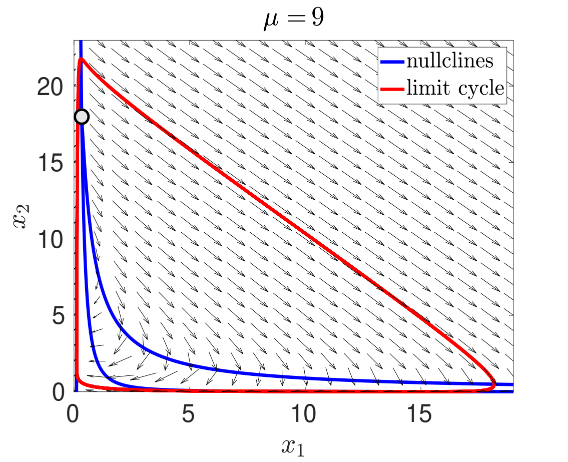
Question 6 on Problem Sheet 1:

$$\mu = 9$$



Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2 \, x_2 + 1 - \mu \, x_1$$

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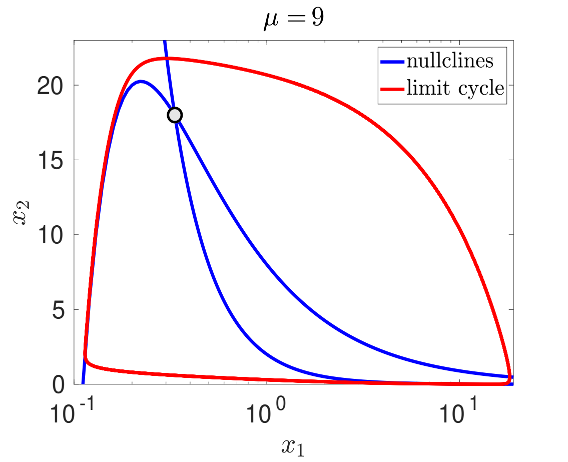
Question 6 on Problem Sheet 1:

$$\mu = 9$$



Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2 x_2 + 1 - \mu x_1$$

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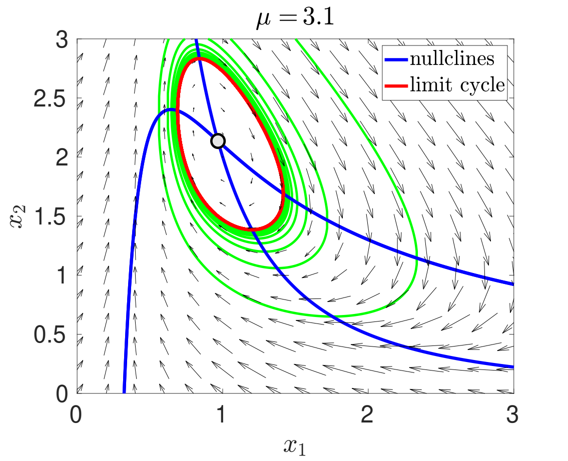
We decrease the value of parameter μ and the limit cycle shrinks.



Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\, x_2 + 1 - \mu\, x_1$$

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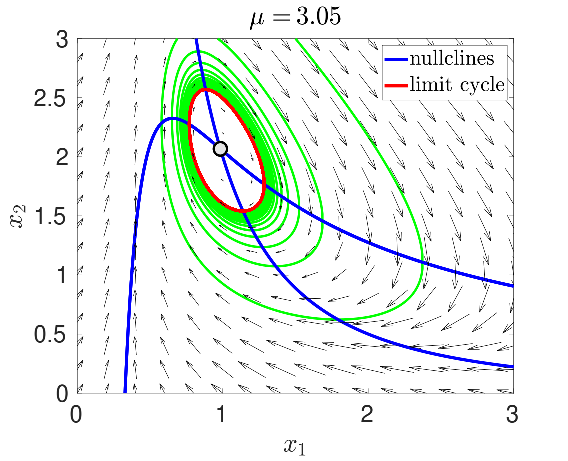

Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2 x_2 + 1 - \mu x_1$$

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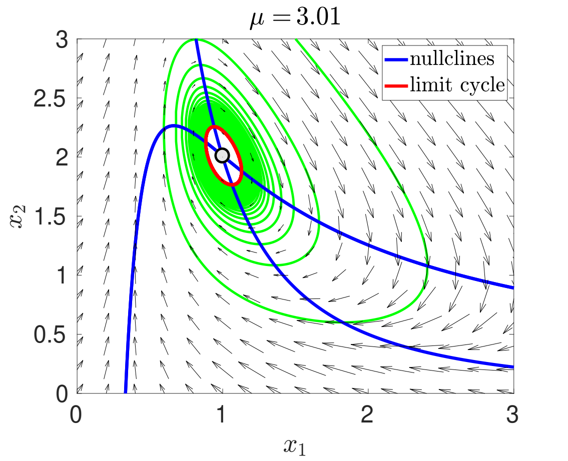
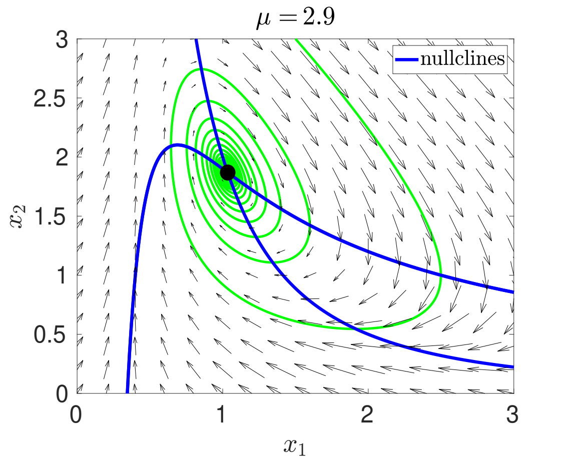


Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\,x_2 + 1 - \mu\,x_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -\,x_1^2\,x_2 + 2$$

There is no limit cycle for $\mu < 3$.



Question 6 on Problem Sheet 1

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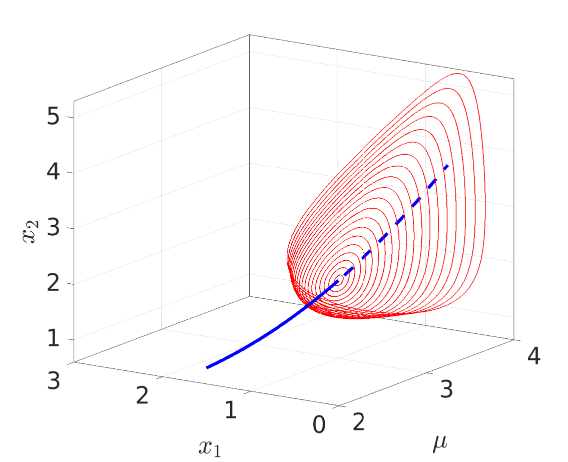
bifurcation diagram

[show 3D animation]



Question 6 on Problem Sheet 1

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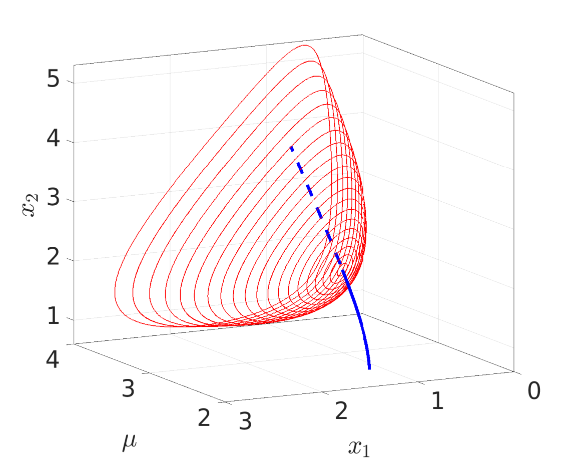
bifurcation diagram

[show 3D animation]



Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2 x_2 + 1 - \mu x_1$$

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fixed point at $\mathbf{x}_c = \left[\frac{3}{\mu},\, \frac{2\mu^2}{9}\right]$

Jacobian $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1x_2 - \mu & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{pmatrix}$

$$D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \mu/3 & 9/\mu^2 \\ -4\mu/3 & -9/\mu^2 \end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{dx_1}{dt} = x_1^2 x_2 + 1 - \mu x_1$$

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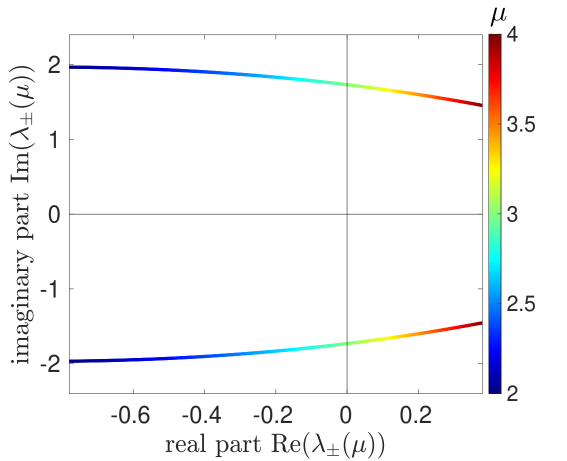

solving $\lambda^2 + \left(\frac{9}{\mu^2} - \frac{\mu}{3}\right)\lambda + \frac{9}{\mu} = 0$, we get $\lambda_\pm = \frac{1}{2}\left(\frac{\mu}{3} - \frac{9}{\mu^2} \pm \sqrt{\frac{\mu^2}{9} + \frac{81}{\mu^4} - \frac{42}{\mu}}\right)$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2 x_2 + 1 - \mu x_1$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1^2 x_2 + 2$$

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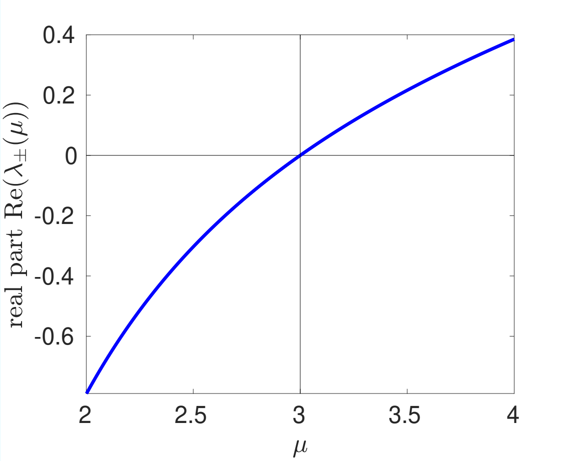

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Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\,x_2 + 1 - \mu\,x_1$$

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fixed point at $\mathbf{x}_c = \left[\frac{3}{\mu},\ \frac{2\mu^2}{9}\right]$

Jacobian $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} 2x_1x_2 - \mu & x_1^2 \\ -2x_1x_2 & -x_1^2 \end{pmatrix}$ at $\mathbf{x}_c$ is $D\mathbf{f}(\mathbf{x}_c) = \begin{pmatrix} \mu/3 & 9/\mu^2 \\ -4\mu/3 & -9/\mu^2 \end{pmatrix}$

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bifurcation at $\mu = 3$, when $\lambda_\pm = \pm\, i\,\sqrt{3}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1^2\, x_2 + 1 - \mu\, x_1$$

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bifurcation at $\mu = 3$, when $\lambda_\pm = \pm\, i\,\sqrt{3}$

using new variables $\overline{x}_1 = x_1 - \frac{3}{\mu}$, $\overline{x}_2 = x_2 - \frac{2\mu^2}{9}$, $\overline{\mu} = \frac{\mu - 3}{3}$, we obtain

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -\,4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

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Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

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$$\frac{\mathrm{d}\overline{x}_2}{\mathrm{d}t} = -4\,(1+\overline{\mu})\,\overline{x}_1 - \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 - 2(1+\overline{\mu})^2\,\overline{x}_1^2 - \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 - \overline{x}_1^2\,\overline{x}_2$$

bifurcation at $\overline{\mu} = 0$, fixed point $\mathbf{0}$ with $D\mathbf{f}(\mathbf{0}) = M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}\overline{x}_1}{\mathsf{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

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denote $g(\overline{x}_1, \overline{x}_2; \overline{\mu}) = 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$ and rewrite the system as:

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} = M(\overline{\mu}) \begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} + g(\overline{x}_1, \overline{x}_2; \overline{\mu}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

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at $\overline{\mu} = 0$, we have $M(0) = \begin{pmatrix} 1 & 1 \\ -4 & -1 \end{pmatrix}$

eigenvalues $\lambda_\pm = \pm\, i\,\sqrt{3}$, eigenvectors $\mathbf{v}_\pm = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \pm i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}$,

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}\overline{x}_1}{\mathrm{d}t} = (1+\overline{\mu})\,\overline{x}_1 + \frac{1}{(1+\overline{\mu})^2}\,\overline{x}_2 + 2(1+\overline{\mu})^2\,\overline{x}_1^2 + \frac{2}{1+\overline{\mu}}\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$$

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eigenvalues $\lambda_\pm = \pm\, i\,\sqrt{3}$, eigenvectors $\mathbf{v}_\pm = \begin{pmatrix}1 \\ -4\end{pmatrix} \pm i\begin{pmatrix}\sqrt{3} \\ 0\end{pmatrix}$, change of variables

$$\begin{pmatrix}\overline{x}_1 \\ \overline{x}_2\end{pmatrix} = \begin{pmatrix} 1 & \sqrt{3} \\ -4 & 0 \end{pmatrix}\begin{pmatrix}y_1 \\ y_2\end{pmatrix} \text{ with inverse } \begin{pmatrix}y_1 \\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix} 0 & -\sqrt{3} \\ 4 & 1 \end{pmatrix}\begin{pmatrix}\overline{x}_1 \\ \overline{x}_2\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1, \overline{x}_2; 0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

change of variables: $\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} = \begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$ and $\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}\overline{x}_1\\ \overline{x}_2\end{pmatrix} + \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \frac{1}{4\sqrt{3}}\begin{pmatrix}0 & -\sqrt{3}\\ 4 & 1\end{pmatrix}M(0)\begin{pmatrix}1 & \sqrt{3}\\ -4 & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}0 & \sqrt{3}\\ -\sqrt{3} & 0\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}g(\overline{x}_1,\overline{x}_2;0)$$

where $g(\overline{x}_1,\overline{x}_2;0) = 2\,\overline{x}_1^2 + 2\,\overline{x}_1\,\overline{x}_2 + \overline{x}_1^2\,\overline{x}_2$

$$= -6\,y_1^2 - 4\,y_1\,y_2\sqrt{3} + 6\,y_2^2 - 4\,y_1^3 - 8\,y_1^2\,y_2\sqrt{3} - 12\,y_2^2\,y_1$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $\quad h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathsf{d}}{\mathsf{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$\alpha(\overline{\mu}) = \frac{1}{2}\left(1 + \overline{\mu} - \frac{1}{(1+\overline{\mu})^2}\right)$ and $\omega(\overline{\mu}) = -\frac{1}{2}\sqrt{\frac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \frac{1}{(1+\overline{\mu})^4}}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1 + \overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

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which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \frac{3}{2}$ and $\omega'(0) = \frac{\sqrt{3}}{2}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$$\alpha(\overline{\mu}) = \frac{1}{2}\left(1+\overline{\mu} - \frac{1}{(1+\overline{\mu})^2}\right) \text{ and } \omega(\overline{\mu}) = -\frac{1}{2}\sqrt{\frac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \frac{1}{(1+\overline{\mu})^4}}$$

which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \dfrac{3}{2}$ and $\omega'(0) = \dfrac{\sqrt{3}}{2}$

normal form in polar coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \alpha'(0)\,\overline{\mu}\,r + a(0)\,r^3 + \mathcal{O}(\overline{\mu}^2 r, \overline{\mu} r^3, r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega(0) + \omega'(0)\,\overline{\mu} + b(0)\,r^2 + \mathcal{O}(\overline{\mu}^2, \overline{\mu} r^2, r^4)$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_2^2\,y_1$

$\overline{\mu}$ close to the bifurcation point $\overline{\mu} = 0$: matrix $M(\overline{\mu}) = \begin{pmatrix} 1+\overline{\mu} & (1+\overline{\mu})^{-2} \\ -4\,(1+\overline{\mu}) & -(1+\overline{\mu})^{-2} \end{pmatrix}$

has eigenvalues $\lambda_\pm(\overline{\mu}) = \alpha(\overline{\mu}) \pm i\,\omega(\overline{\mu})$ where

$$\alpha(\overline{\mu}) = \frac{1}{2}\left(1 + \overline{\mu} - \frac{1}{(1+\overline{\mu})^2}\right) \text{ and } \omega(\overline{\mu}) = -\frac{1}{2}\sqrt{\frac{14}{1+\overline{\mu}} - 1 - 2\,\overline{\mu} - \overline{\mu}^2 - \frac{1}{(1+\overline{\mu})^4}}$$

which implies $\alpha(0) = 0$, $\omega(0) = -\sqrt{3}$, $\alpha'(0) = \dfrac{3}{2}$ and $\omega'(0) = \dfrac{\sqrt{3}}{2}$

normal form in polar coordinates:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r + a(0)\,r^3 + \mathcal{O}(\overline{\mu}^2 r, \overline{\mu} r^3, r^5)$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + b(0)\,r^2 + \mathcal{O}(\overline{\mu}^2, \overline{\mu} r^2, r^4)$$

Calculation of $a(0)$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Calculation of $a(0)$

supercritical Hopf bifurcation: $a(0) < 0$ (periodic orbit is asymptotically stable)

subcritical Hopf bifurcation: $a(0) > 0$ (periodic orbit is unstable)

Lemma: Assume that the ODE system with Hopf bifurcation at $\overline{\mu} = 0$ was transformed to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where $h_1(y_1, y_2)$ and $h_2(y_1, y_2)$ contain only higher-order nonlinear terms that vanish at the origin. Then

$$a(0) = \frac{1}{16}\left(\frac{\partial^3 h_1}{\partial y_1^3} + \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} + \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} + \frac{\partial^3 h_2}{\partial y_2^3}\right) + \frac{1}{16\,\omega(0)}\left[\frac{\partial^2 h_1}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_1}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\right)\right.$$
$$\left. - \frac{\partial^2 h_2}{\partial y_1 \partial y_2}\left(\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_2}{\partial y_2^2}\right) - \frac{\partial^2 h_1}{\partial y_1^2}\frac{\partial^2 h_2}{\partial y_1^2} + \frac{\partial^2 h_1}{\partial y_2^2}\frac{\partial^2 h_2}{\partial y_2^2}\right]$$

where the partial derivatives are evaluated at the origin $\mathbf{0}$.

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

Our equation
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_1\,y_2^2$

is in the form
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where $\omega_0 = -\sqrt{3}$, $h_1(y_1, y_2) = h(y_1, y_2)/2$ and $h_2(y_1, y_2) = \sqrt{3}\,h(y_1, y_2)/2$.

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

Our equation $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} h(y_1, y_2)$$

where $h(y_1, y_2) = -3\,y_1^2 - 2\,y_1\,y_2\,\sqrt{3} + 3\,y_2^2 - 2\,y_1^3 - 4\,y_1^2\,y_2\,\sqrt{3} - 6\,y_1\,y_2^2$

is in the form $$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega(0) \\ \omega(0) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} h_1(y_1, y_2) \\ h_2(y_1, y_2) \end{pmatrix}$$

where $\omega_0 = -\sqrt{3}$, $h_1(y_1, y_2) = h(y_1, y_2)/2$ and $h_2(y_1, y_2) = \sqrt{3}\,h(y_1, y_2)/2$.

Substituting (partial derivatives evaluated at the origin $\mathbf{0}$):

$$\frac{\partial^3 h_1}{\partial y_1^3} = -6, \quad \frac{\partial^3 h_1}{\partial y_1 \partial y_2^2} = -6, \quad \frac{\partial^3 h_2}{\partial y_1^2 \partial y_2} = -12, \quad \frac{\partial^3 h_2}{\partial y_2^3} = 0, \quad \frac{\partial^2 h_1}{\partial y_1^2} = -3,$$

$$\frac{\partial^2 h_1}{\partial y_1 \partial y_2} = -\sqrt{3}, \quad \frac{\partial^2 h_1}{\partial y_2^2} = 3, \quad \frac{\partial^2 h_2}{\partial y_1^2} = -3\sqrt{3}, \quad \frac{\partial^2 h_2}{\partial y_1 \partial y_2} = -3, \quad \frac{\partial^2 h_2}{\partial y_2^2} = 3\sqrt{3}$$

we get $a(0) = -\dfrac{3}{2}$ $\quad\Longrightarrow\quad$ supercritical Hopf bifurcation

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

normal form:

$$\frac{\mathsf{d}r}{\mathsf{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \ldots$$

$$\frac{\mathsf{d}\theta}{\mathsf{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$

and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r \,-\, \frac{3}{2}\,r^3 \,+\, \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$

The limit cycle is $y_1^2 + y_2^2 = \dfrac{\mu - 3}{3}$ which corresponds to an ellipse in x_1 and x_2:

$$\overline{x}_2^2 \,+\, \frac{1}{3}\,(4\,\overline{x}_1 + \overline{x}_2)^2 = \frac{16(\mu - 3)}{3}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$

The limit cycle is $y_1^2 + y_2^2 = \dfrac{\mu - 3}{3}$ which corresponds to an ellipse in x_1 and x_2:

$$\left(x_2 - \frac{2\mu^2}{9}\right)^2 + \frac{1}{3}\left(4\,x_1 + x_2 - \frac{12}{\mu} - \frac{2\mu^2}{9}\right)^2 = \frac{16(\mu - 3)}{3}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

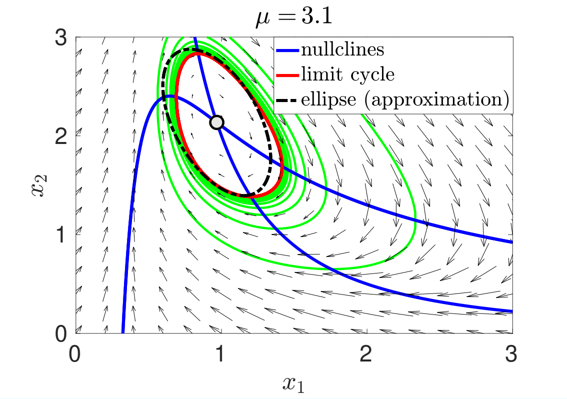
normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \ldots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \ldots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$



The limit cycle is $y_1^2 + y_2^2 = \dfrac{\mu - 3}{3}$ which corresponds to an ellipse in x_1 and x_2:

$$\left(x_2 - \frac{2\mu^2}{9}\right)^2 + \frac{1}{3}\left(4\,x_1 + x_2 - \frac{12}{\mu} - \frac{2\mu^2}{9}\right)^2 = \frac{16(\mu - 3)}{3}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

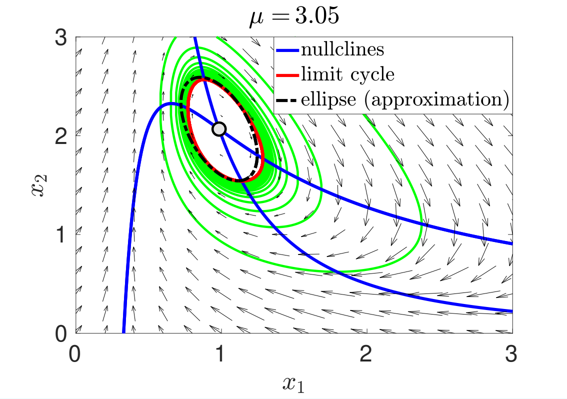
normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \dots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \dots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$



The limit cycle is $y_1^2 + y_2^2 = \dfrac{\mu - 3}{3}$ which corresponds to an ellipse in x_1 and x_2:

$$\left(x_2 - \frac{2\mu^2}{9}\right)^2 + \frac{1}{3}\left(4\,x_1 + x_2 - \frac{12}{\mu} - \frac{2\mu^2}{9}\right)^2 = \frac{16(\mu - 3)}{3}$$

Bifurcation analysis of chemical system in Question 6 on Problem Sheet 1

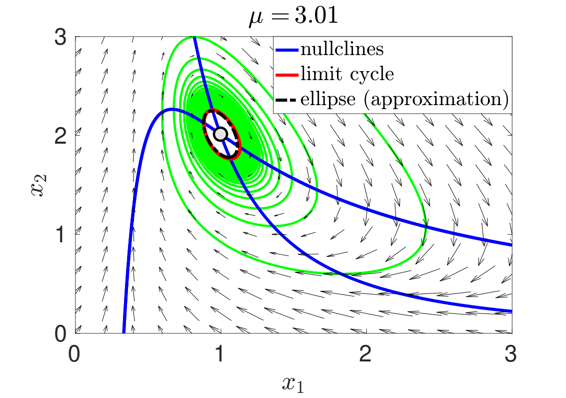
normal form:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{2}\,\overline{\mu}\,r - \frac{3}{2}\,r^3 + \dots$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\sqrt{3} + \frac{\sqrt{3}}{2}\,\overline{\mu} + \dots$$

Origin $\mathbf{0}$ is stable for $\overline{\mu} < 0 \;\Leftrightarrow\; \mu < 3$
and unstable for $\overline{\mu} > 0 \;\Leftrightarrow\; \mu > 3$

A stable limit cycle is born with
amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}}$



The limit cycle is $y_1^2 + y_2^2 = \dfrac{\mu - 3}{3}$ which corresponds to an ellipse in x_1 and x_2:

$$\left(x_2 - \frac{2\mu^2}{9}\right)^2 + \frac{1}{3}\left(4\,x_1 + x_2 - \frac{12}{\mu} - \frac{2\mu^2}{9}\right)^2 = \frac{16(\mu - 3)}{3}$$

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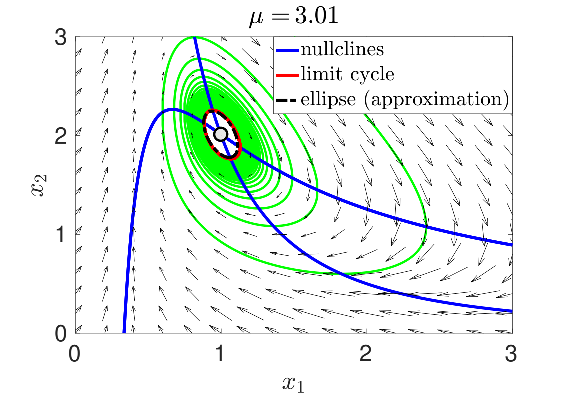

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Additional examples: Questions 1 and 2 on Problem Sheet 3.

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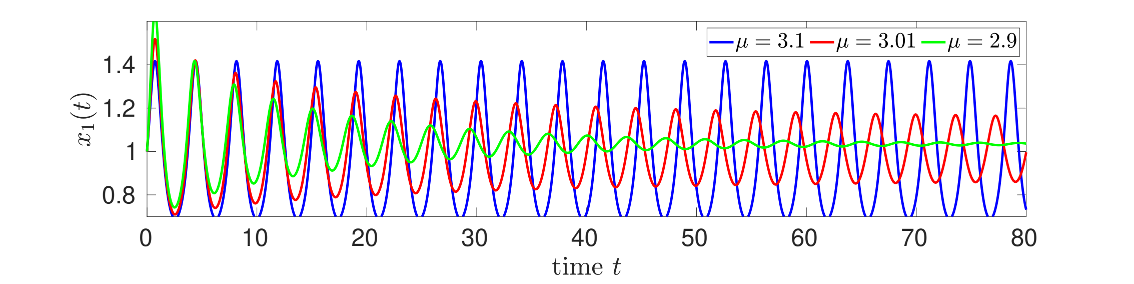

A stable limit cycle is born with amplitude $\sqrt{\dfrac{\mu - 3}{3}}$ and period $\dfrac{2\pi}{\sqrt{3}} \approx 3.6$

Close to the bifurcation point $\mu = \mu_c$, the amplitude is $\mathcal{O}\left(\sqrt{|\mu - \mu_c|}\right)$

TODAY: we will consider global bifucations when the amplitude will satisfy $\mathcal{O}\,(1)$, i.e. the amplitude of the limit cycle does not go to zero as the parameter μ approaches the bifurcation value $\mu = \mu_c$

Bifurcations of (stable) limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
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saddle-node bifurcation of cycles: we have already presented an example when we discussed the subcritical Hopf bifurcation

Example: saddle-node bifurcation of cycles

saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable, or, viewed in the other direction, a stable and unstable cycle collide and disappear as μ decreases through $\mu = -1/4$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu\, r + r^3 - r^5$$

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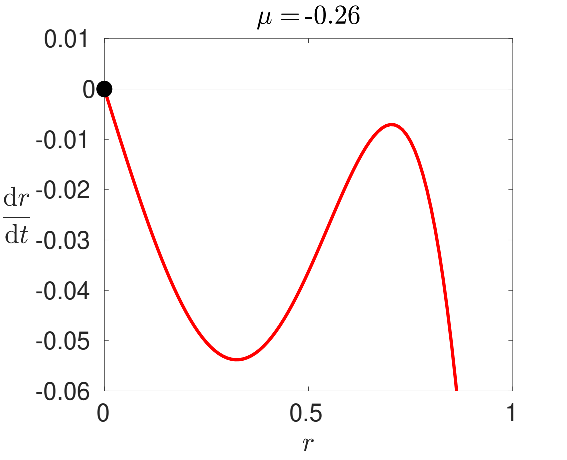

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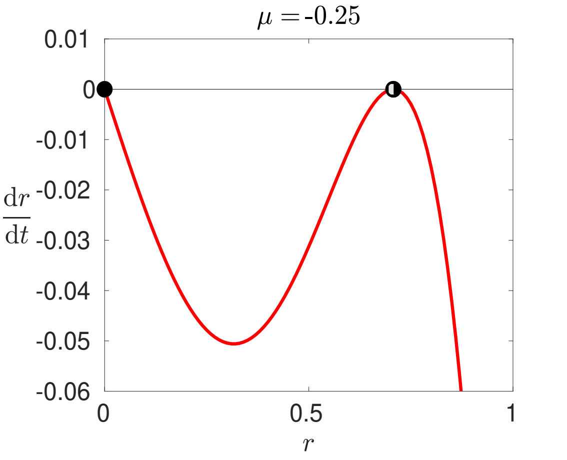

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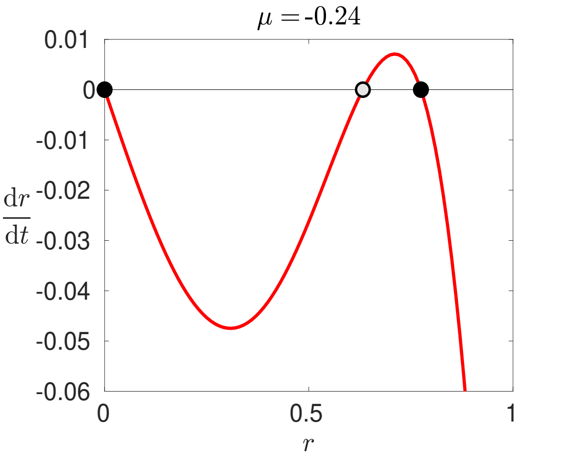

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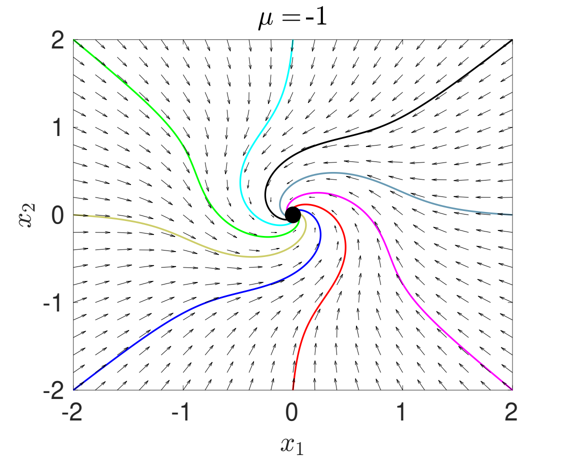

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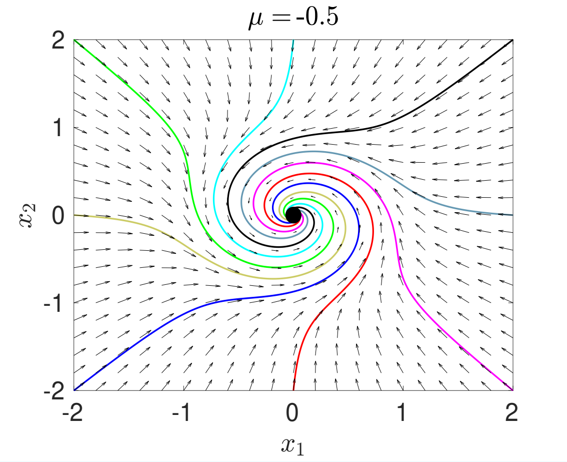

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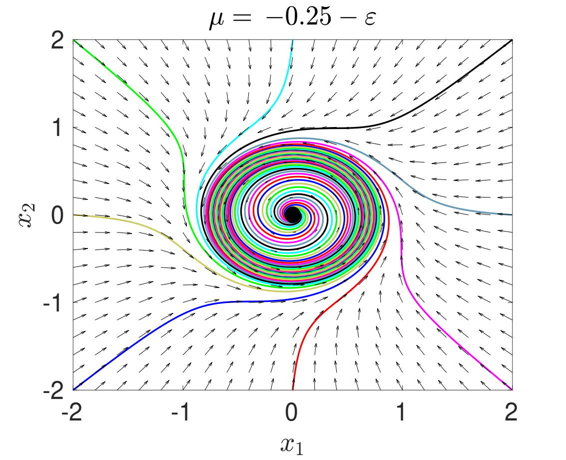

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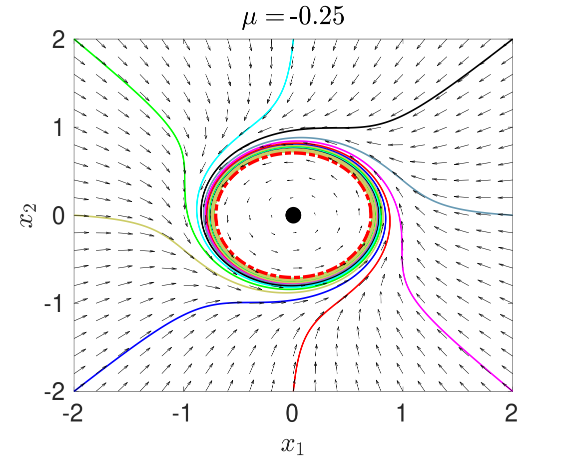

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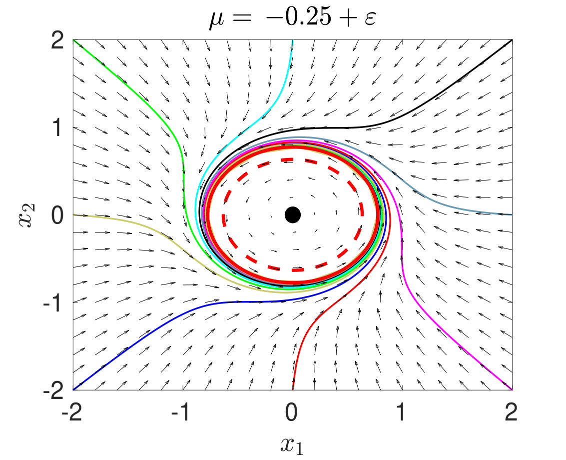

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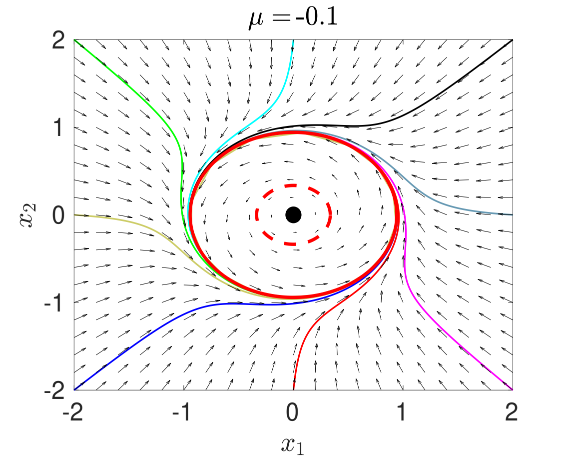
subcritical Hopf bifurcation at $\mu = 0$ (because $a(0) = 1 > 0$)



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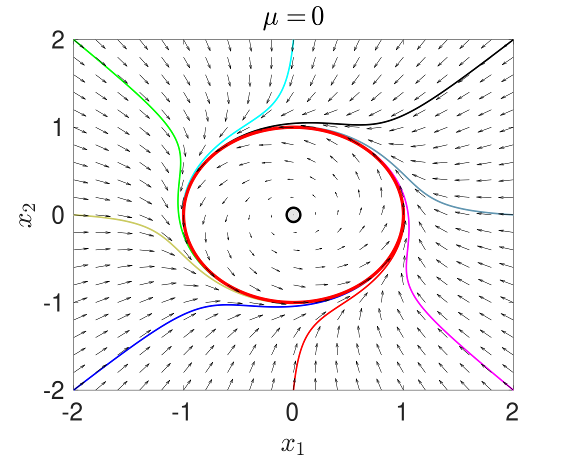

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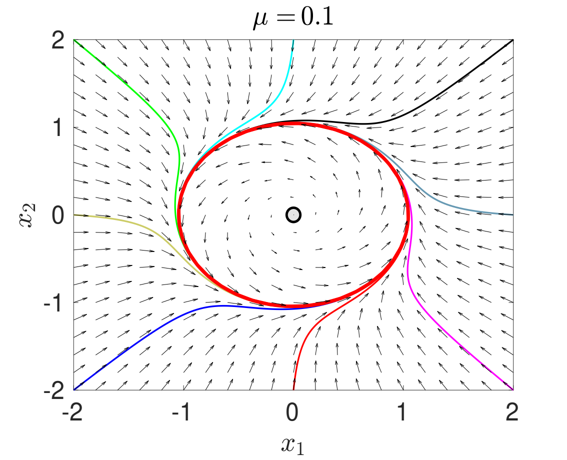

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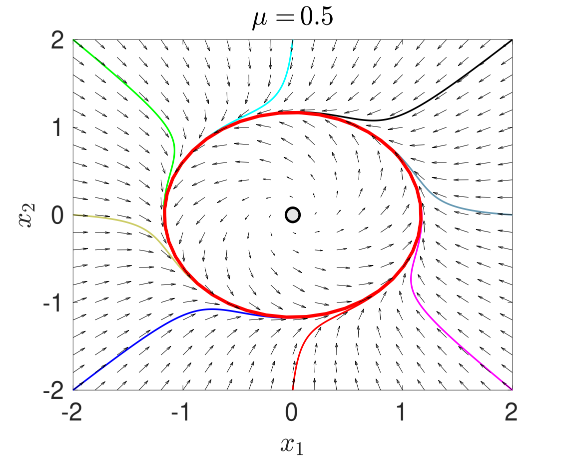

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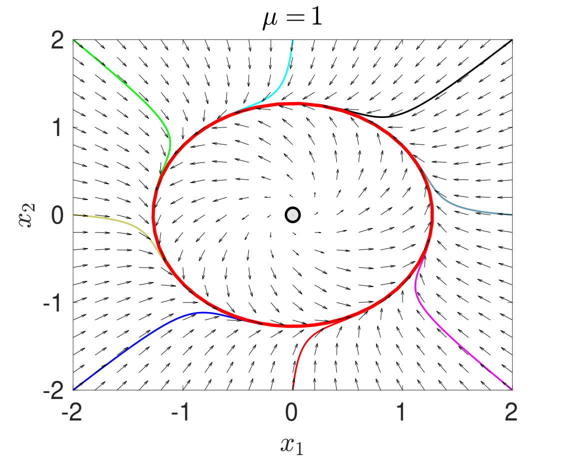

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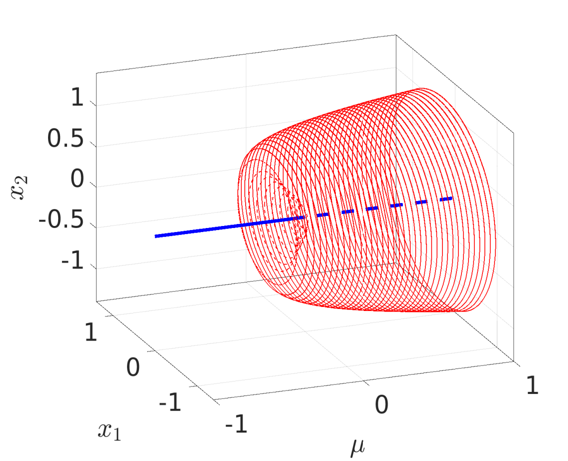

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infinite-period bifurcation: we have already presented an example on Problem Sheet 0

SNIC ... saddle-node bifurcation on invariant circle

SNIPER ... saddle-node infinite-period bifurcation

Example: infinite-period (SNIC, SNIPER) bifurcation

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = x_1 - \mu\, x_2 + x_2^2(1 - x_1) - x_1^3$$

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Problem Sheet 0 Question 5:

$\mu \in (-1, 1)$: three critical points:

$[0, 0]$: unstable spiral eigenvalues: $1 \pm \mu i$

$\left[\sqrt{1-\mu^2}, \mu\right]$: stable node eigenvalues: $-2, -\sqrt{1-\mu^2}$

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$|\mu| > 1$: one critical point:

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saddle-node bifurcations at $\mu = 1$ and $\mu = -1$

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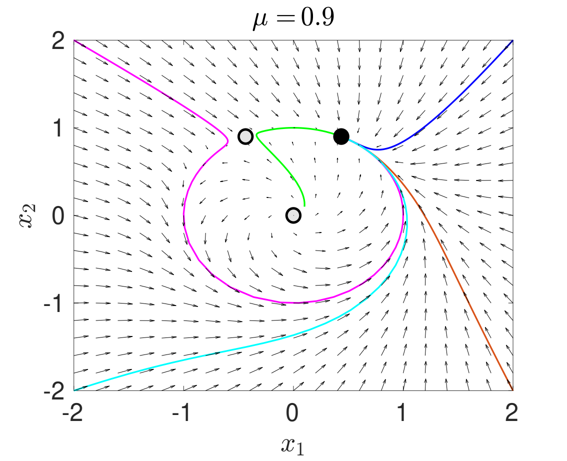

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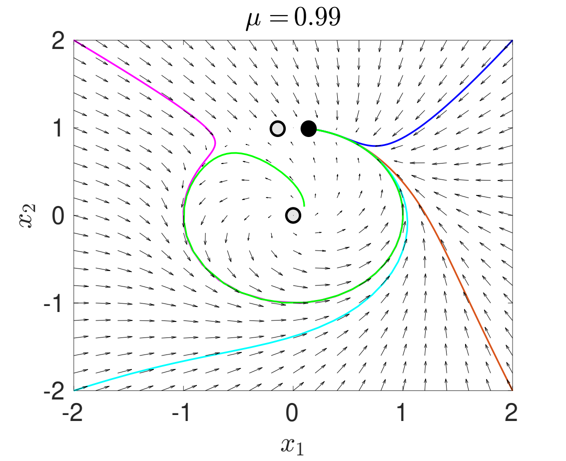

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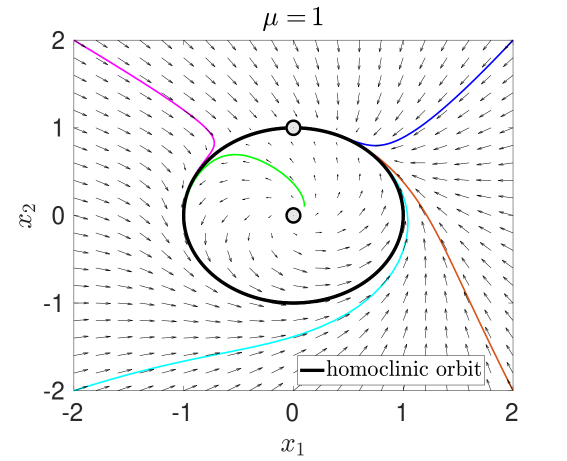

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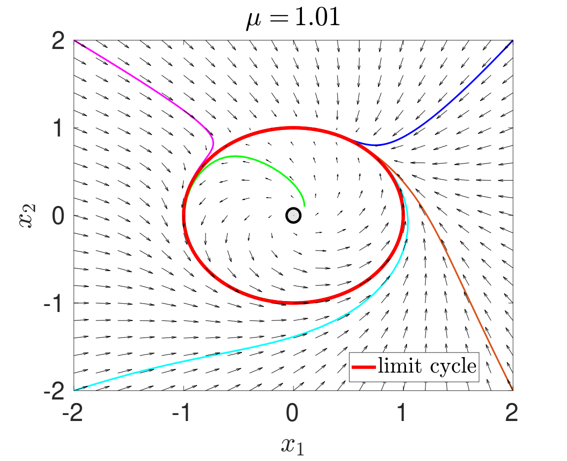

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$\left[-\sqrt{1-\mu^2}, \mu\right]$: saddle

$|\mu| > 1$: one critical point:

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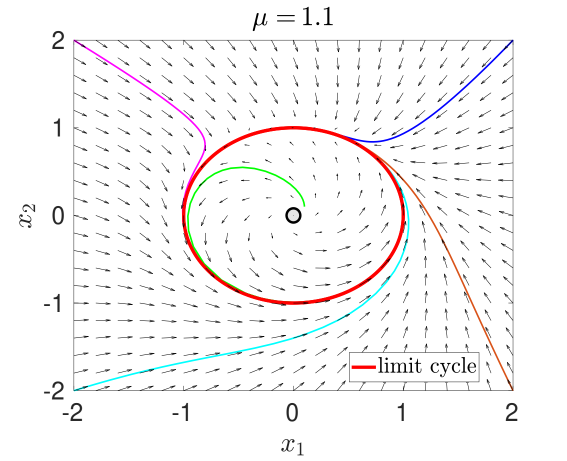
saddle-node bifurcations at $\mu = 1$ and $\mu = -1$



Example: infinite-period (SNIC, SNIPER) bifurcation

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Problem Sheet 0 Question 5:

$\mu \in (-1, 1)$: three critical points:

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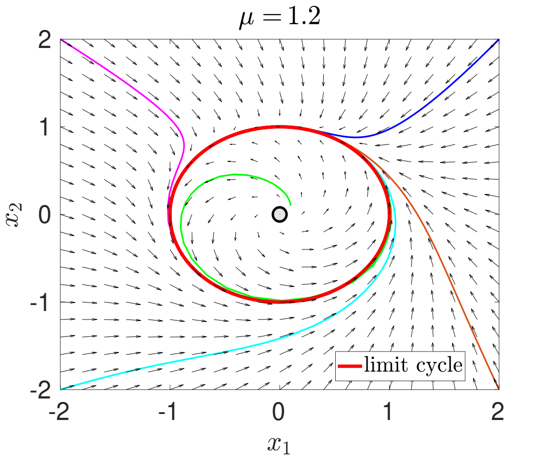

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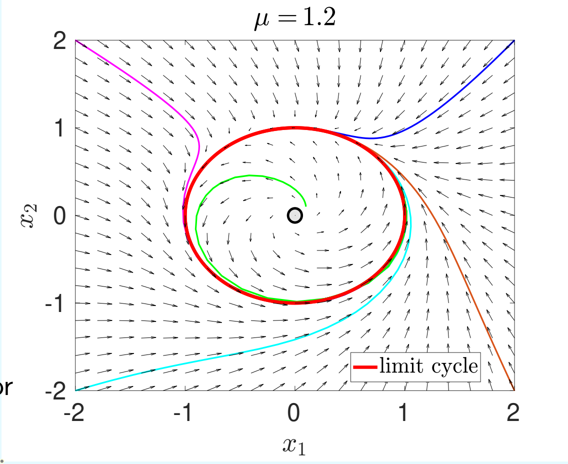

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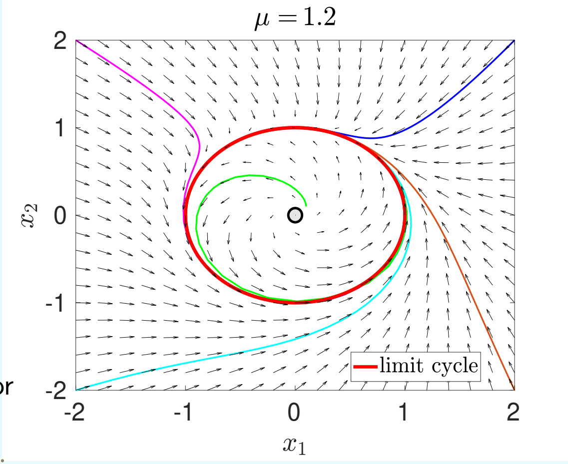

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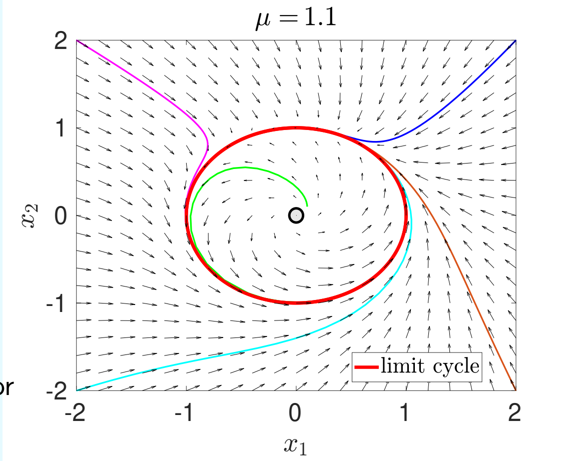

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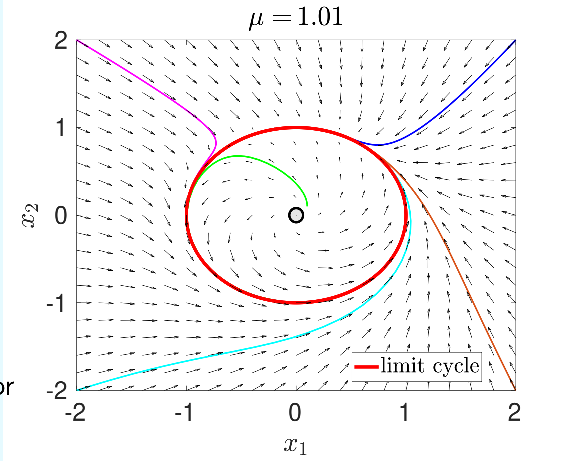

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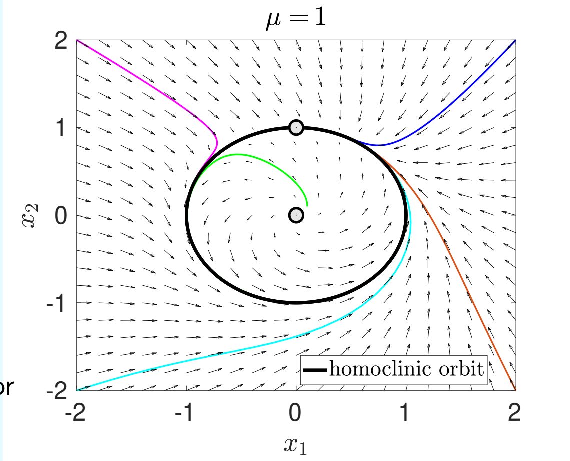

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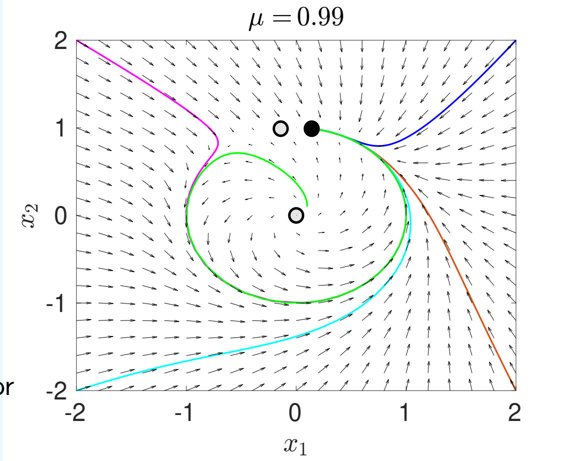

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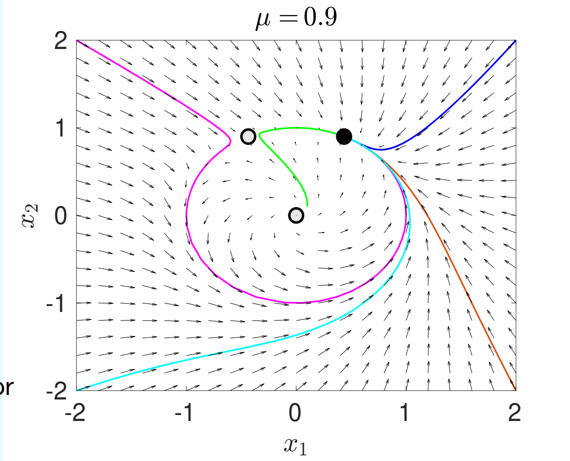

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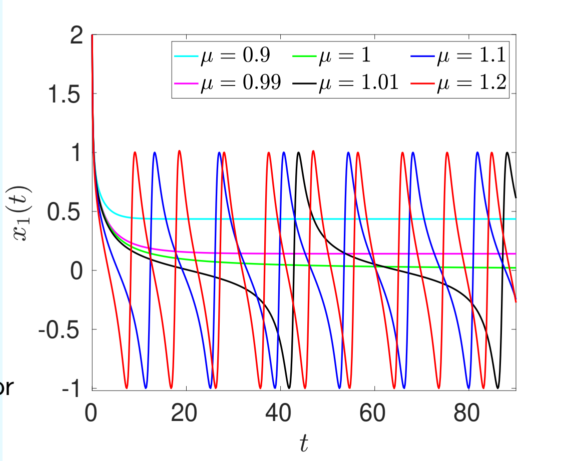

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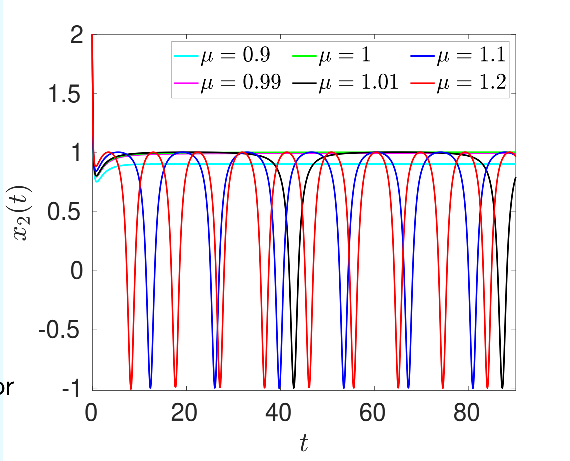

Bifurcations of limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\|\mu - \mu_c\|}\right)$	$\mathcal{O}(1)$
subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\|\mu - \mu_c\|}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\|\mu - \mu_c\|}}\right)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\big\|\log\|\mu - \mu_c\|\big\|\right)$

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infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\lvert \log \lvert\mu - \mu_c\rvert\rvert\right)$

homoclinic bifurcation: another bifurcation when limit cycle is born with infinite period

saddle-loop bifurcation

new example

Example: homoclinic (saddle-loop) bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1$$

Example

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu\, x_1 \,+\, x_2 \,-\, x_2^2 \,-\, x_1\, x_2$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = -x_1$$

two critical points: $\mathbf{x}_{c1} = [0, 0]$ and $\mathbf{x}_{c2} = [0, 1]$

Jacobian matrix is $\mathsf{D}\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \mu - x_2 & 1 - 2x_2 - x_1 \\ -1 & 0 \end{pmatrix}$

Example

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$
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$\mathrm{D}\mathbf{f}(\mathbf{x}_{c1}) = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix}$, eigenvalues $\lambda_\pm = \frac{\mu}{2} \pm \frac{\sqrt{\mu^2 - 4}}{2}$

$\implies$ $\mathbf{x}_{c1}$ is stable for $\mu < 0$ and unstable for $\mu > 0$

$\mathrm{D}\mathbf{f}(\mathbf{x}_{c2}) = \begin{pmatrix} \mu - 1 & -1 \\ -1 & 0 \end{pmatrix}$, eigenvalues $\lambda_\pm = \frac{\mu - 1}{2} \pm \frac{\sqrt{\mu^2 - 2\mu + 5}}{2}$

$\implies$ $\mathbf{x}_{c2}$ is an (unstable) saddle for all $\mu \in \mathbb{R}$

Example

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

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$\mu < 0$:
fixed point $\mathbf{x}_{c1} = [0, 0]$ is a stable spiral

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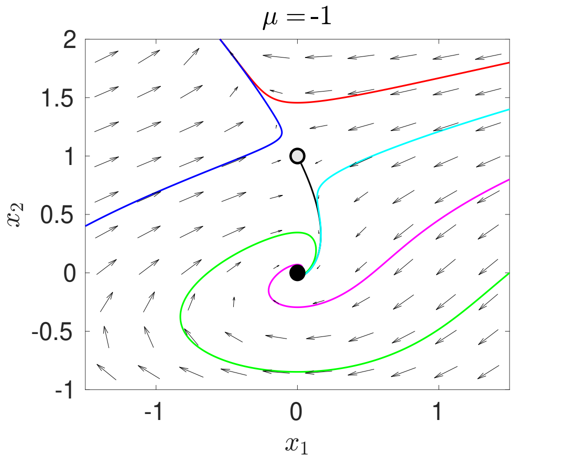

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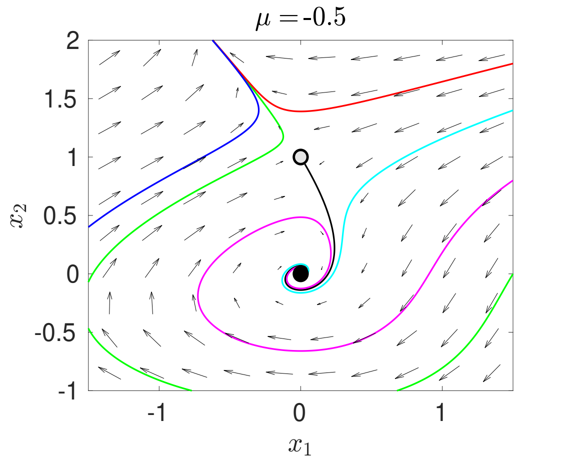

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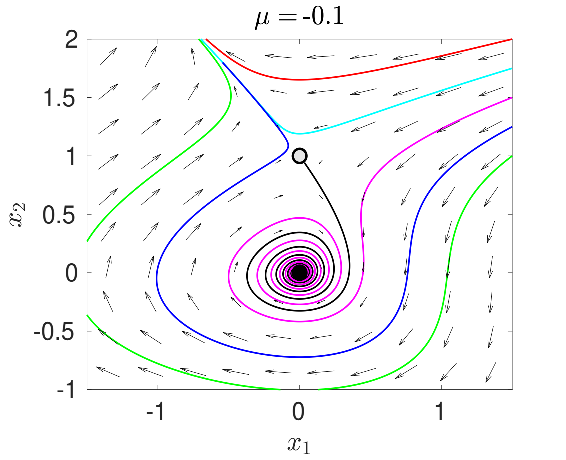

Example: supercritical Hopf bifurcation

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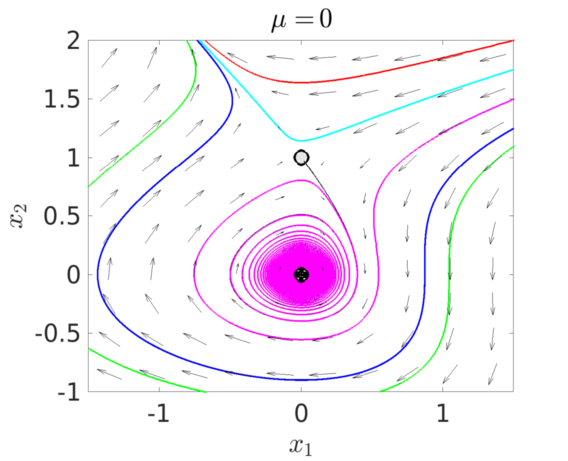

$\mu = 0$: fixed point $\mathbf{x}_{c1} = [0, 0]$ is a still stable spiral, though a very weak one

supercritical Hopf bifurcation at $\mu = 0$

the limit cycle exists in interval $\mu \in (0, 0.135454802155\ldots)$

Example: homoclinic (saddle-loop) bifurcation

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu\, x_1 + x_2 - x_2^2 - x_1\, x_2$$

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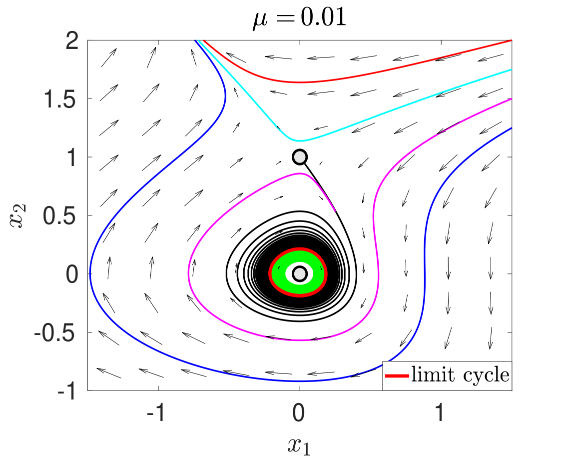
$\mu > 0$: $\mathbf{x}_{c1} = [0, 0]$ is an unstable spiral

the limit cycle exists in interval

$\mu \in (0, 0.135454802155\dots)$



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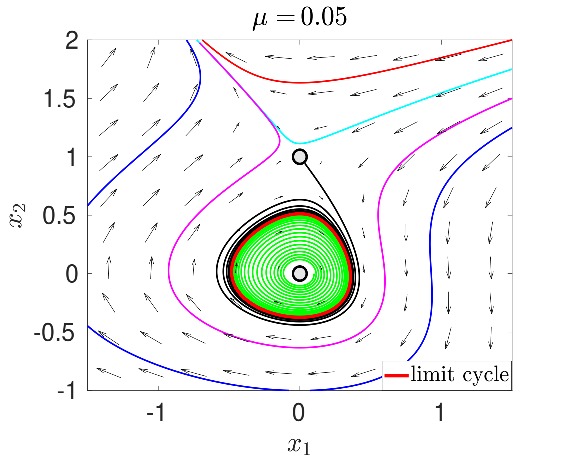

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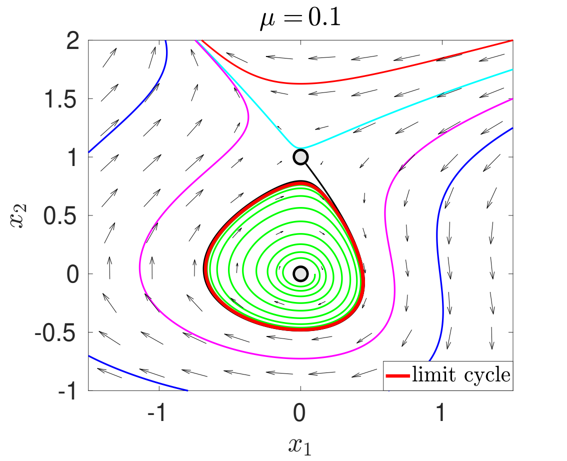

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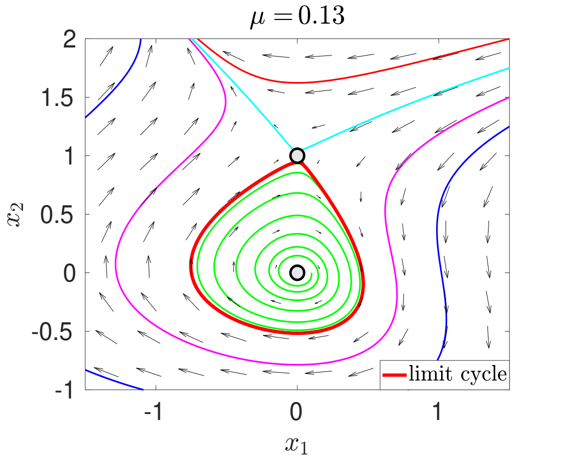

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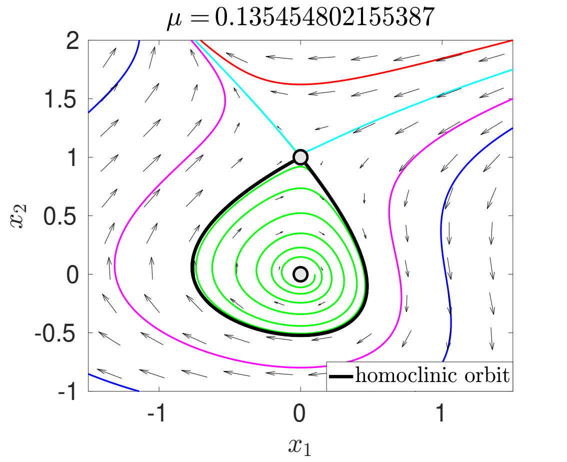
$\mu > 0$: $\mathbf{x}_{c1} = [0, 0]$ is an unstable spiral

the limit cycle exists in interval
$\mu \in (0, 0.13545480215\ldots)$

$\mu = 0.13545480215\ldots$: limit cycle collides with the saddle at $\mathbf{x}_{c2} = [0, 1]$ and it becomes a homoclinic orbit

homoclinic (saddle-loop) bifurcation



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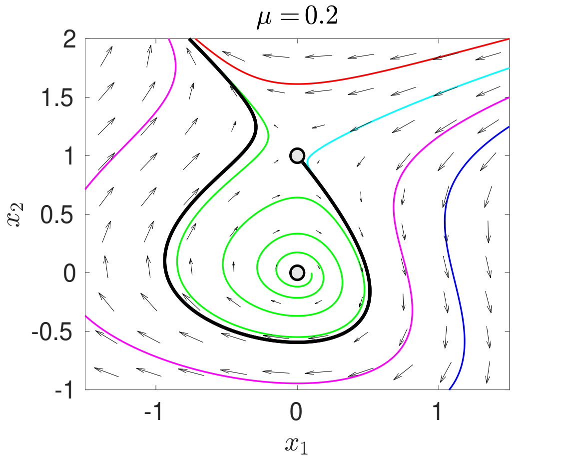
the limit cycle exists in interval
$\mu \in (0, 0.135454802155\ldots)$

$\mu = 0.135454802155\ldots$: limit cycle collides with the saddle at $\mathbf{x}_{c2} = [0, 1]$ and it becomes a homoclinic orbit

homoclinic (saddle-loop) bifurcation

$\mu > 0.135454802155\ldots$: no limit cycle



Example: homoclinic bifurcation and supercritical Hopf bifurcation

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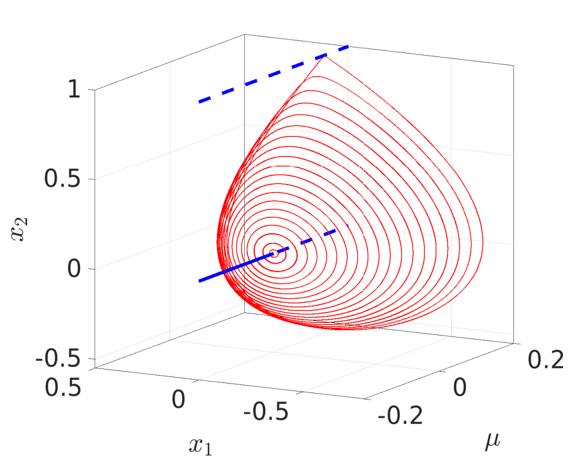
bifurcation diagram

[show 3D animation]



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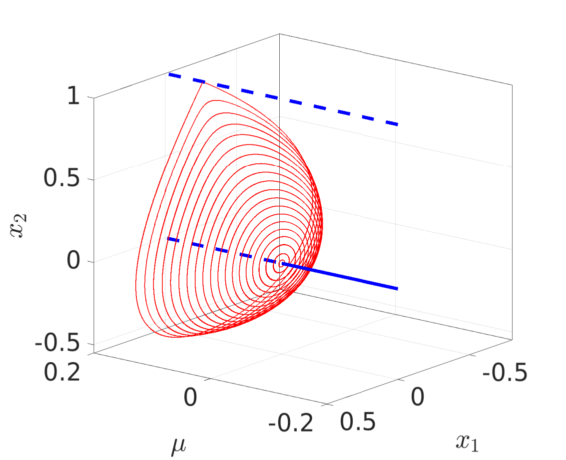
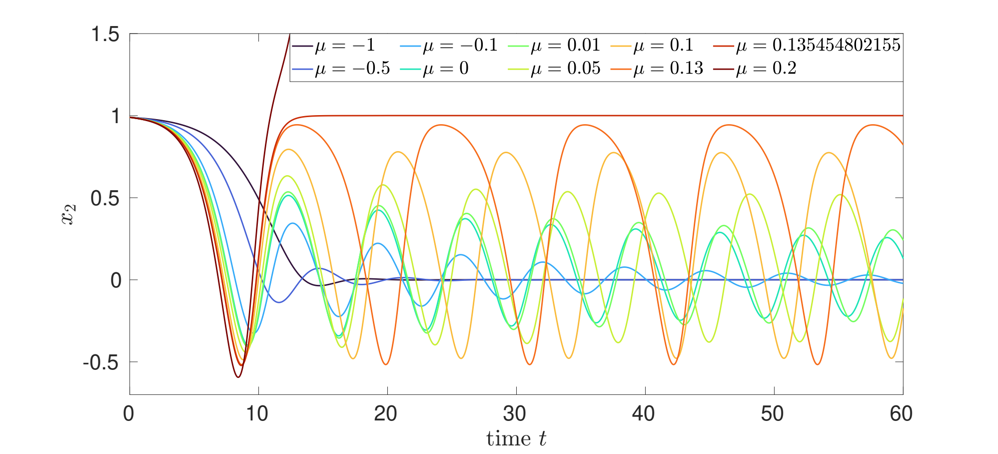


Example: homoclinic bifurcation and supercritical Hopf bifurcation



bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}(\sqrt{\mu - \mu_c})$	$\mathcal{O}(1)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}(\big\vert \log \vert \mu - \mu_c \vert \big\vert)$

Summary: bifurcations of limit cycles

bifurcation at $\mu = \mu_c$	amplitude	period
supercritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
subcritical Hopf bifurcation	$\mathcal{O}\left(\sqrt{\lvert\mu - \mu_c\rvert}\right)$	$\mathcal{O}(1)$
saddle-node bifurcation of cycles	$\mathcal{O}(1)$	$\mathcal{O}(1)$
infinite-period (SNIC, SNIPER)	$\mathcal{O}(1)$	$\mathcal{O}\left(\frac{1}{\sqrt{\lvert\mu - \mu_c\rvert}}\right)$
homoclinic (saddle-loop) bifurcation	$\mathcal{O}(1)$	$\mathcal{O}\left(\big\lvert \log\lvert\mu - \mu_c\rvert\big\rvert\right)$

Additional examples: Questions 1, 4 and 5 on Problem Sheet 3.
They are formulated in a way that the questions do not specify what bifurcations of limit cycles are there.

There are also Questions 2 and 6 on Problem Sheet 3 which ask you to look for a Hopf bifurcation.

Summary of Lecture 12

- weekly nonlinear-oscillators: $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, g\left(x, \dfrac{\mathrm{d}x}{\mathrm{d}t}\right)$ where $0 < \varepsilon \ll 1$
- we have applied the Poincaré-Lindstedt method to examples of both conservative and non-conservative systems
- conservative systems:
  - we have analysed $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, x^3$ (derivation on whiteboard, no slides)
  - additional example $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \varepsilon\, x^2$ is analyzed in Question 3 on Problem Sheet 3 (solutions are available on the course website)
- non-conservative systems: we considered the van der Pol oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -x + \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

  which can be analyzed using the Poincaré-Lindstedt method for $\mu = \varepsilon \ll 1$

Summary of Lecture 12: van der Pol oscillator

$$\frac{\mathsf{d}^2 x}{\mathsf{d}t^2} = -\,x \,+\, \mu\,(1 - x^2)\,\frac{\mathsf{d}x}{\mathsf{d}t}$$

Summary of Lecture 12: van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x + \mu(1 - x^2)\frac{\mathrm{d}x}{\mathrm{d}t}$$

Denoting $y_1 = x$ and $y_2 = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite the van der Pol equation as

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -y_1 + \mu(1 - y_1^2)\, y_2 \end{aligned}$$

Summary of Lecture 12: van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x + \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

Denoting $y_1 = x$ and $y_2 = \frac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite the van der Pol equation as

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- The origin $\mathbf{0} = [0, 0]$ is the only critical point.

Summary of Lecture 12: van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x + \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

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- The origin $\mathbf{0} = [0, 0]$ is the only critical point.
- The Jacobian matrix $D\mathbf{f}(\mathbf{0}) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$ has eigenvalues $\lambda_{\pm} = \frac{\mu}{2} \pm \frac{\sqrt{\mu^2 - 4}}{2}$.

Summary of Lecture 12: van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -x + \varepsilon\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

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- The origin $\mathbf{0} = [0, 0]$ is an unstable spiral for $0 < \mu = \varepsilon \ll 1$.

Summary of Lecture 12: van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathrm{d}^2 x}{\mathrm{d}\tau^2} = -\,x \,+\, \varepsilon\,\omega(\varepsilon)(1-x^2)\,\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Denoting $y_1 = x$ and $y_2 = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite the van der Pol equation as

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_2 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= -\,y_1 \,+\, \mu\,(1-y_1^2)\,y_2\end{aligned}$$

- The origin $\mathbf{0} = [0,0]$ is the only critical point.
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- The origin $\mathbf{0} = [0,0]$ is an unstable spiral for $0 < \mu = \varepsilon \ll 1$.
- To apply the Poincaré-Lindstedt method for $\mu = \varepsilon$, we transformed the time variable as $\tau = \omega(\varepsilon)\,t$ where $2\pi/\omega(\varepsilon)$ is the period of the periodic solution.

Summary of Lecture 12: van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathsf{d}^2 x}{\mathsf{d}\tau^2} = -\,x \,+\, \varepsilon\,\omega(\varepsilon)(1 - x^2)\,\frac{\mathsf{d}x}{\mathsf{d}\tau}$$

Substituting

$$x(\tau;\varepsilon) = x_0(\tau) \,+\, \varepsilon\, x_1(\tau) \,+\, \varepsilon^2\, x_2(\tau) \,+\, \ldots \quad \text{and} \quad \omega(\varepsilon) = \omega_0 \,+\, \varepsilon\,\omega_1 \,+\, \varepsilon^2\,\omega_2 + \ldots$$

and equating coefficients of ε^0 and ε^1, we have obtained $\omega_0 = 1$, $x_0(\tau) = A\,\cos(\tau)$ and

$$\frac{\mathsf{d}^2 x_1}{\mathsf{d}\tau^2} + x_1 = -2\,\omega_1\,\frac{\mathsf{d}^2 x_0}{\mathsf{d}\tau^2} + (1 - x_0^2)\,\frac{\mathsf{d}x}{\mathsf{d}\tau} = 2\,\omega_1\,A\,\cos(\tau) + \left(\frac{A^3}{4} - A\right)\sin(\tau) + \frac{A^3}{4}\,\sin(3\tau)$$

Summary of Lecture 12: van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} = -\,x \;+\; \varepsilon\,\omega(\varepsilon)(1-x^2)\,\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Substituting

$$x(\tau;\varepsilon) = x_0(\tau) \;+\; \varepsilon\,x_1(\tau) \;+\; \varepsilon^2\,x_2(\tau) \;+\; \ldots \quad \text{and} \quad \omega(\varepsilon) = \omega_0 \;+\; \varepsilon\,\omega_1 \;+\; \varepsilon^2\,\omega_2 + \ldots$$

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$$\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2}+x_1 = -2\,\omega_1\,\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2}+(1-x_0^2)\,\frac{\mathrm{d}x}{\mathrm{d}\tau} = 2\,\omega_1\,A\,\cos(\tau)+\left(\frac{A^3}{4}-A\right)\sin(\tau)+\frac{A^3}{4}\,\sin(3\tau)$$

Eliminating the secular terms gives $\omega_1 = 0$ and $A = 2$.

Summary of Lecture 12: van der Pol oscillator

$$\omega^2(\varepsilon)\,\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} = -\,x \,+\, \varepsilon\,\omega(\varepsilon)(1-x^2)\,\frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Substituting

$$x(\tau;\varepsilon) = x_0(\tau) \,+\, \varepsilon\,x_1(\tau) \,+\, \varepsilon^2\,x_2(\tau) \,+\, \ldots \quad \text{and} \quad \omega(\varepsilon) = \omega_0 \,+\, \varepsilon\,\omega_1 \,+\, \varepsilon^2\,\omega_2 + \ldots$$

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Eliminating the secular terms gives $\omega_1 = 0$ and $A = 2$.

We have $x(\tau;\varepsilon) = 2\,\cos(\omega t) \,+\, \varepsilon\,\sin^3(\omega t) \,+ \ldots$ with $\omega = 1 - \varepsilon^2/16 + \ldots$

$\Rightarrow$ the limit cycle is approximately circular with radius 2 for $\mu = \varepsilon \ll 1$

van der Pol oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\,x \,+\, \mu\,(1 - x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t}$$

analysis for $\mu \ll 1$:

Poincaré-Lindstedt method implies that the limit cycle is approximately circular with radius 2 and period $\dfrac{2\pi}{1 - \varepsilon^2/16 + \ldots}$

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intermediate values of μ: we can computationally investigate limit cycles

analysis for $\mu \gg 1$: the limit cycles has period $\mu(3 - 3\log(2))$ as $\mu \to \infty$

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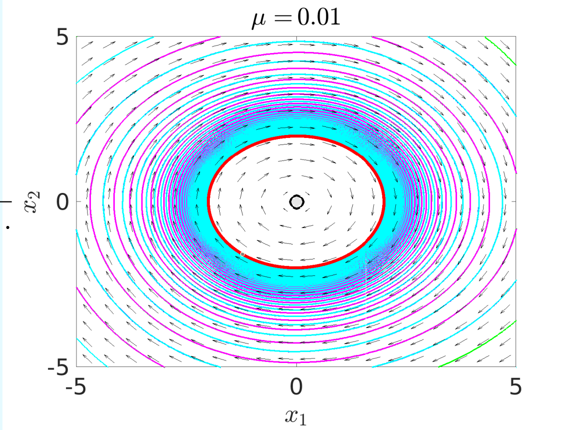

van der Pol oscillator

$$\frac{d^2x}{dt^2} = -x + \mu(1 - x^2)\frac{dx}{dt}$$

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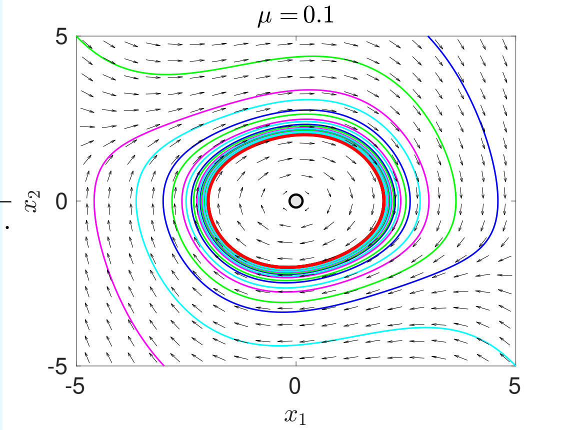

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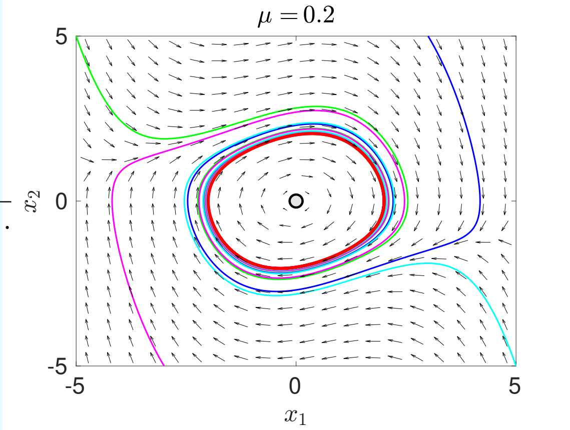

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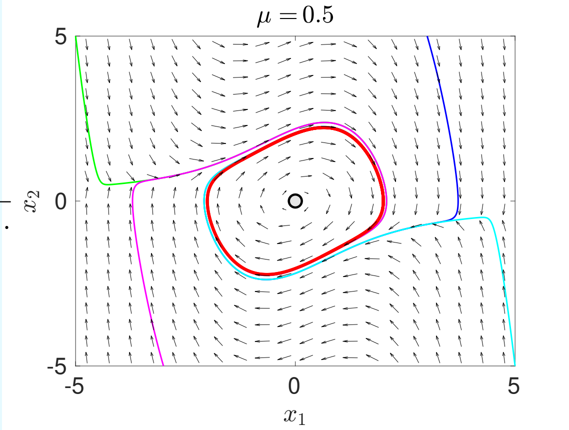

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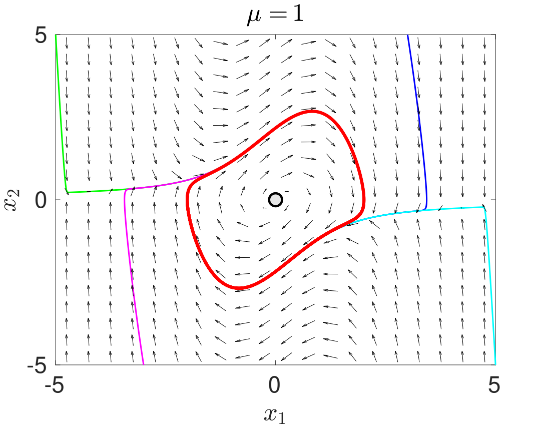

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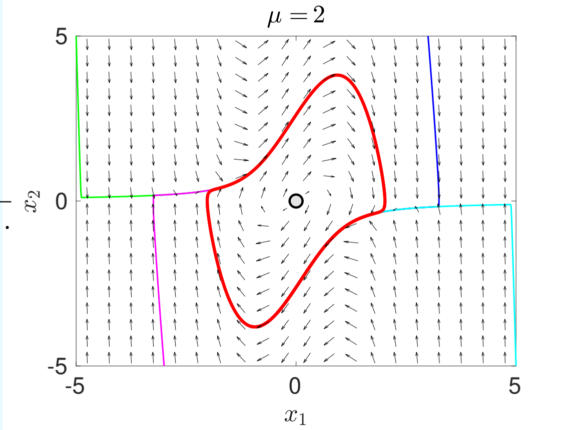

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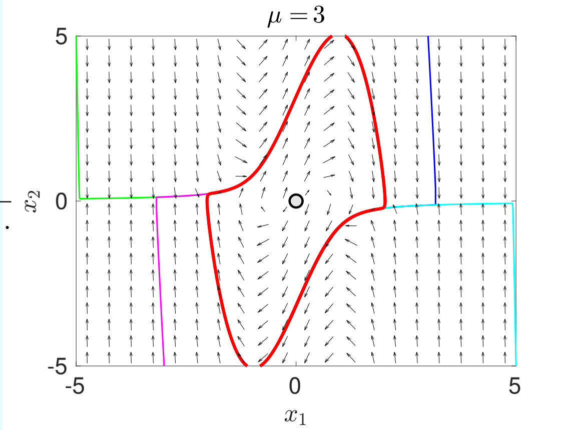

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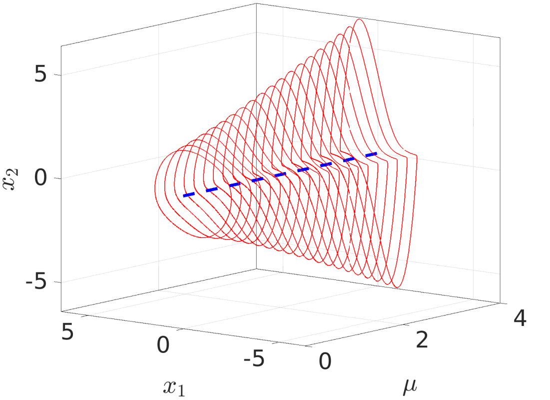

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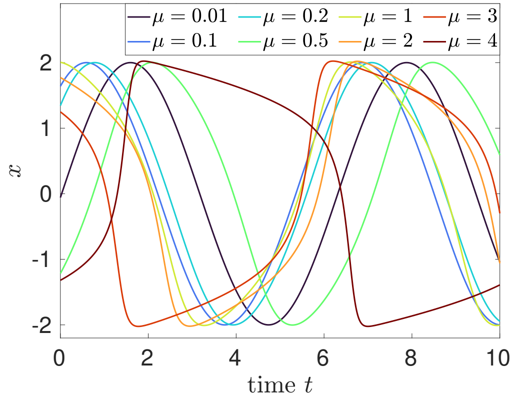

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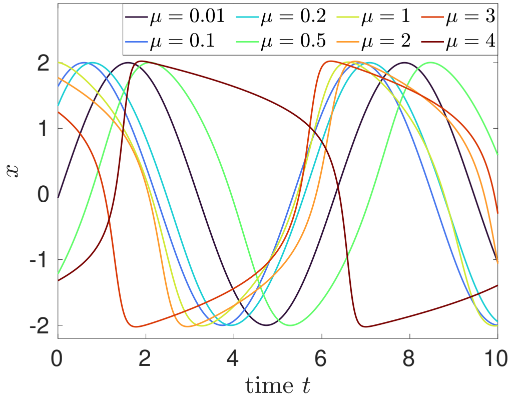


the van der Pol equation is a special case of the Liénard equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -g(x) - f(x)\,\frac{\mathrm{d}x}{\mathrm{d}t} \qquad \text{for} \quad g(x) = x \quad \text{and} \quad f(x) = \mu\,(x^2 - 1)$$

Another Liénard equation

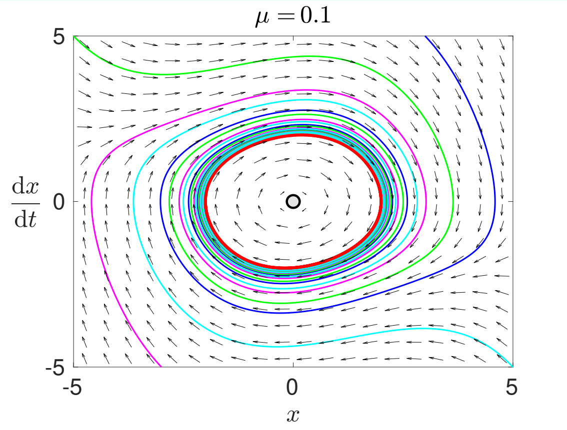
$$\frac{\mathsf{d}^2 x}{\mathsf{d}t} = -\,x \,+\, \mu\, h(x)\, \frac{\mathsf{d}x}{\mathsf{d}t}$$

van der Pol equation: $h(x) = 1 - x^2$

Another Liénard equation

$$\frac{d^2 x}{dt} = -x + \mu\, h(x)\, \frac{dx}{dt}$$

van der Pol equation: $h(x) = 1 - x^2$



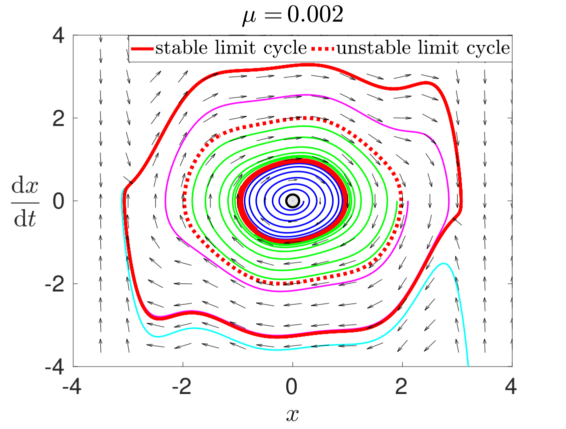
Another Liénard equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t} = -\,x \,+\, \mu\, h(x)\, \frac{\mathrm{d}x}{\mathrm{d}t}$$

van der Pol equation: $h(x) = 1 - x^2$

6-th order polynomial:
$h(x) = 72 \,-\, 392\, x^2 \,+\, 224\, x^4 \,-\, 25\, x^6$

three limit cycles: two limit cycles are stable and one limit cycle is unstable



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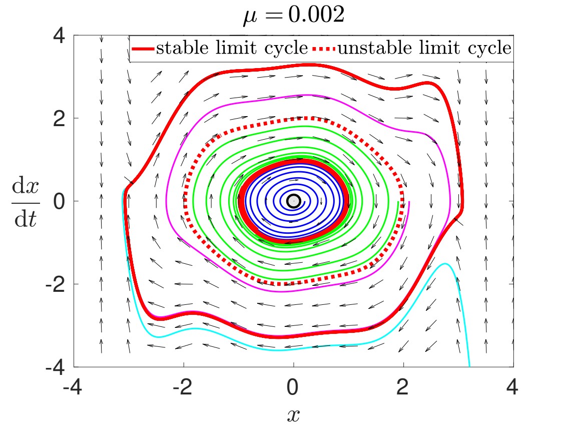
$h(x) = 72 - 392\,x^2 + 224\,x^4 - 25\,x^6$

three limit cycles: two limit cycles are stable and one limit cycle is unstable

Question 7 on Problem Sheet 3

synthetic biology, DNA computing, engineering artificial networks

Lecture 7 of course B5.1 on YouTube:
`https://people.maths.ox.ac.uk/erban/cupbook/`



Lorenz equations: Part 2

$$\frac{\mathsf{d}x_1}{\mathsf{d}t} = \mu_2 \left(x_2 - x_1\right)$$

$$\frac{\mathsf{d}x_2}{\mathsf{d}t} = \mu_1 \, x_1 - x_2 - x_1 \, x_3$$

$$\frac{\mathsf{d}x_3}{\mathsf{d}t} = x_1 \, x_2 - \mu_3 \, x_3$$

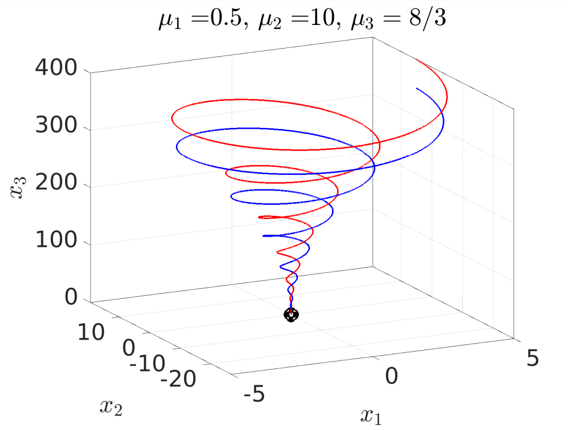
Lorenz equations: summary of Part 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \mu_3\,x_3$$

Lecture 8: we started with a 3D demonstration viewing trajectories in the phase space for different values of parameters μ_1, μ_2, μ_3 and illustrating the convergence to fixed points, limit cycles, chaos, strange attractor and transient chaos

Lorenz equations: summary of Part 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 10\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \frac{8\,x_3}{3}$$



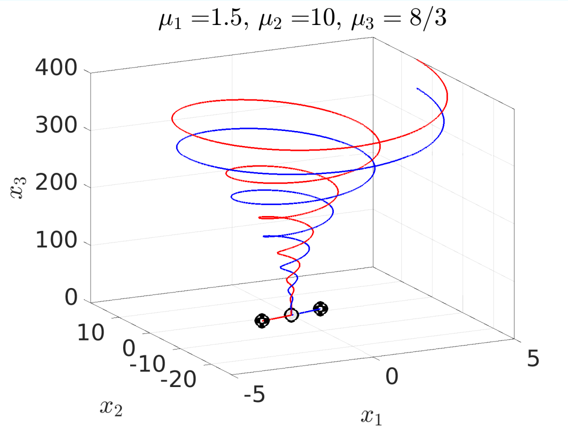
Lecture 8: we started with a 3D demonstration viewing trajectories in the phase space for different values of parameters μ_1, μ_2, μ_3 and illustrating the convergence to fixed points, limit cycles, chaos, strange attractor and transient chaos

we varied μ_1, while we fixed the values of parameters μ_2 and μ_3:

$$\mu_2 = 10 \text{ and } \mu_3 = \frac{8}{3} \qquad \text{(Lorenz used } \mu_1 = 28 \text{ to get chaos)}$$

Lorenz equations: summary of Part 1

$$\frac{dx_1}{dt} = 10\,(x_2 - x_1)$$
$$\frac{dx_2}{dt} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{dx_3}{dt} = x_1\,x_2 - \frac{8\,x_3}{3}$$



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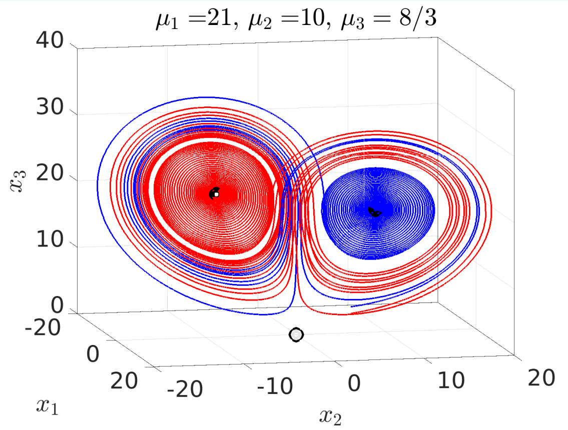
Lorenz equations: summary of Part 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 10\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \frac{8\,x_3}{3}$$



Lecture 8: we started with a 3D demonstration viewing trajectories in the phase space for different values of parameters μ_1, μ_2, μ_3 and illustrating the convergence to fixed points, limit cycles, chaos, strange attractor and transient chaos

we varied μ_1, while we fixed the values of parameters μ_2 and μ_3:

$$\mu_2 = 10 \text{ and } \mu_3 = \frac{8}{3} \qquad \text{(Lorenz used } \mu_1 = 28 \text{ to get chaos)}$$

Lorenz equations: summary of Part 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 10\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
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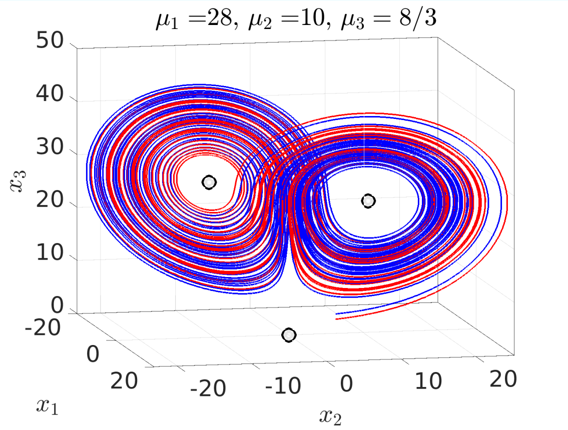

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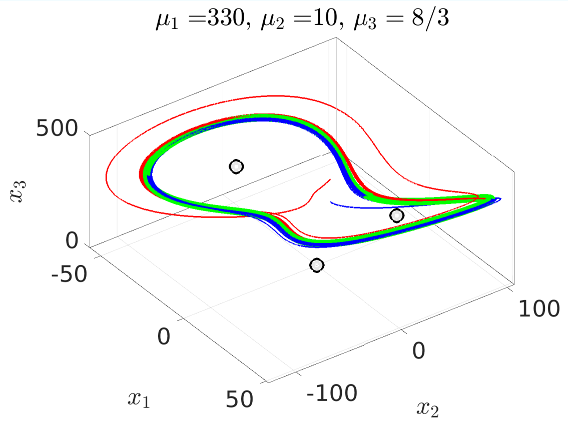

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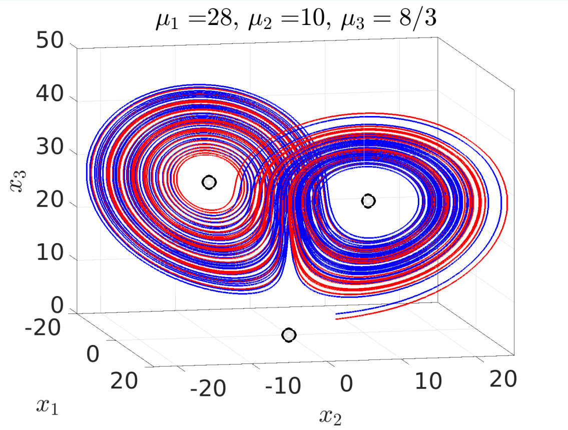

Lecture 8: we used the Lorenz system to further practice some techniques on Problem Sheets 1 and 2

Lorenz equations: summary of Part 1

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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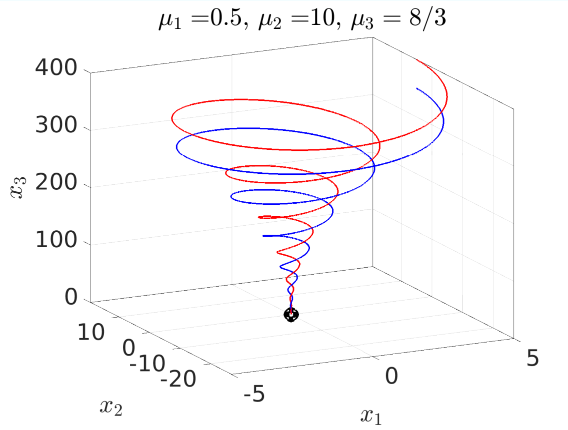

Lecture 8: we used the Lorenz system to further practice some techniques on Problem Sheets 1 and 2 including:

- finding the Lyapunov function to prove the global stability of the fixed point at the origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$

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$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
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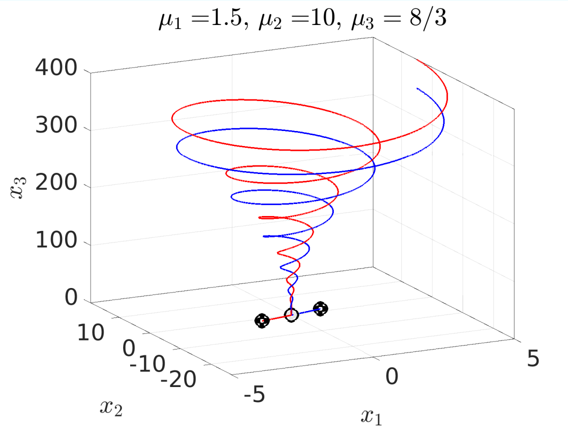

Lecture 8: we used the Lorenz system to further practice some techniques on Problem Sheets 1 and 2 including:

- finding the Lyapunov function to prove the global stability of the fixed point at the origin $\mathbf{0} = [0, 0, 0]$ for $\mu_1 < 1$
- using the extended center manifold theory to analyze the supercritical pitchfork bifurcation at $\mu_1 = 1$, calculating the center manifold and the dynamics on it

Lorenz equations: Question 6 on Problem Sheet 3

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - x_3$$

- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0, 0, 0]$

  $\mathbf{x}_{c2} = \left[\sqrt{\mu_1 - 1}, \sqrt{\mu_1 - 1}, \mu_1 - 1\right]$

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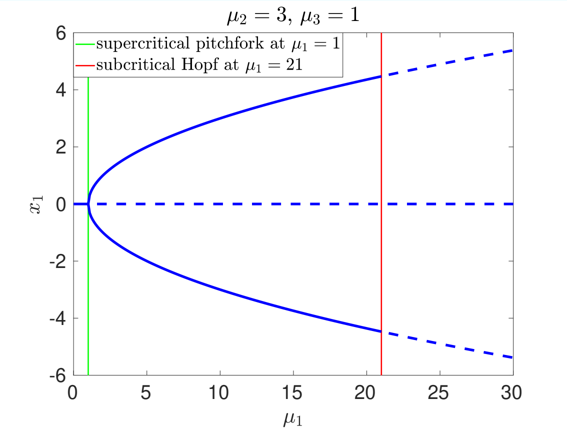

Lorenz equations: Question 6 on Problem Sheet 3

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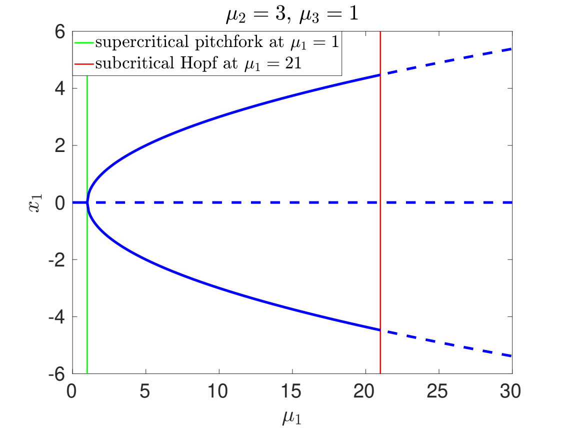

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- supercritical pitchfork bifurcation
  at $\mu_1 = 1$ ($\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1$)

Lorenz equations: Question 6 on Problem Sheet 3

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 3\,(x_2 - x_1)$$
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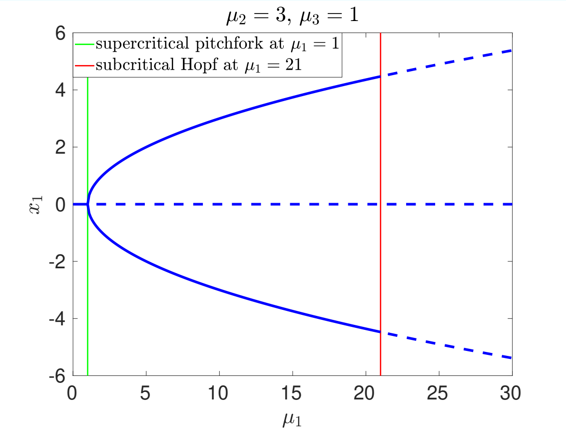

- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0,0,0]$
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  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ only exist for $\mu_1 > 1$
- supercritical pitchfork bifurcation
  at $\mu_1 = 1$ ($\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1$)
- subcritical Hopf bifurcation at $\mu_1 = 21$
  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ are stable for $\mu_1 < 21$ and unstable for $\mu_1 > 21$

Lorenz equations

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
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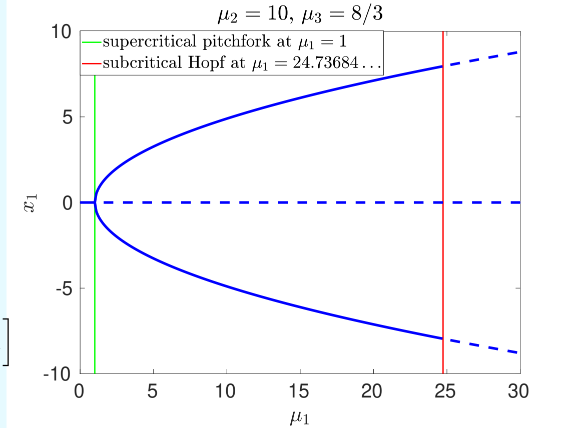

- fixed points $\mathbf{x}_{c1} = \mathbf{0} = [0,0,0]$
  $\mathbf{x}_{c2} = \left[\sqrt{\mu_3(\mu_1-1)}, \sqrt{\mu_3(\mu_1-1)}, \mu_1-1\right]$
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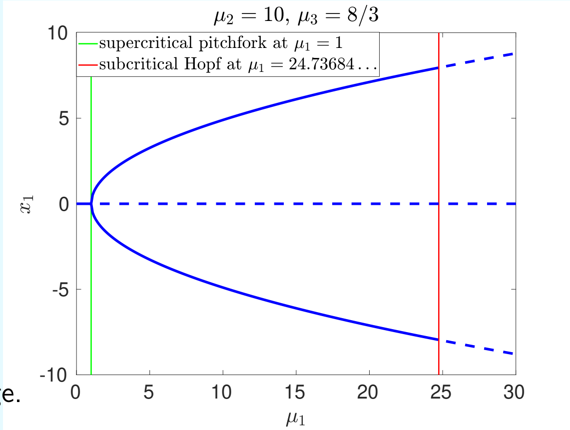
- supercritical pitchfork bifurcation
  at $\mu_1 = 1$ ($\mathbf{x}_{c1} = \mathbf{0}$ is stable for $\mu_1 < 1$ and unstable for $\mu_1 > 1$)
- subcritical Hopf bifurcation at $\mu_1 = \mu_c = \mu_2(\mu_2 + \mu_3 + 3)/(\mu_2 - \mu_3 - 1)$
  $\mathbf{x}_{c2}$ and $\mathbf{x}_{c2}$ are stable for $\mu_1 < \mu_c$ and unstable for $\mu_1 > \mu_c$

Lorenz equations: trapping region

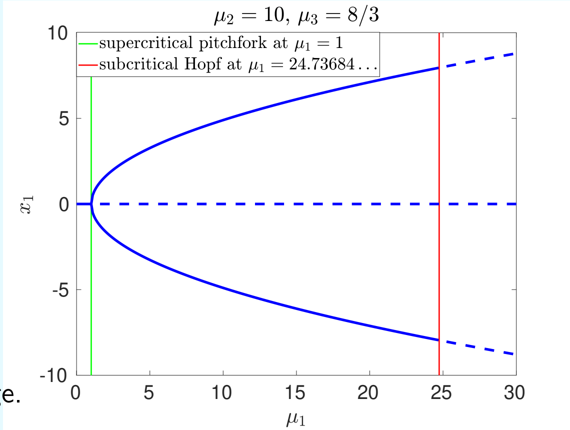
$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = x_1\,x_2 - \mu_3\,x_3$$

Questions 6 on Problem Sheet 4:
All trajectories eventually enter and remain inside a large sphere of the form
$x_1^2 + x_2^2 + (x_3 - \mu_1 - 3)^2 = C(\mu_1)$
where constant $C(\mu_1)$ is sufficiently large.



Lorenz equations: trapping region and volume contraction

$$\frac{dx_1}{dt} = \mu_2 (x_2 - x_1)$$
$$\frac{dx_2}{dt} = \mu_1 x_1 - x_2 - x_1 x_3$$
$$\frac{dx_3}{dt} = x_1 x_2 - \mu_3 x_3$$



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All trajectories eventually enter and remain inside a large sphere of the form
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where constant $C(\mu_1)$ is sufficiently large.

Let $U \equiv U(0) \subset \mathbb{R}^3$ be a compact connected subset of initial conditions.

Let $U(t) = \phi_t(U)$ and $v(t) = |U(t)| = |\phi_t(U)|$ be the volume of $U(t)$. Then

$$\lim_{t\to\infty} v(t) = 0$$

Lorenz equations: Lorenz map

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2\,(x_2 - x_1)$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = \mu_1\,x_1 - x_2 - x_1\,x_3$$

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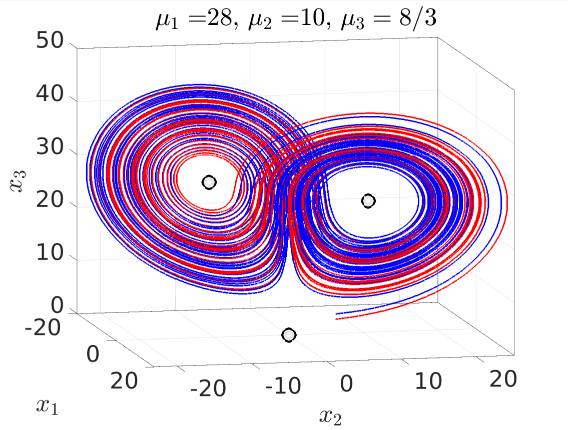
Lorenz map: we investigate chaos using a discrete-time dynamical system



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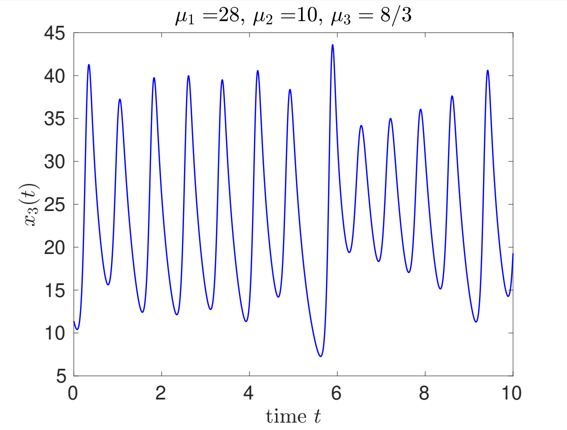

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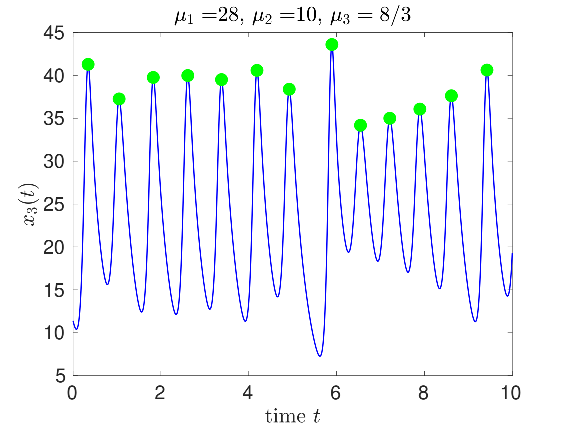

Lorenz map: we investigate chaos using a discrete-time dynamical system

Consider local maxima z_n, $n = 1, 2, \ldots$ of $x_3(t)$ and define Lorenz map by:

$$z_{n+1} = F(z_n)$$

Then a closed orbit corresponds to an N-cycle $\{z_0, z_1, z_2, \ldots, z_{N-1}\}$ of the Lorenz map.

Lorenz equations: Lorenz map

$$\frac{dx_1}{dt} = \mu_2 (x_2 - x_1)$$
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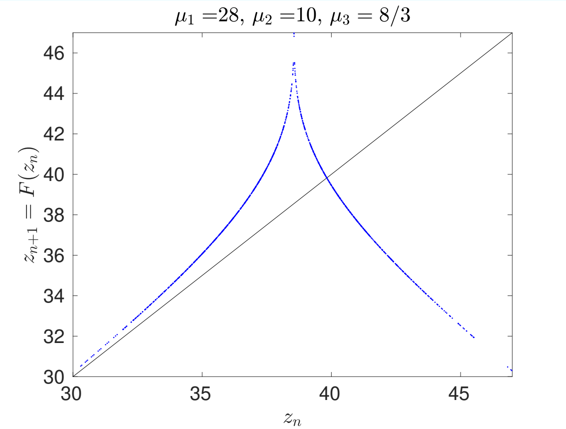

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Lecture 5: N-cycle is *unstable* if $\left|F'(z_0)\, F'(z_1)\, \ldots\, F'(z_{N-1})\right| > 1$

There are no stable fixed points or limit cycles for: $\mu > \mu_c = \dfrac{\mu_2(\mu_2 + \mu_3 + 3)}{\mu_2 - \mu_3 - 1}$

Lorenz equations: Lorenz map

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \mu_2 (x_2 - x_1)$$
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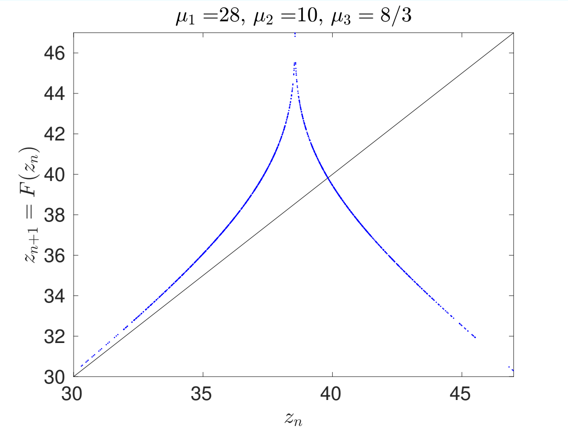

Poincaré map: we investigate ODEs using a discrete-time dynamical system

Poincaré map : saddle-node bifurcation of cycles example

saddle-node bifurcation of cycles at $\mu = -1/4$: a half-stable cycle appears, it splits into a pair of limit cycles for $\mu > -1/4$, one stable, one unstable

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \mu\, r + r^3 - r^5$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1$$

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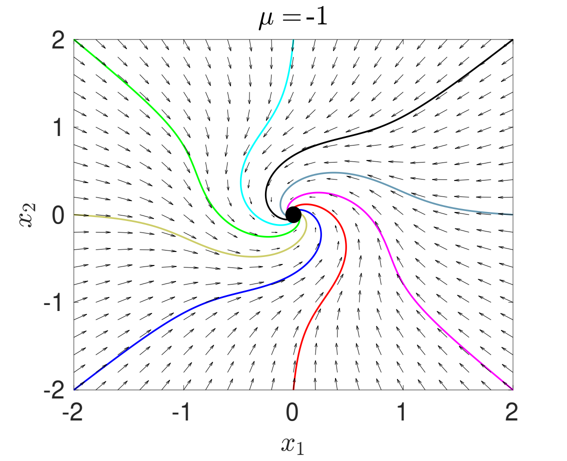

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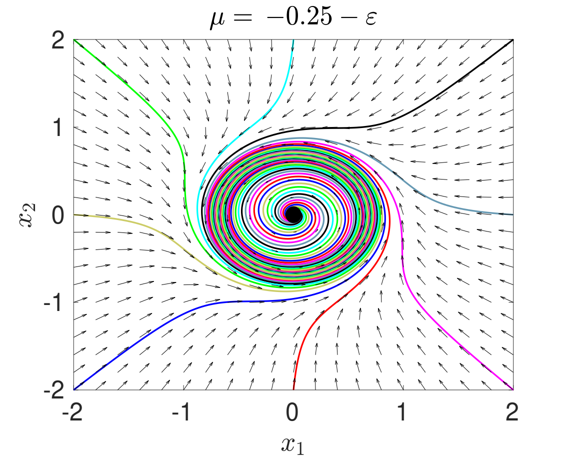

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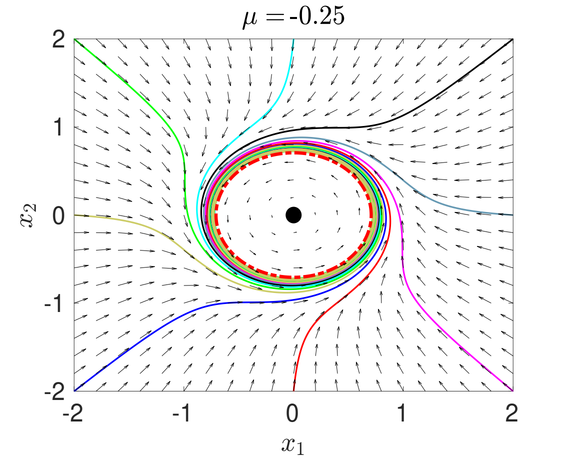

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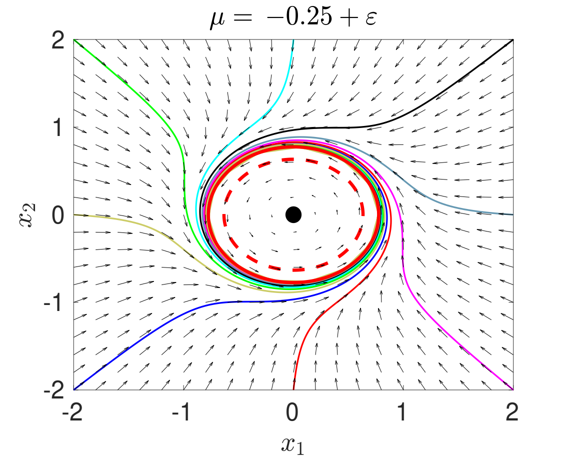

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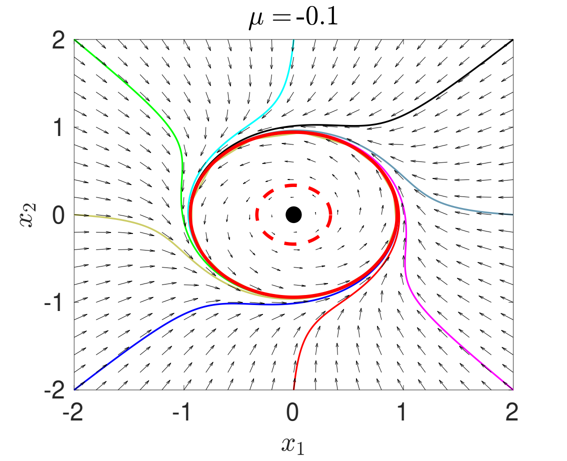
Poincaré section

$$\Sigma = \left\{[x_1, 0] \in \mathbb{R}^2 \,|\, x_1 > 0\right\}$$

Poincaré map: $P_\mu : \Sigma \to \Sigma$, where $P_\mu(x_1) = P(x_1;\mu)$ is defined such that the positive semi-orbit of $[x_1, 0]$ intersects Σ for the first time at $[P_\mu(x_1), 0]$.



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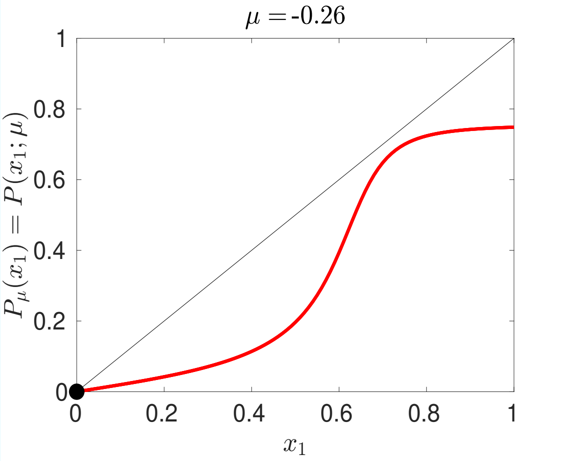

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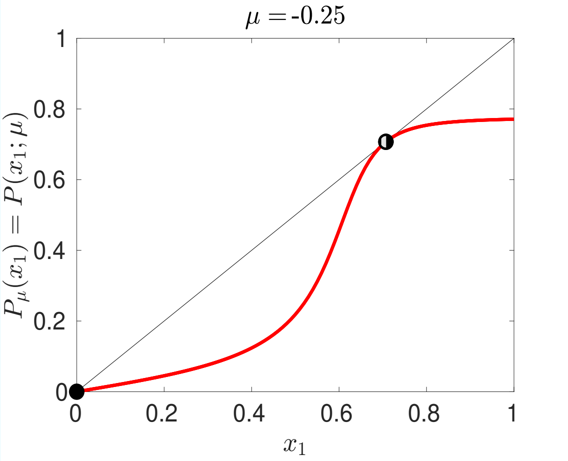

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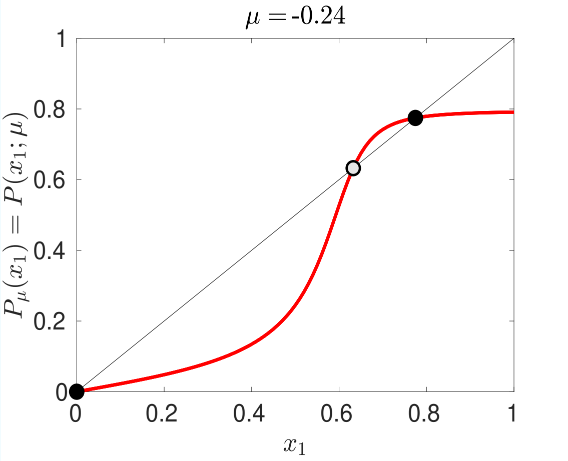

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Poincaré section

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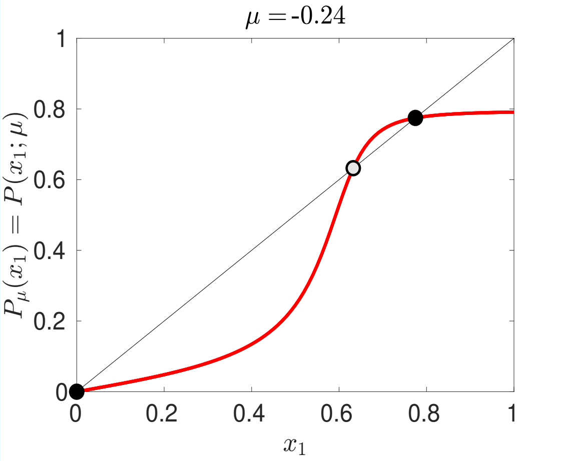
Poincaré map: $P_\mu : \Sigma \to \Sigma$, where $P_\mu(x_1) = P(x_1; \mu)$ is defined such that the positive semi-orbit of $[x_1, 0]$ intersects Σ for the first time at $[P_\mu(x_1), 0]$.



Another example: Question 1 on Problem Sheet 4

Lecture 15: summary

- we discussed chaos, symbolic dynamics and the Bernoulli shift map (whiteboard lecture, no slides)
- we studied dynamical systems associated with function $F : \mathbb{M} \to \mathbb{M}$, where $\mathbb{M}$ is a metric space, *i.e.* a set with metric (distance) $d : \mathbb{M} \times \mathbb{M} \to [0, \infty)$

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- $F : \mathbb{M} \to \mathbb{M}$ is called *transitive* if there exists $x_0 \in \mathbb{M}$ such that orbit $O(x_0) = \big\{x_0, F(x_0), F^{(2)}(x_0), F^{(3)}(x_0), \dots\big\}$ is a dense subset of $\mathbb{M}$ (a *transitive point* of F is a point $x_0 \in \mathbb{M}$ which has a dense orbit $O(x_0)$ under F)
- $F : \mathbb{M} \to \mathbb{M}$ has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that for all $x \in \mathbb{M}$ and any open set $U \subset \mathbb{M}$ satisfying $x \in U$, there is a point $y \in U$ and $j \in \mathbb{N}$ with $d\big(F^{(j)}(x), F^{(j)}(y)\big) > \delta$

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- if $F : \mathbb{M} \to \mathbb{M}$ is continuous and $\mathbb{M}$ is not a finite set, then (i) and (ii) imply (iii)

Lecture 15: summary – Bernoulli shift map

$\mathbb{M}_{01} = \Big\{ (a_1, a_2, a_3, a_4, \dots) \,\Big|$ such that $a_j = 0$ or $a_j = 1$ for $j = 1, 2, 3, 4, \dots \Big\}$

$\mathbb{M}_{01}$ is a metric space with metric defined by

$d(x, y) = \displaystyle\sum_{j=1}^{\infty} \frac{|a_j - b_j|}{2^j}$ for $x = (a_1, a_2, a_3, a_4, \dots)$ and $y = (b_1, b_2, b_3, b_4, \dots)$

Bernoulli shift map: $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ where $\sigma((a_1, a_2, a_3, a_4, \dots)) = (a_2, a_3, a_4, a_5 \dots)$

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we stated and proved some of properties of the shift map, namely:

- fixed points are $(0, 0, 0, 0, \dots)$ and $(1, 1, 1, 1, \dots)$
  2-cycle is $\big\{ (0, 1, 0, 1, 0, 1, 0, 1, \dots), (1, 0, 1, 0, 1, 0, 1, 0, \dots) \big\}$
- shift map $\sigma : \mathbb{M}_{01} \to \mathbb{M}_{01}$ is continuous and chaotic (we proved several lemmas showing the continuity and properties (i), (ii) and (iii) in our definition of chaos)

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we also discussed that we could obtain the same properties if we worked with the metric space of bi-infinite sequences of 0's and 1's, *i.e.* where

$x = \big(\dots, a_{-j}, \dots, a_{-2}, a_{-1} \,|\, a_0, a_1, a_2, \dots, a_j, \dots \big)$ and

$y = \big(\dots, b_{-j}, \dots, b_{-2}, b_{-1} \,|\, b_0, b_1, b_2, \dots, b_j, \dots \big)$ have distance $d(x, y) = \displaystyle\sum_{j=-\infty}^{\infty} \frac{|a_j - b_j|}{2^{|j|}}$

Question 3 on Problem Sheet 4

Let $x_0 \in [0,1)$ and $F : [0,1) \to [0,1)$. Define sequence $x_k \in [0,1)$, $k = 0,1,2,\ldots,$ iteratively by $x_{k+1} = F(x_k)\,,$ where

$$F(x) = \begin{cases} 2x & \text{for } x \in [0,1/2) \\ 2x-1 & \text{for } x \in [1/2,1) \end{cases}$$

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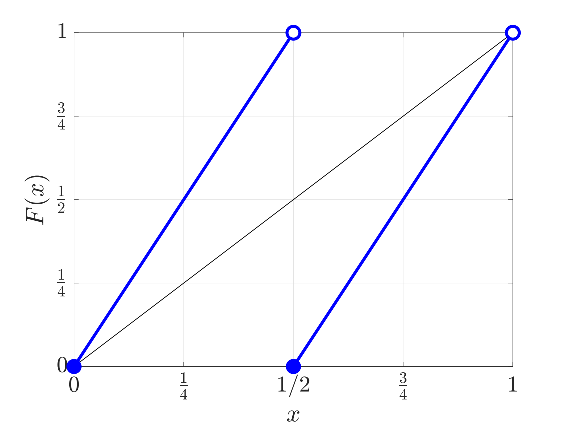

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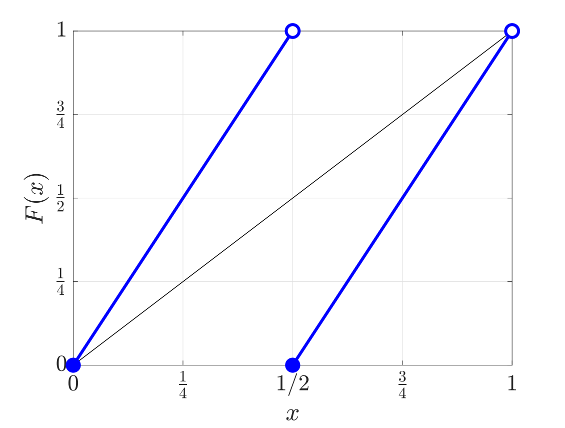

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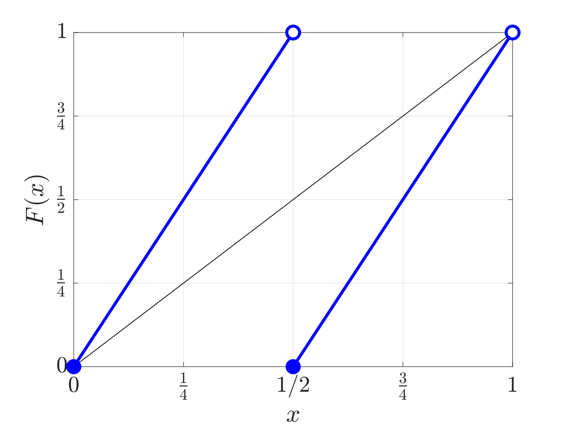

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Q3(a): if $x_0 \in [0,1)$ is not a dyadic rational, then $F^{(k)}(x) = 0.a_{k+1}a_{k+2}a_{k+3}a_{k+4}\dots$

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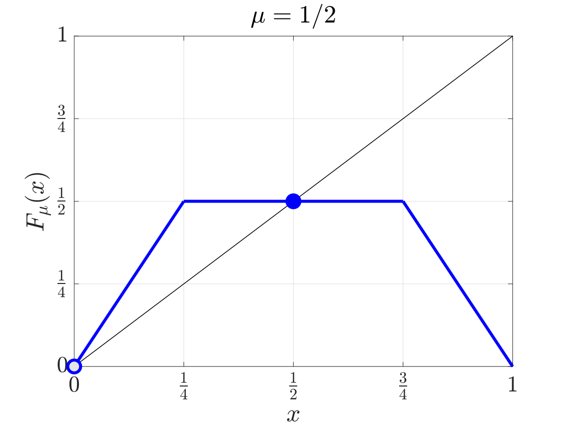

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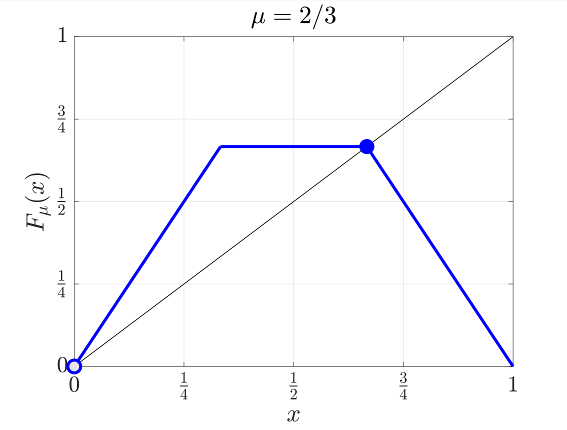

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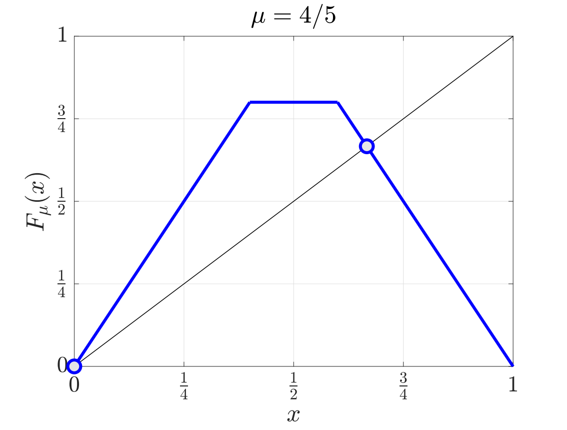

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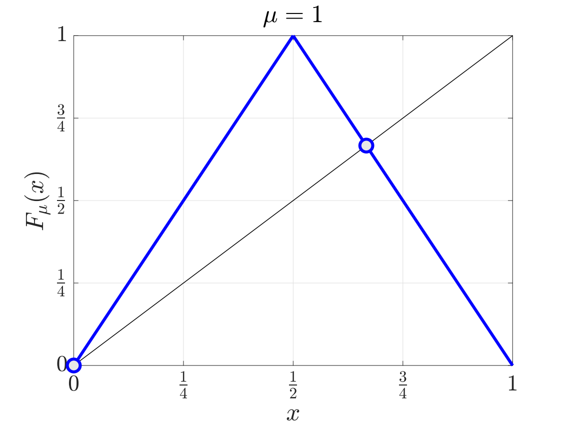

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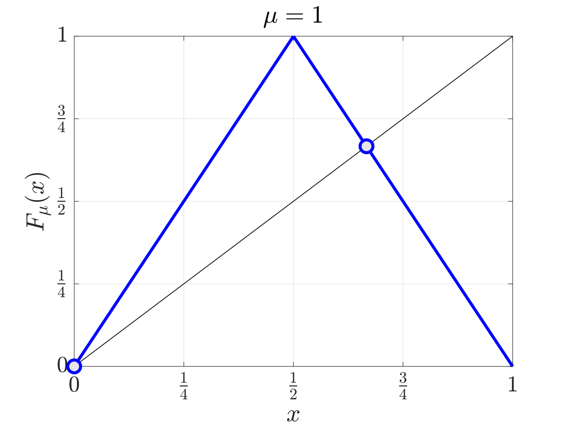

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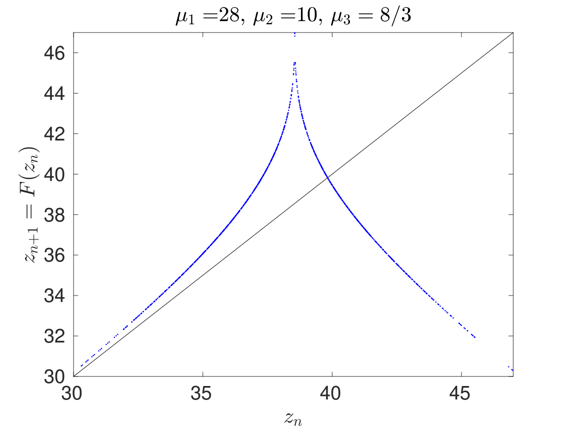

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some maps look 'similar' to the tent map: Lorenz map

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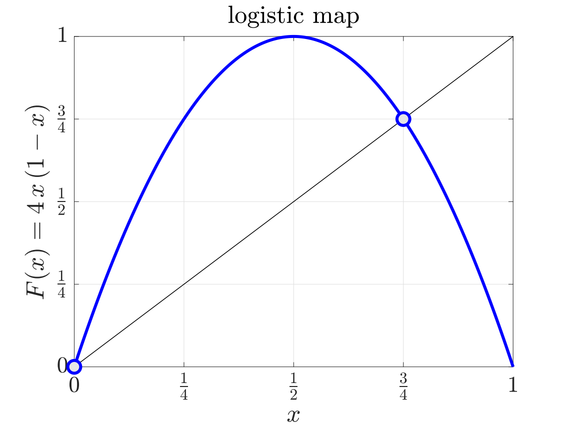

some maps look 'similar' to the tent map: logistic map $F(x) = 4\,x\,(1-x)$

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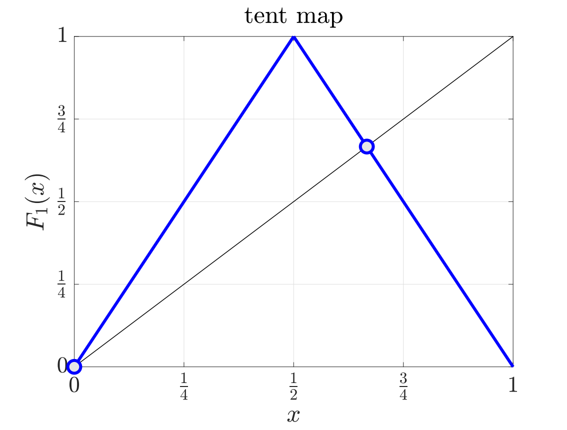

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some maps look 'similar' to the tent map and tent map $F_1(x)$ is chaotic

Homeomorphism

Definition: Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two metric spaces. A function $h : \mathbb{M}_1 \to \mathbb{M}_2$ is a *homeomorphism* if: (i) h is continuous;

(ii) h is one-to-one, *i.e.* if $h(x) = h(y)$, then $x = y$;

(iii) h is onto, *i.e.* $\forall y \in \mathbb{M}_2$ there exists $x \in \mathbb{M}_1$ such that $h(x) = y$;

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Theorem: Let $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ be continuous maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, and assume that there is a conjugacy $h : \mathbb{M}_1 \to \mathbb{M}_2$ with $h \circ F_1 = F_2 \circ h$. Then F_1 is chaotic if and only if F_2 is chaotic.

Conjugate maps

Definition: Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two metric spaces. A function $h : \mathbb{M}_1 \to \mathbb{M}_2$ is a *homeomorphism* if: (i) h is continuous;

(ii) h is one-to-one, *i.e.* if $h(x) = h(y)$, then $x = y$;

(iii) h is onto, *i.e.* $\forall y \in \mathbb{M}_2$ there exists $x \in \mathbb{M}_1$ such that $h(x) = y$;

(iv) the inverse mapping $h^{-1} : \mathbb{M}_2 \to \mathbb{M}_1$ is continuous.

Definition: Let $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ be maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. Then F_1 and F_2 are said to be *conjugate* if there is a homeomorphism $h : \mathbb{M}_1 \to \mathbb{M}_2$ such that $h \circ F_1 = F_2 \circ h$.

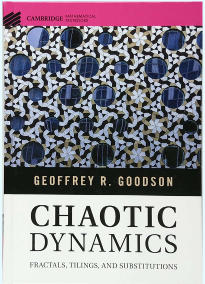
Theorem: Let $F_1 : \mathbb{M}_1 \to \mathbb{M}_1$ and $F_2 : \mathbb{M}_2 \to \mathbb{M}_2$ be continuous maps of metric spaces $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively, and assume that there is a conjugacy $h : \mathbb{M}_1 \to \mathbb{M}_2$ with $h \circ F_1 = F_2 \circ h$. Then F_1 is chaotic if and only if F_2 is chaotic.

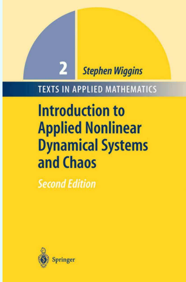
Example: tent map $F_1(x) = \begin{cases} 2x & \text{for } x \in [0, 1/2] \\ 2 - 2x & \text{for } x \in [1/2, 1] \end{cases}$ is conjugate to the logistic map $F_2(x) = 4\,x\,(1 - x)$ with conjugacy $h : [0, 1] \to [0, 1]$ given as $h(x) = \sin^2(\pi x/2)$

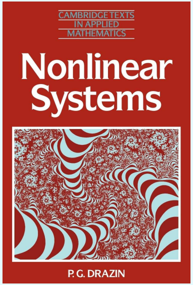
$\Longrightarrow$ logistic map $F_2(x) = 4\,x\,(1 - x)$ is chaotic

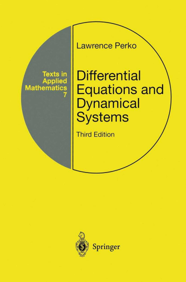
Further reading and exam preparation

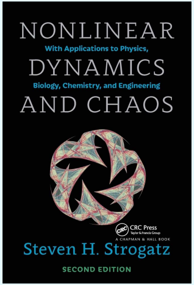
There are 6 books in the Reading List which include a lot of additinal examples:

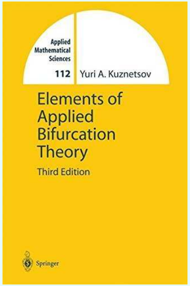












Past papers are available here:
www.maths.ox.ac.uk/members/students/undergraduate-courses/examinations-assessments/past-papers

Please note that this course was called B5.6 *Nonlinear Systems* in previous years. It was renamed to B5.6 *Nonlinear Dynamics, Bifurcations and Chaos* to give a clearer sense of what the course covers.