

# **Further Partial Differential Equations**

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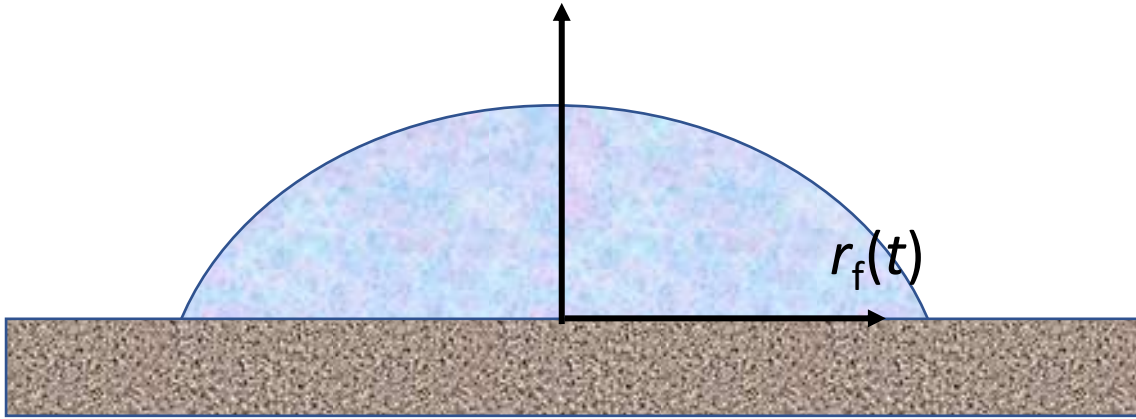
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## Problem sheet classes

- Problem sheet classes will take place on Tuesday of Week 3, 5, 7 and 9 at 1–2pm and 2–3pm with Dominic Vella (tutor) and Grace Curtis and Torin Fastnedge (TAs).
- Hand-in time: Friday 1pm Weeks 2, 4, 6, and 8.

## Similarity solutions

- Consider the position of the front  $r_f$  as a function of time  $t$



- The spreading behaviour is the same for all liquids.
- We see similar kinds of behaviour in other scenarios, e.g., trapping  $\text{CO}_2$  underground or spreading of honey on porous toast.
- Can we understand the shape this evolves to more easily than solving the full problem?

## Spreading of liquid on a vertical liquid-coated wall

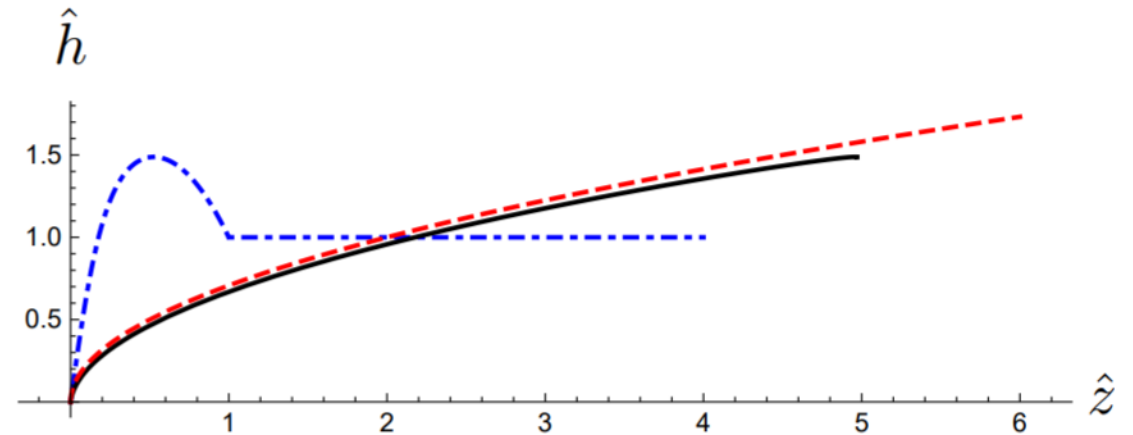
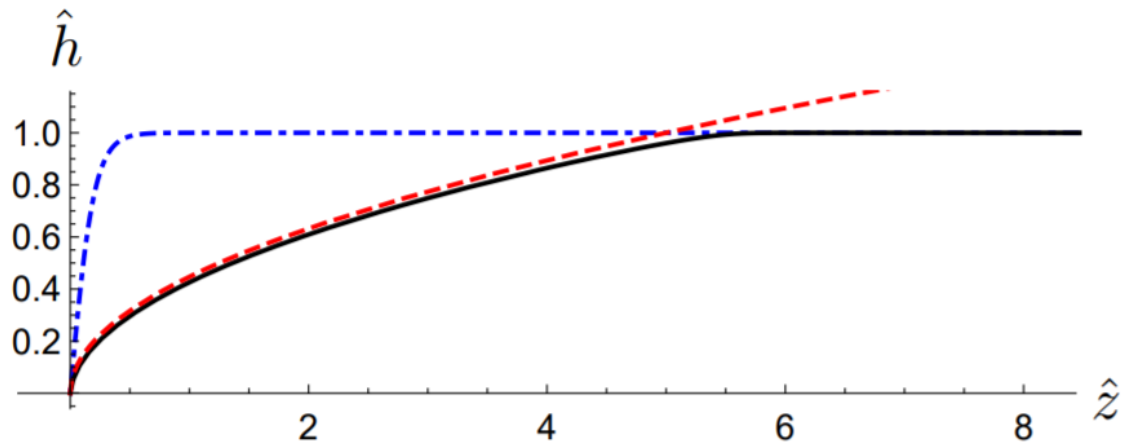
$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{z}} = 0, \quad \hat{Q} = \frac{\rho g \hat{h}^3}{3\mu}$$

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu} \frac{\partial}{\partial \hat{z}} \left( \hat{h}^3 \right) = 0.$$

- What type of equation is this?
- First order hyperbolic equation
- What extra conditions do we need to solve this?
- One initial condition and one boundary condition

$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\rho g}{3\mu} \frac{\partial}{\partial \hat{z}} \left( \hat{h}^3 \right) = 0 \quad \hat{h}(\hat{z}, 0) = \hat{h}_0(\hat{z}) \quad \hat{h}(0, \hat{t}) = 0$$

- We can solve this problem analytically (see problem sheet 1).
- For any initial condition, we appear to evolve to the same final shape.



- Can we extract what this universal shape is?
- First, non-dimensionalize:

$$\hat{z} = \hat{z}_0 z, \quad \hat{t} = \hat{t}_0 t, \quad \hat{h} = \hat{h}_0 h,$$

- There is no natural length or height scale.
- We could choose these using the initial conditions, but there is nothing inherent in the problem to specify these.

$$\frac{\partial h}{\partial t} + \frac{1}{3} \frac{\partial}{\partial z} (h^3) = 0$$

- Try a solution of the form

$$h(z, t) = f(\eta) \quad \eta = \frac{z}{t^\alpha}$$

- Substituting in gives

$$f^2 f' - \frac{z}{t} f' = 0$$

- If we choose  $\alpha=1$  then we turn the equation into one solely in terms of  $\eta$

$$(f^2 - \eta) f' = 0$$

$$f = \eta^{1/2} = \left(\frac{z}{t}\right)^{1/2}$$

$$\hat{h} = \left( \frac{\mu}{\rho g} \right)^{1/2} \left( \frac{\hat{z}}{\hat{t}} \right)^{1/2}$$

- This is precisely the shape that we see all the profiles evolving towards.
- However, in getting this solution, we did not use the initial condition to find this similarity solution. This highlights the fact that our similarity solution ‘forgets’ the initial information.
- Although in our case we could obtain an analytic solution, in many cases we cannot.
- This similarity solution provides key information about the system behaviour without having to solve the full equation.
- Before we move onto the next example, we will show how we can predict the form of the similarity variables.



- We approximate derivatives as  $\frac{\partial y}{\partial x} \sim \frac{Y}{X}$
- This is called a **scaling law** approach.
- Substitute this into our partial differential equation:

$$\frac{\partial h}{\partial t} + \frac{1}{3} \frac{\partial}{\partial z} (h^3) = 0$$

$$H \sim \left( \frac{3\mu}{\rho g} \right)^{1/2} \left( \frac{Z}{T} \right)^{1/2}$$

- Rearranging gives 
$$H \sim \left( \frac{3\mu}{\rho g} \right)^{1/2} \left( \frac{Z}{T} \right)^{1/2}$$

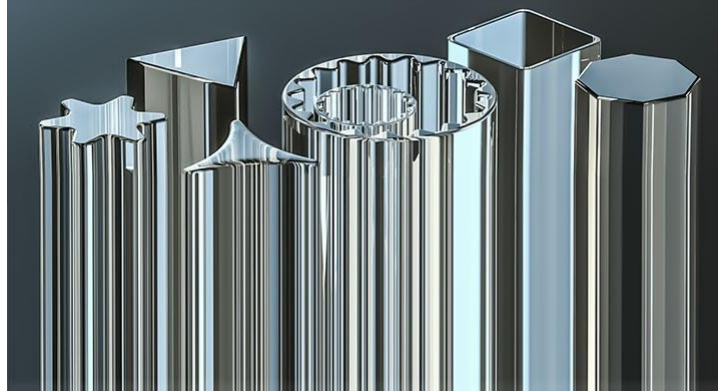
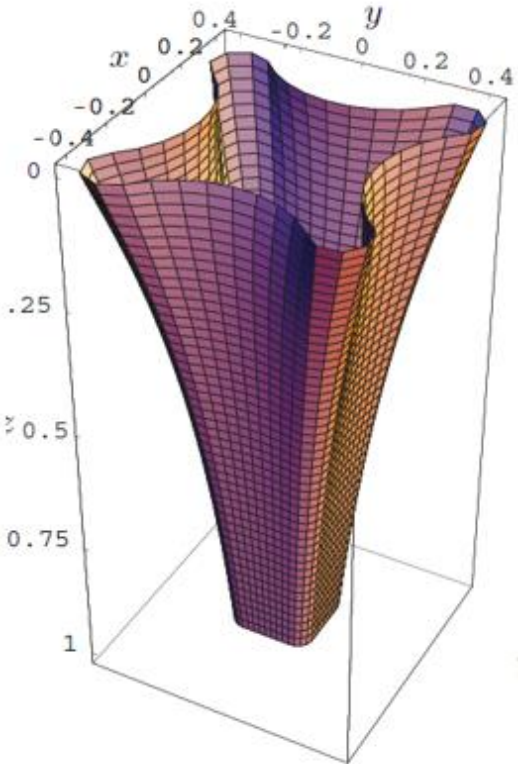
- This has given us almost as much information as the similarity solution.
  - It gives us the right functional form and parametric dependence.
  - It doesn't give us the right prefactor.

- The **scaling law gives** the long time **functional**  $(\hat{z}/\hat{t})^{1/2}$  and **parametric**  $(\mu/\rho g)^{1/2}$  dependence.
- The **similarity solution** gives the **long time solution**.
- The **analytic solution** gives the **correct behaviour for all time**.

**Question: What shape cookie cutter should you choose to bake the perfect gingerbread man?**

The shape needs to take into account the dough spreading in the oven before it sets.

**Application:** Mathematics can tell us what shape hole to pump glass through to make **exotically shaped** glass tubes, bottles and Nestlé Alphabet cereal:



## Summary of lecture 1

- Scaling laws can give the long time functional and parametric dependence
- Similarity solutions give the long time solution
- Analytic solutions give the correct behaviour for all time

## The heat equation

$$\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \quad \hat{Q} = -D \frac{\partial \hat{T}}{\partial \hat{x}}$$

$$\frac{\partial \hat{T}}{\partial \hat{t}} = D \frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

- What kind of equation is this?
- Second order parabolic equation.
- What extra conditions do we need to solve this?
- One initial condition and two boundary conditions.

- We non-dimensionalize the equation:

$$\hat{x} = \hat{x}_0 x, \quad \hat{t} = \frac{\hat{x}_0^2}{D} t, \quad \hat{T} = \left( \hat{T}_{+\infty} - \hat{T}_{-\infty} \right) T + \hat{T}_{-\infty}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x, t) \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty$$

$$T(x, t) \rightarrow 1 \quad \text{as} \quad x \rightarrow +\infty$$

- First let's try the scaling argument:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

- This gives us that our similarity variable is  $\eta = \frac{x}{t^{1/2}}$  but nothing more.
- So this time we try a similarity solution of the form  $T = T(\eta)$  with  $\eta = \frac{x}{t^\alpha}$

$$f'' + \alpha \frac{x}{t^\alpha} t^{2\alpha-1} f' = 0$$

- Our governing equation and boundary conditions then become:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x, t) \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty$$

$$T(x, t) \rightarrow 1 \quad \text{as} \quad x \rightarrow +\infty$$

- This has solution  $f = \frac{1}{2} (1 + \operatorname{erf}(\eta/2))$  where  $\operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-s^2} ds$

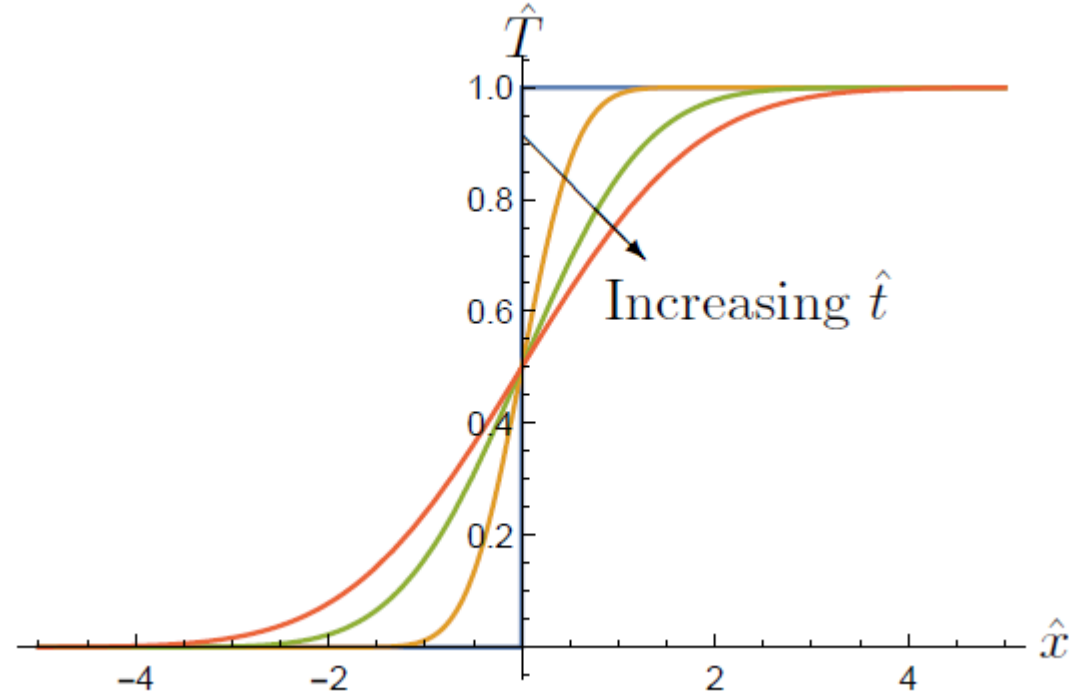
- In terms of dimensional variables, this is

$$\hat{T} = \frac{(\hat{T}_{+\infty} - \hat{T}_{-\infty})}{2} \left( 1 + \operatorname{erf} \left( \frac{\hat{x}}{2\sqrt{D\hat{t}}} \right) \right) + \hat{T}_{-\infty} \quad (*)$$



$$\hat{T} = \frac{(\hat{T}_{+\infty} - \hat{T}_{-\infty})}{2} \left( 1 + \operatorname{erf} \left( \frac{\hat{x}}{2\sqrt{D\hat{t}}} \right) \right) + \hat{T}_{-\infty} \quad (*)$$

- As in the previous example for flow down a vertical wall, we did not use the initial condition to obtain this solution, so the result only holds for long time...
- ...unless the initial condition happens to satisfy (\*) at  $t=0$



- This example was fairly similar to the previous example. However, now let's suppose that  $\hat{T}_{-\infty} = \hat{T}_{+\infty}$
- In this case we must non-dimensionalize  $\hat{T}$  differently.
- The dimensionless equation is the same as before but the boundary conditions are different:
- Question: what happens if we try the same similarity ansatz as before?



- This means we need to seek a more general similarity ansatz if we are to obtain a non-trivial solution.

- We try  $T = t^\beta f(\eta)$  with  $\eta = \frac{x}{t^\alpha}$

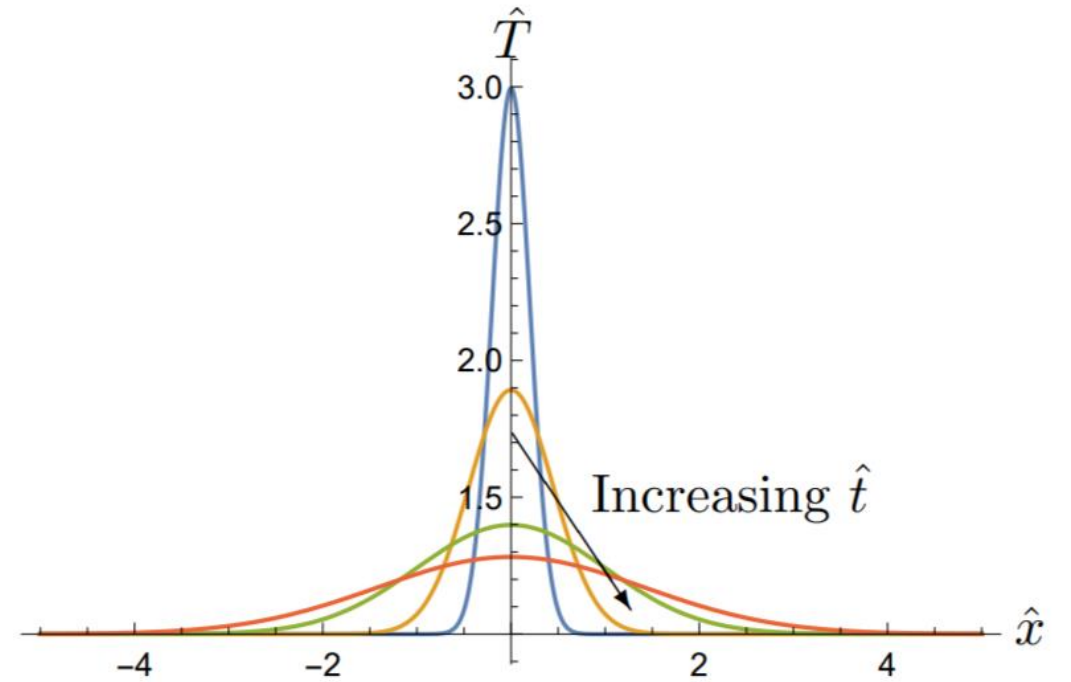
- Substituting into the governing equation and the energy constraint tells us that we need  $\alpha=1/2$  and  $\beta=-1/2$ .

- Then, we get

- This has solution  $f = \frac{E}{2\sqrt{\pi}} e^{-\eta^2/4}$

- In original dimensional terms, this solution is

$$\hat{T} = \frac{\hat{E}}{2\sqrt{\pi D\hat{t}}} \exp\left(-\frac{\hat{x}^2}{4D\hat{t}}\right) + \hat{T}_{+\infty} \quad (*)$$



- Again, we have not imposed an initial condition to obtain this result, but if our initial condition satisfied (\*) for  $t=0$  then this would be the solution for all time.

- But where did the similarity form  $T = t^{-1/2} f\left(\frac{x}{t^{1/2}}\right)$  come from?

- If we make a transformation

$$x = a\tilde{x}, \quad t = b\tilde{t}, \quad T = c\tilde{T}.$$

and substitute into the equation and boundary condition then we find that if

$$\frac{x}{t^{1/2}} = \frac{\tilde{x}}{\tilde{t}^{1/2}} \quad Tt^{1/2} = \tilde{T}\tilde{t}^{1/2}$$

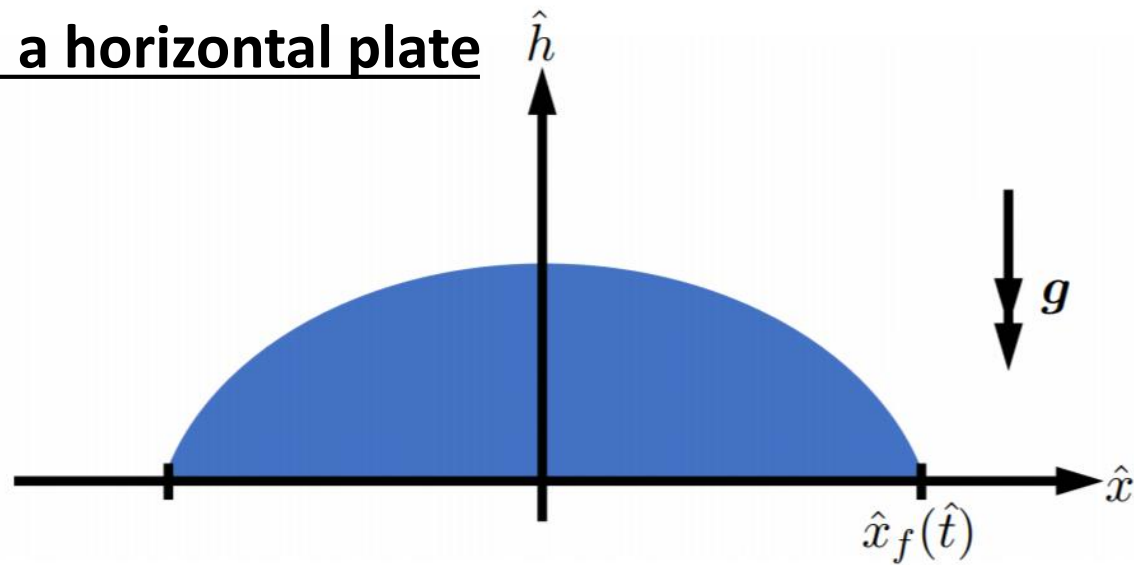
then the equations in terms of the new variables are identical.

- This indicates that  $\frac{x}{t^{1/2}}$  and  $Tt^{1/2}$  are **invariants** of the system.
- This tells us that the solution depends on  $x/t^{1/2} = \eta$  and has the form  $f(\eta) = t^{1/2}T(x, t)$ . which is exactly what we have seen.

## Summary of lecture 2

- We can get **different similarity solutions from the same equation** by applying different boundary conditions.
- **Invariants** of the system tell us the form of the similarity solution.

## Spreading of liquid on a horizontal plate



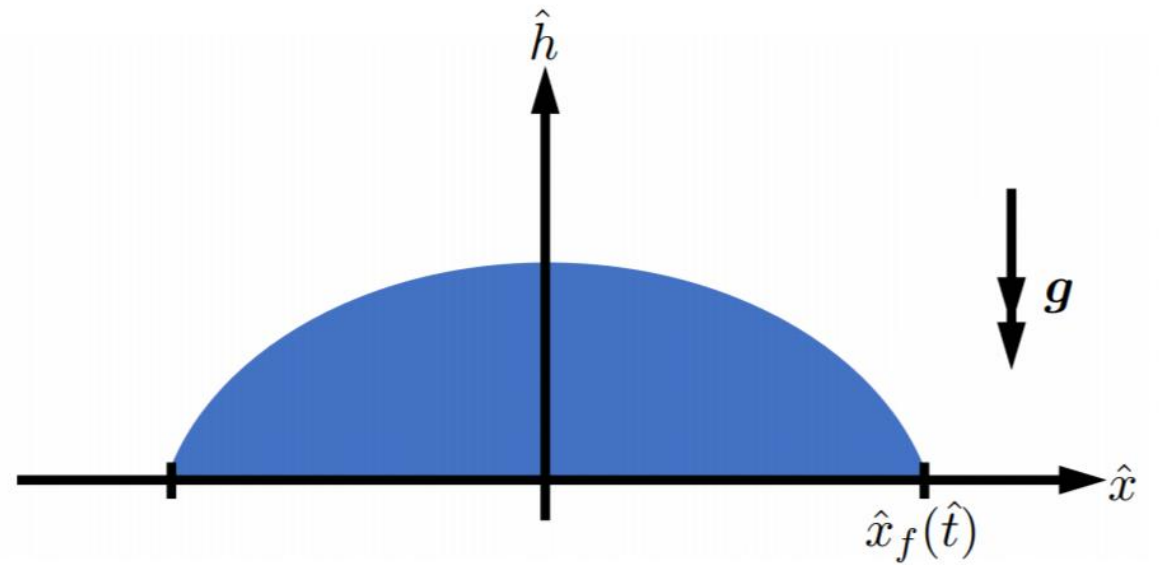
$$\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0$$

$$\hat{Q} = -\frac{\Delta \rho g}{3\mu} \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}}$$

$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta \rho g}{3\mu} \frac{\partial}{\partial \hat{x}} \left( \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \right) = 0$$



$$\frac{\partial \hat{h}}{\partial \hat{t}} - \frac{\Delta \rho g}{3\mu} \frac{\partial}{\partial \hat{x}} \left( \hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \right) = 0. \quad (1)$$



- What extra conditions do we need to solve this?
- One initial condition and two boundary conditions

- One boundary condition is  $\hat{h}(\hat{x}_f(\hat{t}), \hat{t}) = 0$ . (A)

- Volume conservation: 
$$\hat{V}(\hat{t}) = \int_0^{\hat{x}_f(\hat{t})} \hat{h}(\hat{x}, \hat{t}) d\hat{x} \quad (\text{B})$$

- The final condition to close the system is 
$$\hat{h}^3 \frac{\partial \hat{h}}{\partial \hat{x}} \rightarrow 0 \quad \text{as } \hat{x} \rightarrow \hat{x}_f \quad (\text{C})$$

- For numerics it can be easier to work with boundary conditions rather than **integral constraints** like (B) and **regularity conditions** like (C).
- But we can transform (B) and (C) into boundary conditions.
- We integrate (1) over  $0 < \hat{x} < \hat{x}_f$  and apply (B) and (C) to get

$$-\frac{\Delta\rho g}{3\mu}\hat{h}^3\frac{\partial\hat{h}}{\partial\hat{x}} = \hat{V}'(\hat{t}) \quad \text{on } \hat{x} = 0, \quad (\text{B1})$$

- Now rescale into a local region near the moving front:

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon \hat{\xi}, \quad \hat{t} = \hat{\tau}$$

- And rescale the height:

$$\hat{h}(\hat{x}, \hat{t}) = \epsilon^{1/3} \hat{H}(\hat{\xi}, \hat{\tau})$$

- Then (1) becomes

$$\epsilon \frac{\partial \hat{H}}{\partial \hat{\tau}} - \frac{d\hat{x}_f}{d\hat{\tau}} \frac{\partial \hat{H}}{\partial \hat{\xi}} - \frac{\Delta \rho g}{3\mu} \frac{\partial}{\partial \hat{\xi}} \left( \hat{H}^3 \frac{\partial \hat{H}}{\partial \hat{\xi}} \right) = 0$$

- At leading order this gives

- Integrating gives

- In terms of the original variables this gives

$$\hat{h}^2 \frac{\partial \hat{h}}{\partial \hat{x}} \sim -\frac{3\mu}{\Delta\rho g} \frac{d\hat{x}_f}{d\hat{t}} \quad \text{as } \hat{x} \rightarrow \hat{x}_f \quad (\text{C1})$$

- (B1) and (C1) are easier boundary conditions for numerics than (B) and (C).

- Here we will study just  $\hat{V} = \text{constant}$
- Analytical solutions are not possible so we proceed directly to the similarity solution.
- A scaling law analysis gives
- This scaling law gives the parametric dependence.

- Now consider the full similarity solution.

- First, non-dimensionalize:

$$\hat{x} = \hat{x}_0 x, \quad \hat{x}_f = \hat{x}_0 x_f, \quad \hat{t} = \hat{t}_0 t, \quad \hat{h} = \hat{h}_0 h$$

- Dimensionless problem:

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left( h^3 \frac{\partial h}{\partial x} \right) = 0,$$

$$h = 0,$$

on  $x = x_f(t)$

$$\int_0^{x_f} h \, dx = 1,$$

$$h^2 \frac{\partial h}{\partial x} \sim -\frac{dx_f}{dt}$$

as  $x \rightarrow x_f$ ,



- Dimensionless problem:

$$(f^3 f')' + \frac{1}{5} (f + \eta f') = 0,$$

$$f = 0,$$

on  $\eta = \eta_f$ ,

$$\int_0^{\eta_f} f \, d\eta = 1,$$

$$f^2 f' \sim -\eta_f$$

as  $\eta \rightarrow \eta_f$ ,

with  $\eta_f = \frac{x_f(t)}{t^{1/5}}$

- A final change of variable decouples the spreading domain:

$$s = \frac{\eta}{\eta_f}, \quad k(s) = \frac{f(\eta_f s)}{\eta_f^{2/3}}$$

- This leads to the system

$$(k^3 k')' + \frac{1}{5} (k + s k') = 0,$$

$$k(1) = 0,$$

$$\eta_f = \left( \int_0^1 k(s) ds \right)^{-3/5},$$

$$k^2 k' = -1 \quad \text{as } s \rightarrow 1.$$

- The solution to this system is

$$k = \left(\frac{3}{10}\right)^{1/3} (1 - s^2)^{1/3}$$

$$\eta_f = \left[ \left(\frac{3}{10}\right)^{1/3} \frac{\pi^{1/2} \Gamma(1/3)}{5 \Gamma(5/6)} \right]^{-3/5}$$

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

- In dimensional terms:

$$\hat{h}(\hat{x}, \hat{t}) = \frac{\eta_f^{2/3}}{\hat{t}^{1/5}} \left( \frac{\Delta\rho g}{3\mu\hat{V}^2} \right)^{-1/5} k \left( \frac{\hat{x}}{\hat{x}_f} \right),$$

$$\hat{x}_f(\hat{t}) = \eta_f \hat{t}^{1/5} \left( \frac{\Delta\rho g \hat{V}^3}{3\mu} \right)^{1/5}.$$

- The **scaling analysis** provided the **time dependence** and the **parametric dependence** but **lacked the shape** that this similarity solution provides.

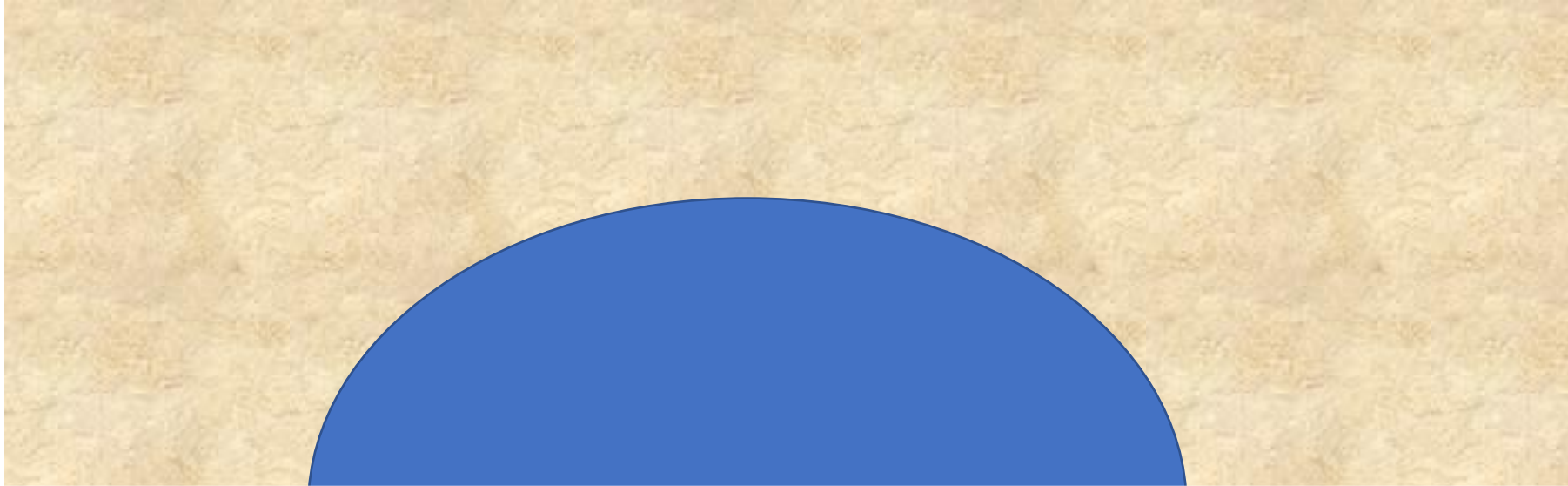
## Summary of lecture 3

- Spreading of liquid on a horizontal plate.
- We use the **governing equation** and **mass conservation** to determine the similarity solution.
- The **position of the front** is also an unknown we must solve for.
- We can transform the **mass-conservation** integral relation into a **boundary condition**.
- We can transform the **regularity condition** at the moving front into a **boundary condition**.
- We scale the variables so that we solve for the height of the profile first and determine the position of the front afterwards.

- In all the examples we have considered so far we have been able to determine the form of the required solution by a scaling argument.
- Examples of this kind are called **similarity solutions of the first kind**.
- We will now consider an example where scaling analysis alone does not determine the solution.
- These are called **similarity solutions of the second kind**.

## Spreading of a groundwater mound

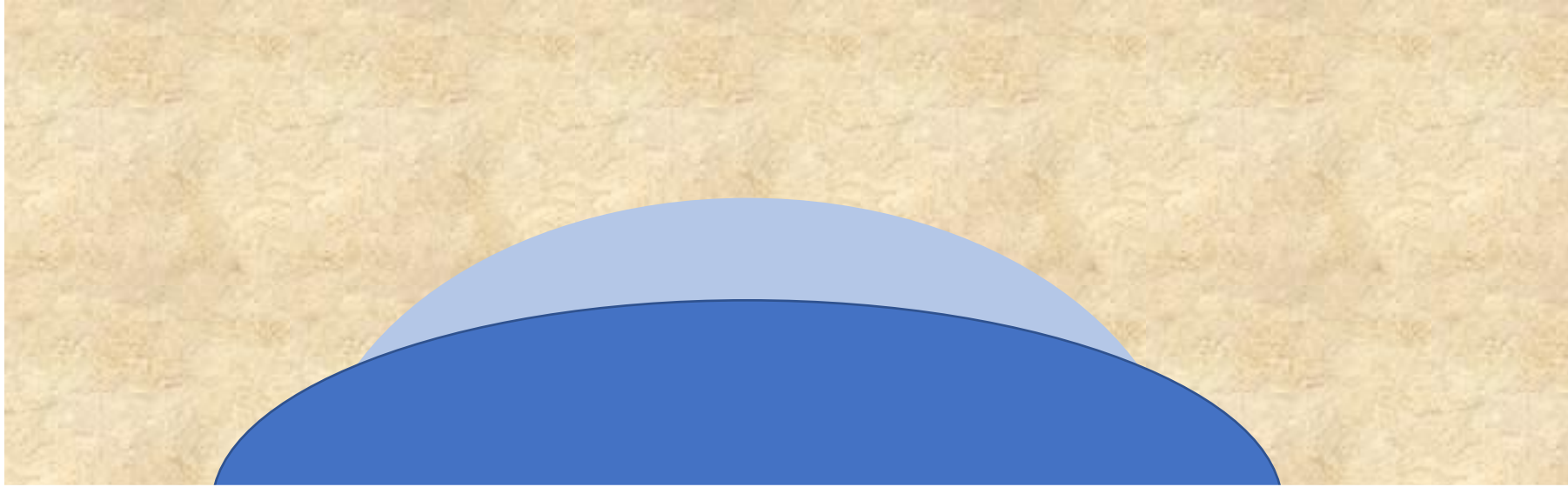
- Now let's consider the spreading in a porous medium rather than in air:



- Suppose that the liquid leaves behind a residue.

## Spreading of a groundwater mound

- Now let's consider the spreading in a porous medium rather than in air:



- Suppose that the liquid leaves behind a residue.



- We now split up the problem into two regions.
- **Region 1: Liquid invading dry porous medium**  $\partial \hat{h} / \partial \hat{t} > 0$

- Here the flow is governed by the porous medium equation

$$\phi \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \quad \hat{Q} = -\frac{\Delta \rho g K}{\mu} \hat{h} \frac{\partial \hat{h}}{\partial \hat{x}}$$

- **Region 2: Liquid draining**  $\partial \hat{h} / \partial \hat{t} < 0$ .

- Here the flow is governed by a modified porous medium equation due to the partially saturated residue region:

$$(1 - s)\phi \frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\partial \hat{Q}}{\partial \hat{x}} = 0, \quad \hat{Q} = -\frac{\Delta \rho g K}{\mu} \hat{h} \frac{\partial \hat{h}}{\partial \hat{x}}$$

- $s$  = fraction of space occupied by liquid in drained region.  
 $s = 0$  recovers the original porous medium equation. As  $s$  increases,  $\partial \hat{h} / \partial \hat{t}$  gets larger due to the liquid in the drained region pushing down.

- Denote the position of the front by  $\hat{x}_f(\hat{t})$  and the position of the joint separating Regions 1 and 2 by  $\hat{x}_s(\hat{t})$ .

- We have two second-order partial differential equations with two additional unknowns ( $\hat{x}_f$  and  $\hat{x}_s$ ) so we require initial conditions for the two regions plus six boundary conditions (four for the two equations and two for the unknowns).

- We apply:

$$\hat{h}(\hat{x}_f, \hat{t}) = 0$$

Definition of  $\hat{x}_f$

$$\frac{\partial \hat{h}}{\partial \hat{t}}(\hat{x}_s, \hat{t}) = 0$$

Definition of  $\hat{x}_s$

$$\hat{h} \frac{\partial \hat{h}}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0,$$

No flux at the centre

$$\hat{h} \frac{\partial \hat{h}}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = \hat{x}_f,$$

No flux at the front

Continuity in  $\hat{h}$  and  $\hat{h} \partial \hat{h} / \partial \hat{x}$  at the joint  $\hat{x} = \hat{x}_s$

- As for the case of a spreading liquid, (D) doesn't look like it gives any more information than (A). However, it does enforce a regularity condition on  $\hat{h}$  at the front.
- We can transform this into a stronger condition by performing a local analysis near the front. We rescale:

$$\hat{x} = \hat{x}_f(\hat{\tau}) + \epsilon \hat{\xi}, \quad \hat{t} = \hat{\tau}, \quad \hat{h}(\hat{x}, \hat{t}) = \epsilon \hat{H}(\hat{\xi}, \hat{\tau}).$$

- Substituting into (1) and considering the result at leading order gives

$$\frac{\partial \hat{H}}{\partial \hat{\xi}} = -\frac{\mu\phi}{\Delta\rho g K} \frac{d\hat{x}_f}{d\hat{\tau}}$$

which in terms of the original variables gives

$$\frac{\partial \hat{h}}{\partial \hat{x}}(\hat{x}_f, \hat{t}) = -\frac{\mu\phi}{\Delta\rho g K} \frac{d\hat{x}_f}{d\hat{t}}$$

- We non-dimensionalize via

$$\hat{x} = \hat{x}_0 x, \quad \hat{x}_s = \hat{x}_0 x_s, \quad \hat{x}_f = \hat{x}_0 x_f, \quad \hat{t} = \hat{t}_0 t, \quad \hat{h} = \hat{h}_0 h,$$

- The resulting dimensionless system is then

$$\frac{\partial h}{\partial t} - \kappa \frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) = 0, \quad 0 \leq x \leq x_s(t), \quad (1)$$

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) = 0, \quad x_s(t) \leq x \leq x_f(t), \quad (2)$$

$$h(x_f, t) = 0, \quad (A)$$

$$\frac{\partial h}{\partial t}(x_s, t) = 0, \quad (B)$$

$$\frac{\partial h}{\partial x}(0, t) = 0, \quad (C)$$

$$\frac{\partial h}{\partial x}(x_f, t) = -\frac{dx_f}{dt} \quad (D)$$

plus continuity in  $h$  and  $h\partial h/\partial x$  at  $x = x_s$ . (E)

- Since the liquid leaves behind a residue we can't apply a simple global conservation law like in previous cases.
- We try a similarity solution of the form

$$h = t^\beta f(\eta) \quad \text{with} \quad \eta = \frac{x}{t^\alpha}.$$

This gives

$$\begin{aligned} \kappa(ff')' + \alpha\eta f' - (2\alpha - 1)f &= 0, & 0 \leq \eta \leq \eta_s, \\ (ff')' + \alpha\eta f' - (2\alpha - 1)f &= 0, & \eta_s \leq \eta \leq \eta_f, \end{aligned}$$

where  $\eta_f = x_f/t^\alpha$  and  $\eta_s = x_s/t^\alpha$  are defined by

$$\begin{aligned} f(\eta_f) &= 0, \\ (2\alpha - 1)f(\eta_s) - \alpha\eta_s f'(\eta_s) &= 0, \end{aligned}$$

- We also apply

$$\begin{aligned}f'(0) &= 0, \\f'(\eta_f) &= -\alpha\eta_f,\end{aligned}$$

as well as continuity in  $f$  and  $f'$  at  $\eta_s$ .

- Unlike in our previous examples, we have no further information to determine  $\alpha$ . So, we press on.



- Make a change of variable  $z = \eta/\eta_f$  and  $k(z) = f(\eta_f z)/\eta_f^2$ .

$$\kappa (kk')' + \alpha z k' - (2\alpha - 1)k = 0, \quad 0 \leq z \leq z_s,$$

$$(kk')' + \alpha z k' - (2\alpha - 1)k = 0, \quad z_s \leq z \leq 1,$$

$$k(1) = 0,$$

$$(2\alpha - 1)k(z_s) - \alpha z k'(z_s) = 0,$$

$$k'(0) = 0,$$

$$k'(1) = -\alpha,$$

And continuity in  $k$  and  $k'$  at  $z=z_s$ .

- This is two second-order ordinary differential equations plus one unknown ( $z_s$ ).
- How many boundary conditions does this require?

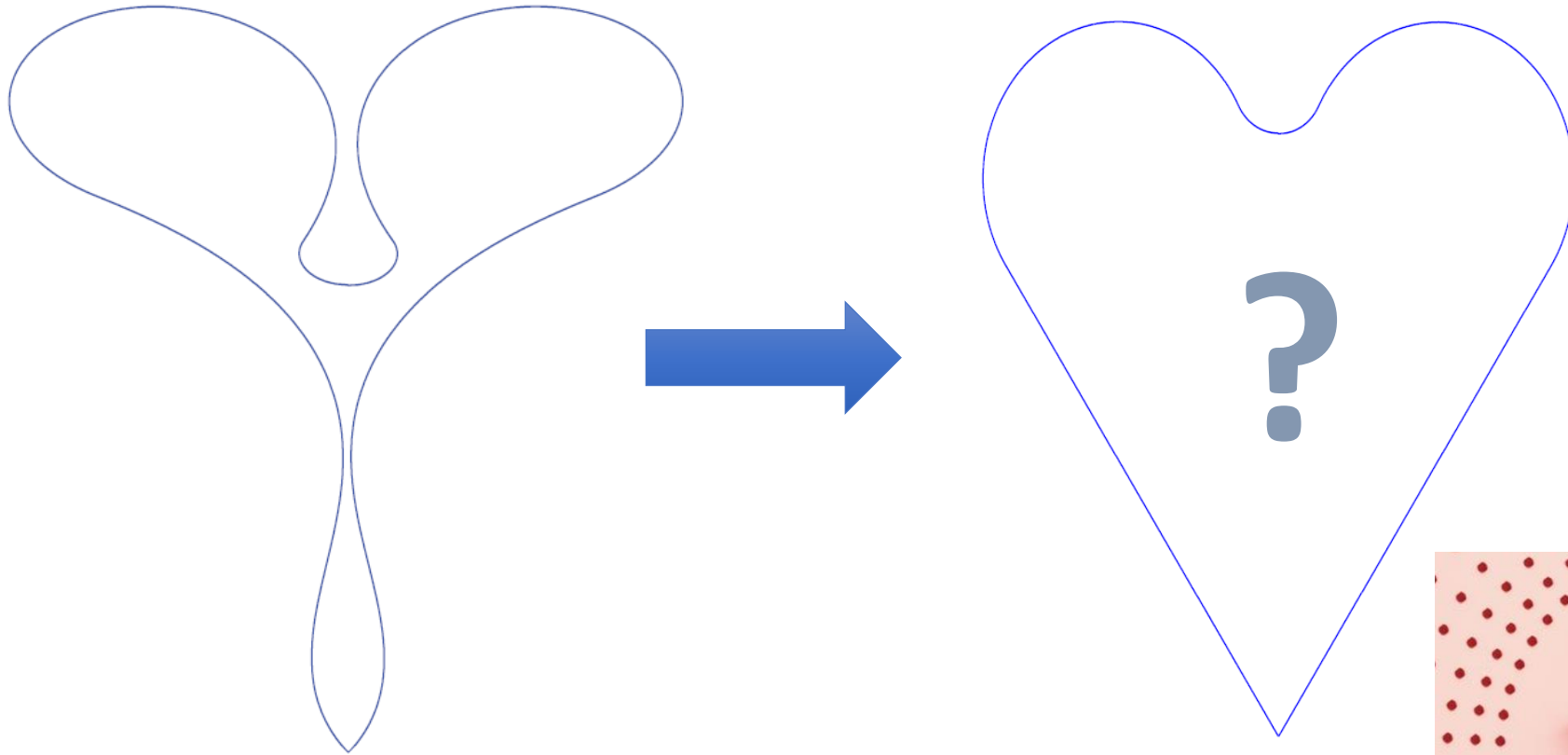
- We have six boundary conditions. The sixth condition determines  $\alpha$ .
- This is an **eigenvalue problem**.
- For a given value of  $\kappa$  there is an associated  $\alpha$  that gives a solution.
- The natural similarity variables did not emerge from a scaling argument but instead as part of the solution.
- Such problems are called **similarity solutions of the second kind**.

- Importantly, in this case, the similarity solution only provides the shape of the interface up to a constant. Our final scaling moved the unknown  $\eta_f$  to determine  $\alpha$  so we are still missing one piece of information. (In our previous example of spreading liquid on a horizontal plate, this came from mass conservation.)
- To determine the appropriate scaling we must compare our similarity solution with the full numerical solution at one point in time.

## Summary

- **Scaling laws** can provide a significant amount of information about the solution of a partial differential equation system.
- **Similarity solutions of the first kind** provide the full behaviour for long time but does not (always) capture the early-time behaviour (unless this happens to satisfy the initial condition).
- **Similarity solutions of the second kind** arise when we cannot fully determine the form of the similarity solution from the governing equation and boundary conditions. This forms an eigenvalue problem. The similarity solution provides the shape up to a scaling constant.

**Question: What shape biscuit would this cookie cutter make?**



## Summary of lecture 4

- Similarity solutions of the **second kind** arise when we **cannot fully determine the form** of the similarity solution from the governing equation and boundary conditions.
- This forms an **eigenvalue problem**.
- The similarity solutions **provides the shape up to a scaling constant**.

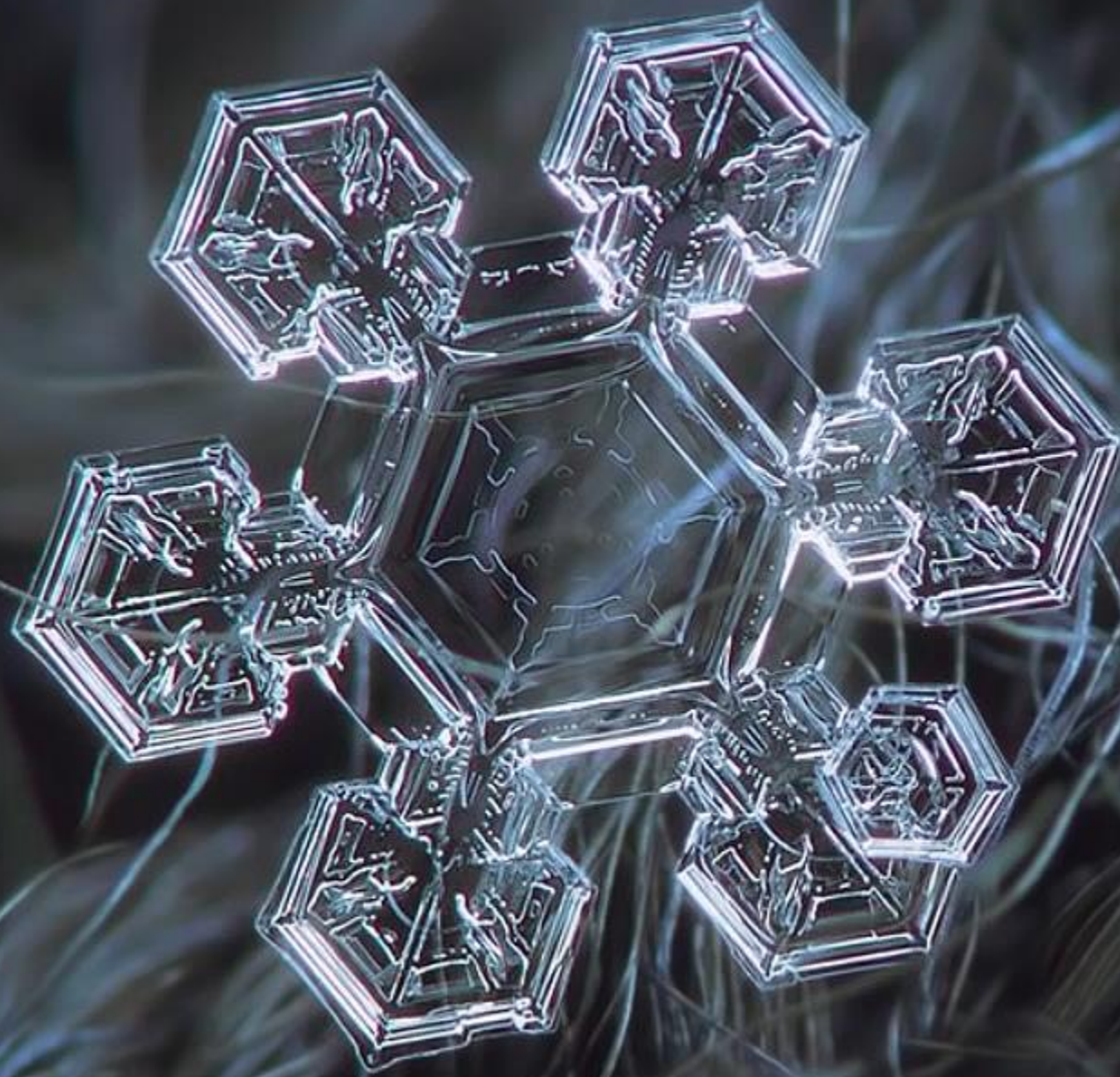
## Free boundary problems

- A free boundary problem occurs when the region in which the problem is to be solved is unknown in advance and must be found as part of the solution.



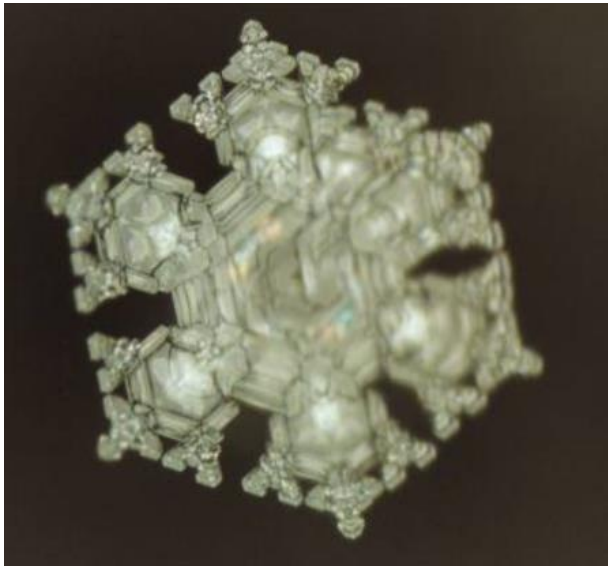
- One example concerns the freezing or melting of ice. Here, we have to solve for the free boundary between the two phases as well as for the temperature field in each phase.

- The shape of a free boundary as it is formed can be quite exotic.

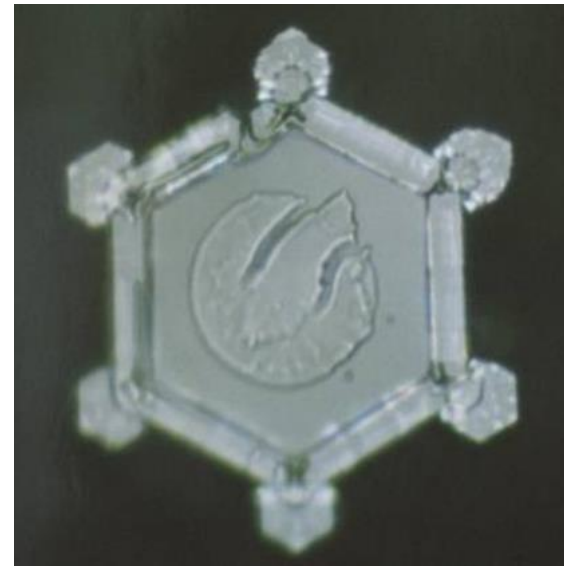




# *Masaru Emoto*



Shimanto River, Japan



Mount Cook Glacier,  
New Zealand



Playing Imagine by John  
Lennon to the water



Showing the water a  
photo of an elephant

## Industrial problem: electric welding

- Steel plates are welded together by attaching electrodes to the edges and then passing a current through.
- This heats the metal, causing the plates to melt. When the current is switched off, the liquid metal solidifies and the two plates are welded together.
- How long should the current be applied to ensure the metal melts without melting the entire plates?
- How long should the plates be left once the current has been switched off to ensure that they have solidified completely?



## One-dimensional Stefan problem

- Consider a slab of initially solid material occupying the region  $0 < x < a$ . The temperature  $T(x,t)$  in the slab satisfies the heat equation:

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right)$$

- Suppose the slab is initially at a uniform temperature  $T_0$  and the face  $x = a$  is insulated, while the temperature of the face at  $x = 0$  is suddenly increased to a temperature  $T_1 > T_0$ :

$$T(x, 0) = T_0, \quad 0 < x < a; \quad T(0, t) = T_1, \quad t > 0; \quad \frac{\partial T}{\partial x}(a, t) = 0, \quad t > 0.$$

- If  $T_1$  is high enough the solid will melt and become liquid in some neighbourhood of  $x = 0$ .

- If the slab melts, we must seek a solution in which the slab is liquid in  $0 < x < s(t)$  and solid in  $s(t) < x < a$ , where  $x = s(t)$  is the free boundary separating the liquid and solid phases.
- $s(t)$  is unknown in advance and must be found as part of the solution.
- We have to solve the heat equation in both solid and liquid phases, with different values of the parameters  $\rho$ ,  $c$  and  $k$  in each phase.
- We also need to impose conditions at the free boundary  $x = s(t)$ .

- First we assume that the phase change happens at a known melting temperature so

$$T(s(t), t) = T_m$$

- Since the temperature is continuous, this applies on both sides of the free boundary.

- The second condition comes from an energy balance. At a fixed boundary, the heat flux coming in from one side must equal the heat flux exiting from the other side, i.e.,  $q = -k\partial T / \partial x$  must be continuous.
- However, at a moving interface, we must also consider the energy associated with the phase change, known as the latent heat  $L$ .
- The latent heat acts as an energy source or sink at a moving solid–liquid interface, and the resulting boundary condition

$$k \frac{\partial T}{\partial x} \Big|_{x=s(t)+} - k \frac{\partial T}{\partial x} \Big|_{x=s(t)-} = \rho L \frac{ds}{dt}$$

- This is called the **Stefan condition**.

## Assumptions

- We assumed that the heat flux is purely due to thermal conductivity. If the liquid flows at all, then there will also be a contribution due to convection.
- We assumed that the solid and liquid have the same density. If there is a change in density as the material melts, this would induce a velocity.



## One-phase Stefan problem

- Assume that  $\rho$ ,  $c$  and  $k$  are all constant on either side of the solid–liquid interface.
- Suppose that the initial temperature  $T_0$  is equal to the melting temperature  $T_m$ . This means that the solid region  $s(t) < x < a$  is always at temperature  $T_m$  and we only need to solve the heat equation in the liquid phase  $0 < x < s(t)$ .
- This is a **one-phase Stefan problem**.

- Non-dimensionalize:

$$x = ax', \quad t = \left( \frac{\rho L a^2}{k(T_1 - T_m)} \right) t', \quad T = T_m + (T_1 - T_m)u.$$

$$\text{St} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u = 1$$

$$0 < x < s(t), \quad t > 0,$$

$$x = 0, \quad t > 0$$

$$u = 0, \quad -\frac{\partial u}{\partial x} = \frac{ds}{dt}$$

$$s(0) = 0.$$

$$x = s(t), \quad t > 0$$

- Seek a similarity solution:

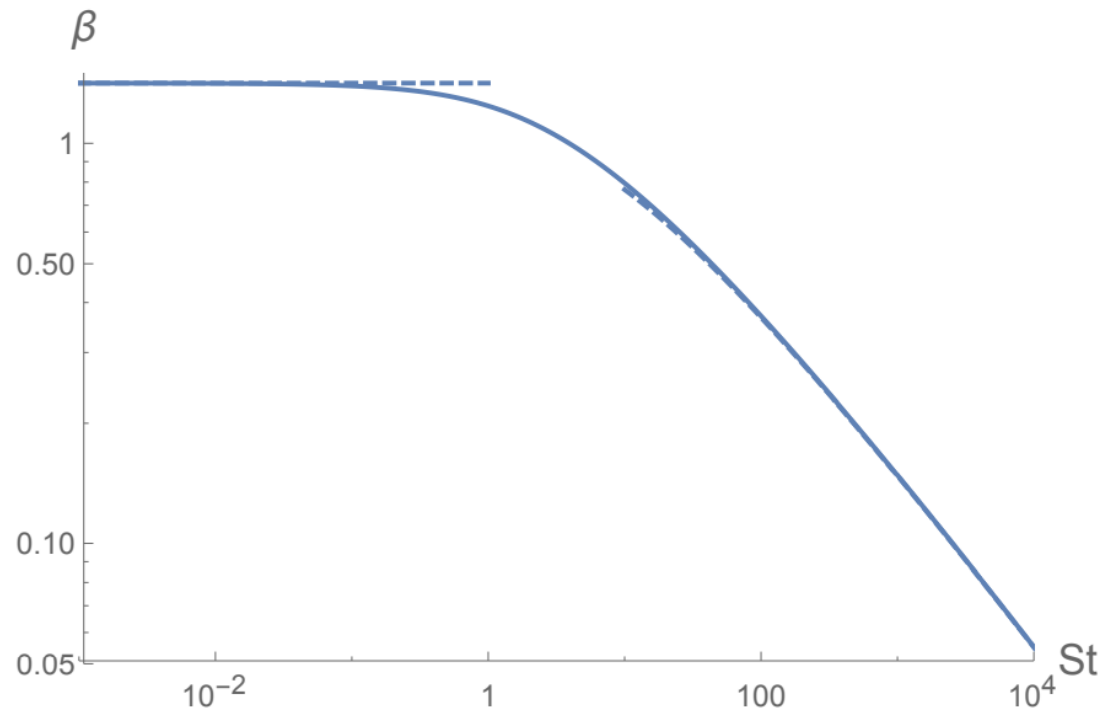
$$s(t) = \beta\sqrt{t} \quad \text{and} \quad u(x, t) = f(\eta), \quad \text{where} \quad \eta = \frac{x}{\sqrt{t}}$$

$$\begin{aligned} \frac{d^2 f}{d\eta^2} + \frac{St}{2} \eta \frac{df}{d\eta} &= 0 & 0 < \eta < \beta, \\ f &= 1 & \eta = 0, \\ f = 0, \quad \frac{df}{d\eta} &= -\frac{\beta}{2} & \eta = \beta. \end{aligned}$$

- The solution is

$$f(\eta) = 1 - \frac{\operatorname{erf}\left(\eta\sqrt{\operatorname{St}}/2\right)}{\operatorname{erf}\left(\beta\sqrt{\operatorname{St}}/2\right)}$$

$$\frac{\sqrt{\pi}\beta e^{\operatorname{St}\beta^2/4} \operatorname{erf}\left(\beta\sqrt{\operatorname{St}}/2\right)}{2\sqrt{\operatorname{St}}} = 1$$



$$\beta \sim \begin{cases} \sqrt{2} & \text{as } \operatorname{St} \rightarrow 0, \\ \frac{2}{\sqrt{\operatorname{St}}} \sqrt{\log\left(\frac{\operatorname{St}}{\sqrt{\pi}}\right)} & \text{as } \operatorname{St} \rightarrow \infty \end{cases}$$

- The entire slab has melted when  $s(t) = 1$ , and the dimensionless time taken is thus given by  $1/\beta^2$ .
- By reversing the non-dimensionalization we find approximations for the melting time  $t_m$ :

$$t_m \sim \begin{cases} \frac{\rho L a^2}{2k(T_1 - T_m)} & \text{St} \ll 1, \\ \frac{\rho c a^2}{4k \log(\text{St}/\sqrt{\pi})} & \text{St} \gg 1. \end{cases}$$

- When  $\text{St} \ll 1$ , the melting of the slab is limited by the latent heat required.
- When  $\text{St} \gg 1$ , the main barrier is the energy needed to heat up the slab, and the melting time depends only weakly on the latent heat  $L$ .



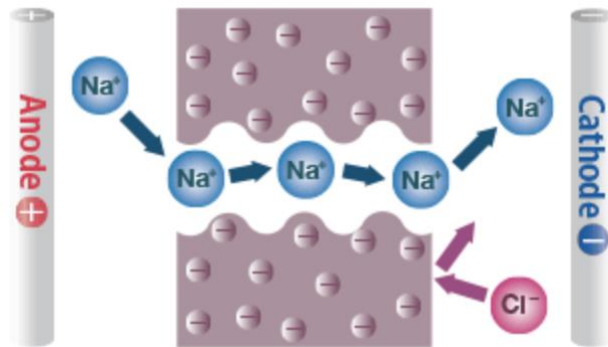
# Introducing Gore

- **W. L. Gore & Associates, Inc.** is an American manufacturing company specializing in **fluoropolymers**.
- Best known for **Gore Tex waterproof breathable products**.
- Also specialize in: **electrochemical materials, fibres, filtration products** and **medical products**.

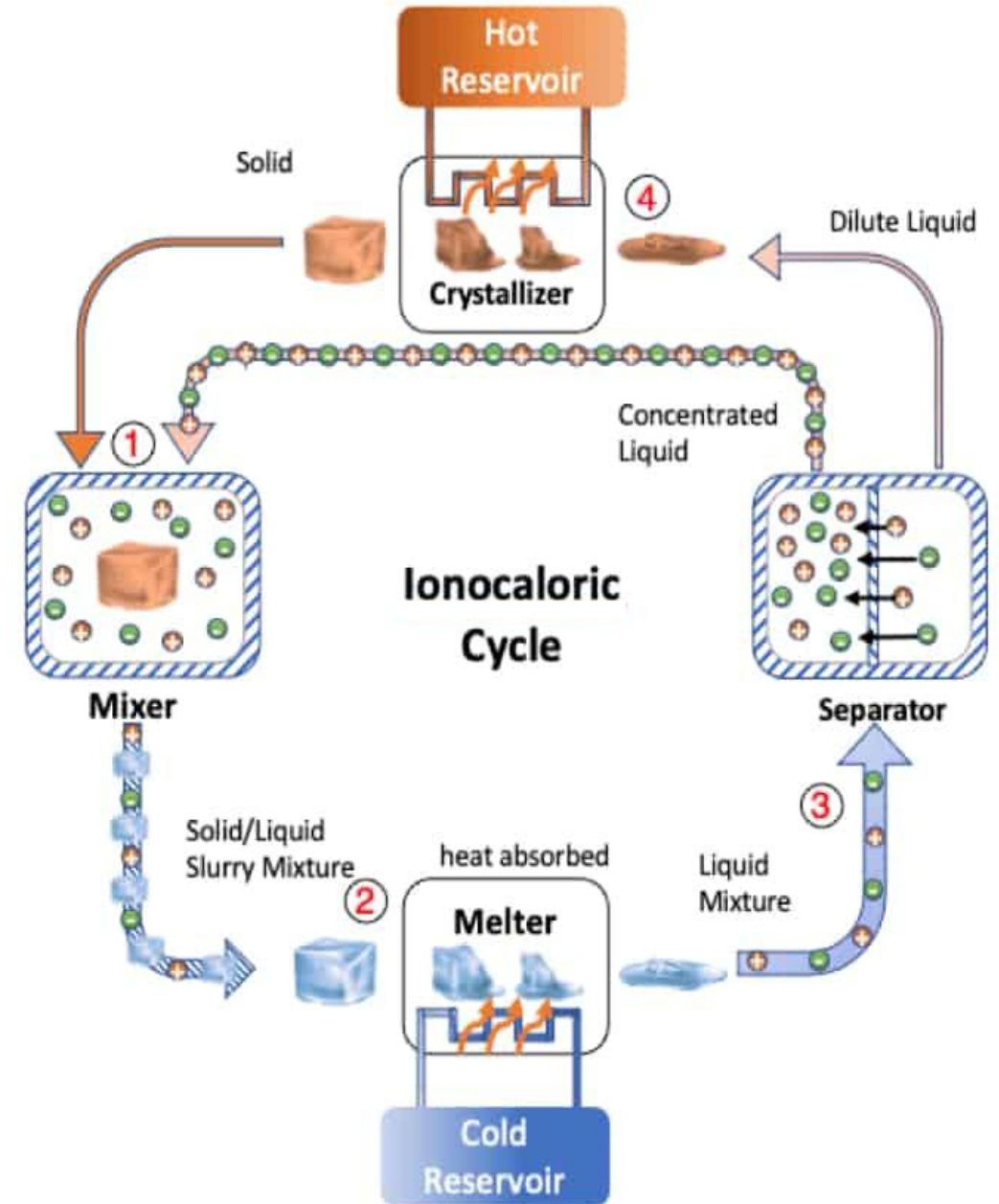


# MMSC project 1: Modelling the ionocaloric cycle and device

Ian Griffiths and Chris Breward (Oxford)  
Vasudevan Venkateshwaran (Gore)



- We will model the ion-exchange membranes using PDEs to understand what controls performance.

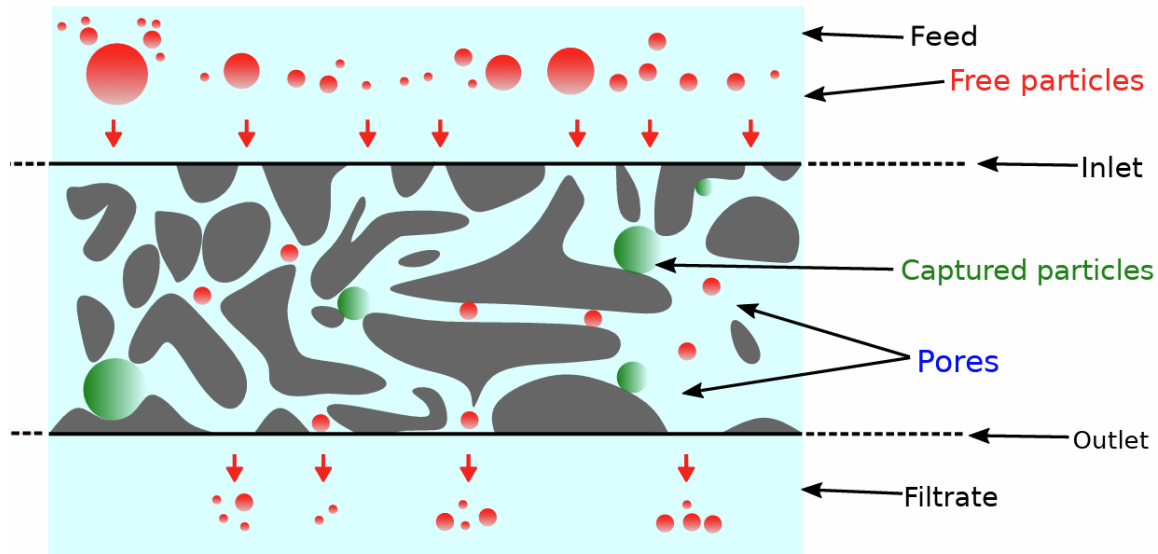


# MMSC project 2: Use of Machine Learning for Filtration Optimization

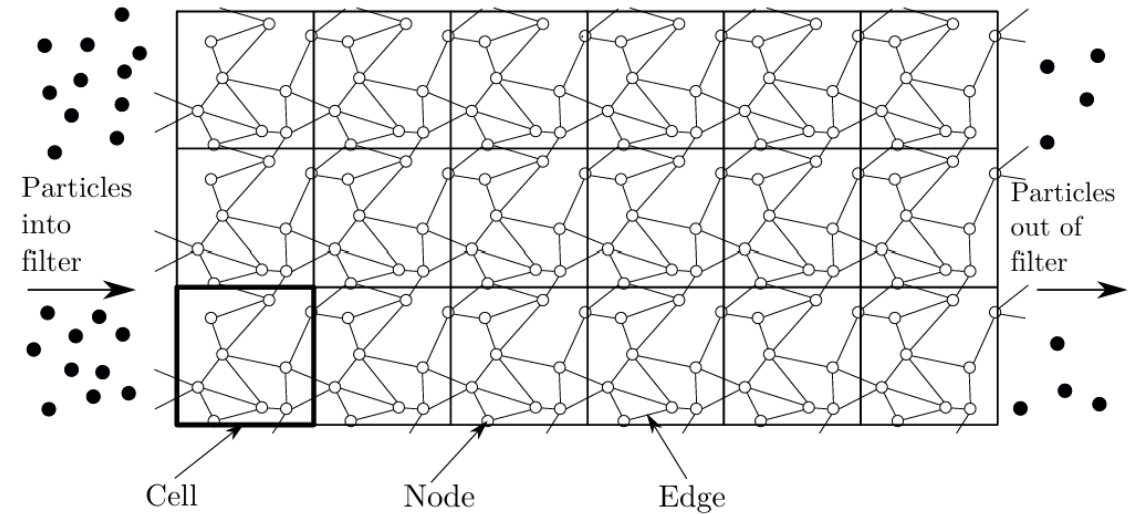


Ian Griffiths, Chris Breward and Arkady Wey (Oxford)  
Uwe Beuscher (Gore)

- Size structured filtration



- Network homogenization



- We will use machine learning tools to understand the filtration parameter space.



## Summary of lecture 5

- A **free boundary problem** occurs when the region in which the problem is to be solved is unknown in advance and must be found as part of the solution.
- Stefan problems describe melting or freezing. At the interface we apply a **Stefan condition**:

$$k \frac{\partial T}{\partial x} \Big|_{x=s(t)+} - k \frac{\partial T}{\partial x} \Big|_{x=s(t)-} = \rho L \frac{ds}{dt}$$

- A **one-phase Stefan problem** considers solid at the melting point temperature of the liquid.
- The interface position follows the power law  $s(t)=\beta t^{1/2}$ .



Jožef Stefan  
(1835-1893)  
Slovene physicist

## Two-phase Stefan problem

$$\text{St} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$0 < x < s(t), \quad t > 0,$$

$$\frac{\text{St}}{\kappa} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$s(t) < x < 1, \quad t > 0,$$

$$u = 1$$

$$x = 0, \quad t > 0,$$

$$\frac{\partial u}{\partial x} = 0$$

$$x = 1, \quad t > 0,$$

$$u = 0, \quad K \left[ \frac{\partial u}{\partial x} \right]^+ - \left[ \frac{\partial u}{\partial x} \right]^- = \frac{ds}{dt}$$

$$x = s(t), \quad t > 0$$

$$u = -\theta, \quad s = 0$$

$$t = 0,$$

$$\kappa = \frac{c_1 k_2}{c_2 k_1},$$

$$K = \frac{k_2}{k_1},$$

$$\theta = \frac{T_m - T_0}{T_1 - T_m}$$

- When  $St \ll 1$ :

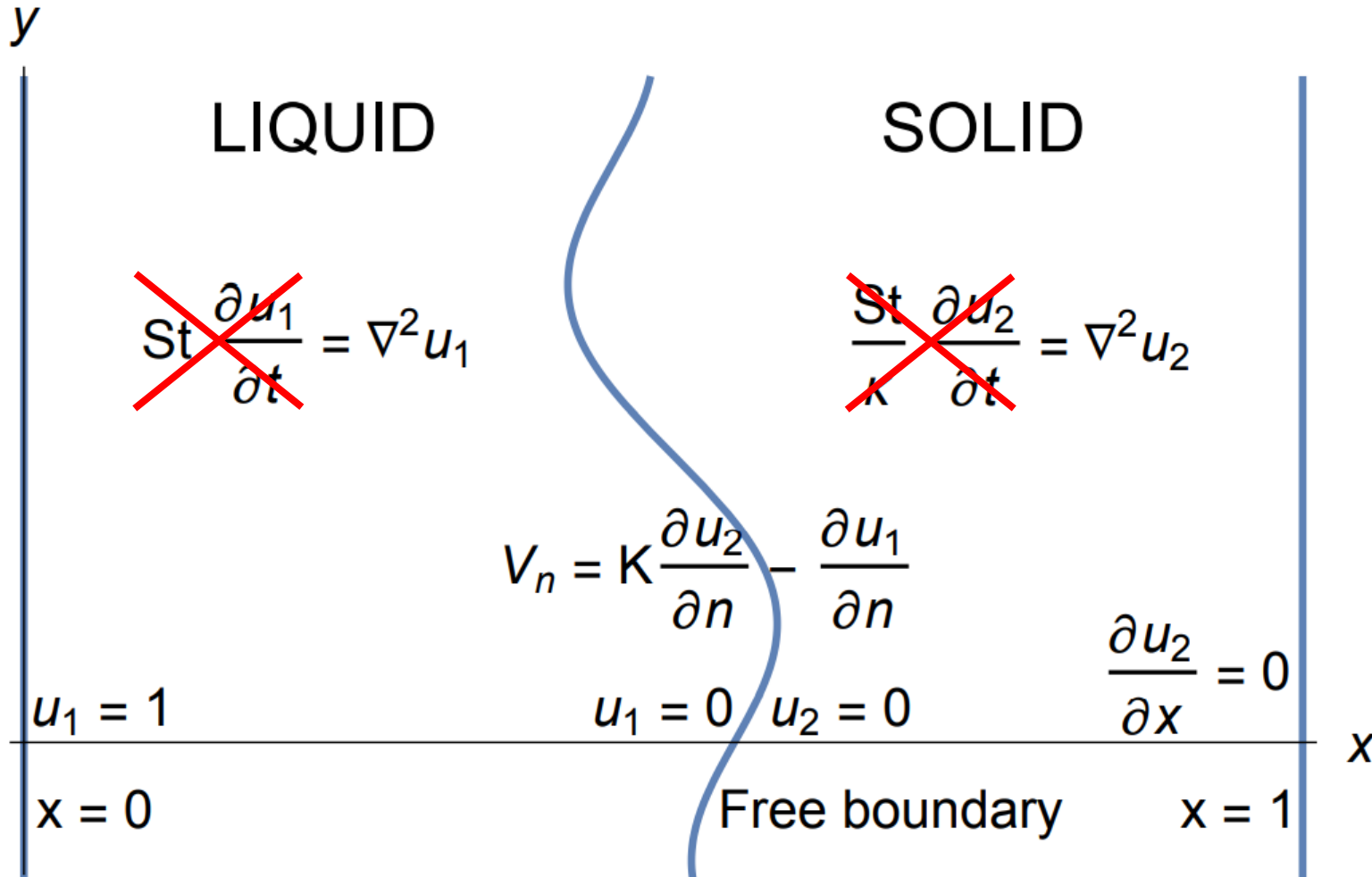
$$u(x, t) = \begin{cases} 1 - \frac{x}{s(t)} & 0 < x < s(t) < 1 \\ 0 & 0 < s(t) < x < 1 \end{cases} \quad (*)$$

$$\frac{ds}{dt} = \frac{1}{s} \quad \Rightarrow \quad s(t) = \sqrt{2t},$$

- This is equivalent to the one-phase small  $St$  Stefan problem ( $\beta = \sqrt{2}$ .)

- The approximate solution (\*) does not satisfy the initial condition (except in the special case when  $\theta = 0$ , i.e.,  $T_0 = T_m$ ).
- We resolve this by considering a small- $t$  boundary layer in which  $u$  adjusts from the initial value  $-\theta$  to the outer solution.

## Two-dimensional Stefan problem



- We study the small Stefan number limit again:  $St \ll 1$ .
- We then have a [Hele-Shaw problem](#).

- We examine the linear stability.
- For simplicity, consider the one-phase Stefan problem again so  $u_2=0$ .
- Assume that the interface is moving at speed  $V$  and there is a temperature gradient  $-\lambda$ . (We expect  $\lambda>0$  if the left region is liquid and the right is solid.) The solution is then

$$u_1^{(0)} = -V(x - Vt) \quad x = Vt$$

- Consider a small perturbation about this solution

$$u_1(x, y, t) = u_1^{(0)}(x, t) + \epsilon u_1^{(1)}(x, y, t) \quad x = Vt + \epsilon \xi(y, t) \quad \epsilon \ll 1$$

- This leads to the problem:

$$\begin{aligned}\nabla^2 \tilde{u} &= 0 & x < Vt, \\ \tilde{u} - \lambda \xi &= 0 & x = Vt, \\ -\frac{\partial \tilde{u}}{\partial x} &= \frac{\partial \xi}{\partial t} & x = Vt.\end{aligned}$$

- This is not a closed problem – we need one more boundary condition since  $\xi$  is also unknown.

- Try a wave-like solution:

$$u_1^{(1)} = Ae^{\sigma t + ik y + k(x - V t)} \quad \xi(y, t) = Be^{\sigma t + ik y}$$

Here,  $k$  is the wavenumber and  $\sigma$  is the linear growth rate.

- Substituting into the governing equations gives the linear system

$$\begin{pmatrix} 1 & -\lambda \\ k & \sigma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \mathbf{0}.$$

- Non-trivial solutions require the determinant of the matrix to be zero, giving

$$\sigma = -Vk$$



$$\sigma = -\lambda k = -V k$$

### If $V > 0$

- The **solid is melting**.
- $\sigma < 0$  and so the perturbations decay with time – **the interface is stable**.

### If $V < 0$

- The **liquid is freezing**.
- $\sigma > 0$  and so the perturbations grow with time – **the interface is unstable**.
- The smallest wavelength modes grow the fastest – this is an **ill-posed problem**.
  - The solution does not depend continuously on the data.
  - an arbitrarily small perturbation of the initial conditions may cause an arbitrarily large change in the solution.
  - This is physically unacceptable, and such ill posed problems are also effectively impossible to solve numerically.
- $\lambda < 0$  which means that the temperature in the liquid is lower than the melting temperature. The liquid is **supercooled**.

- The results we have found tie in with what we observe physically
  - An ice cube melts smoothly in a drink.
  - But as water freezes, dendrites form whose structure are unpredictable.

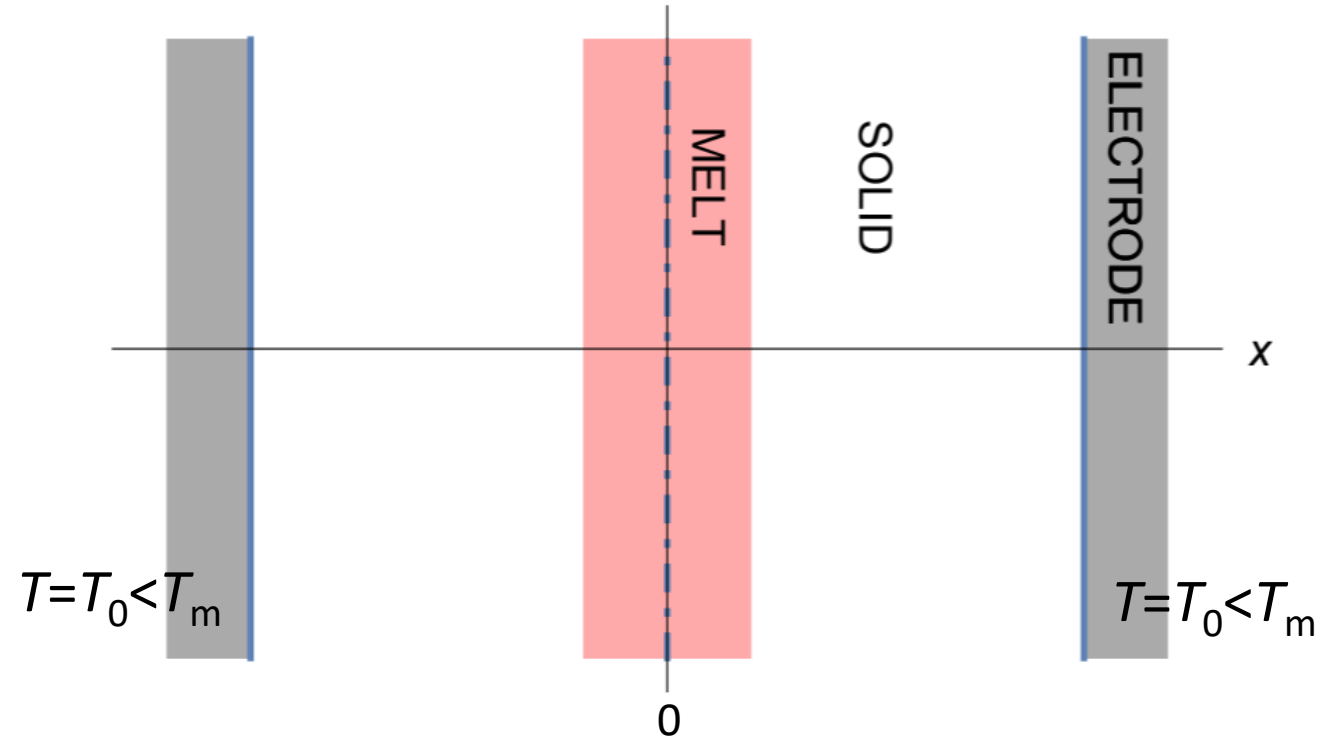


- It is reasonable for a freezing boundary to be unstable, but it is not acceptable for the problem to be ill posed.
- Some additional physics must become important to regularize the problem by suppressing arbitrarily high wavenumbers, e.g.,:
  - **Surface energy** effects penalise high curvature variations in the free boundary.
  - **Kinetic undercooling** – the temperature need not be continuous if the free boundary evolves rapidly enough to be out of thermodynamic equilibrium.
  - Model the complex **dendrites**.
- But we shall choose a simpler strategy
  - to model the dendritic region as a “mushy region”



## One-dimensional welding problem

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{J^2}{\sigma}$$



- Assume that the parameters  $\rho$ ,  $c$ ,  $k$  and  $\sigma$  are constant and the same in both phases.
- Assume symmetry about  $x=0$ .

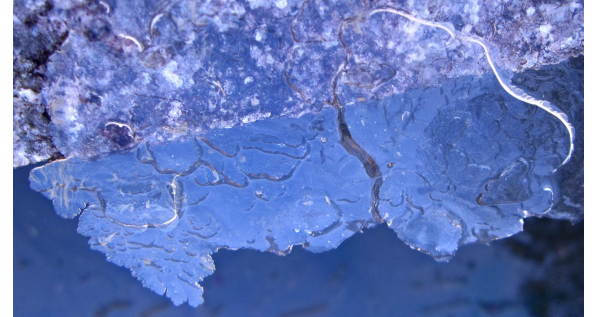
- Let's draw how we expect the temperature to evolve:

- This is an ill-posed problem.

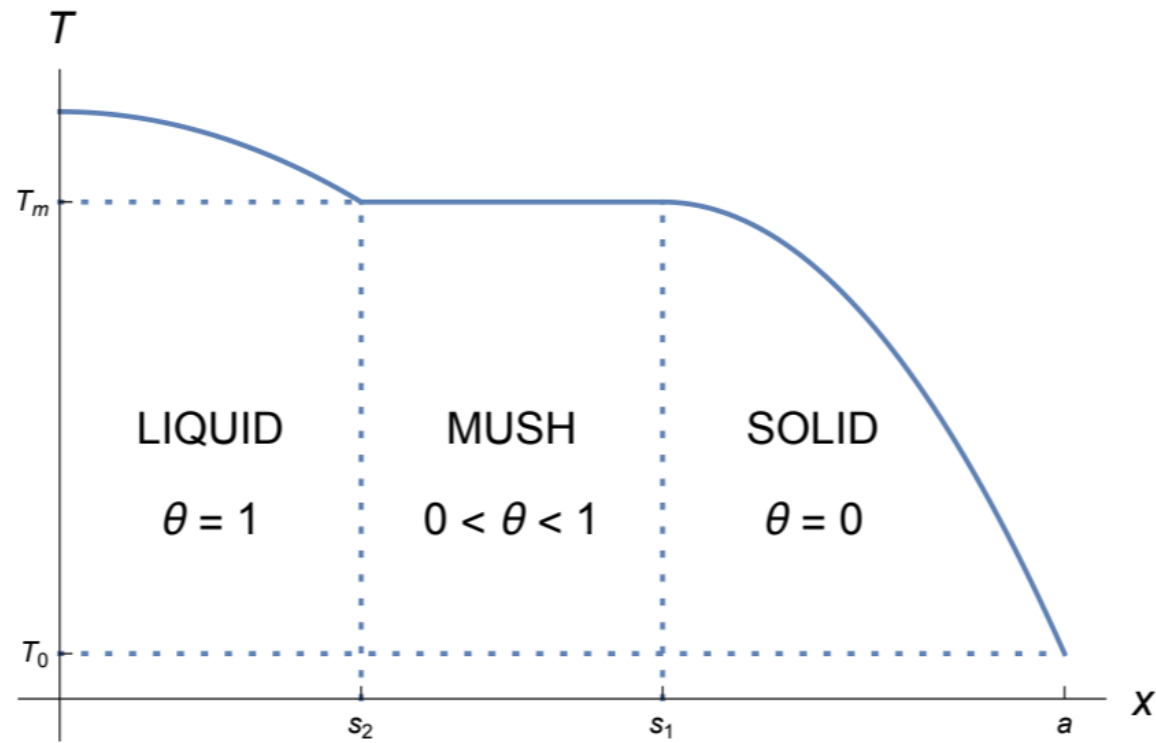
# Summary of lecture 6

## Two-dimensional Stefan problem

- Linear stability demonstrates that:
  - solid melting is stable
  - liquid freezing is unstable – this is an ill-posed problem.
- We need a model for liquid freezing.



- What happens instead?
- There is not a simple free boundary between pure liquid and pure solid phases.
- Instead, there is a “mushy region” in which both solid and liquid phases coexist, with the solid existing in a very fine dendritic crystalline structure.
- Rather than try to resolve this highly complicated structure, we construct a homogenised model that describes the net macroscopic behaviour of the mixture as a whole.



- We have **two free boundaries**  $x = s_1(t)$  and  $x = s_2(t)$ ,  
with **solid material** in  $s_1 < x < a$ ,  
**mush** in  $s_2 < x < s_1$   
**liquid** in  $0 < x < s_2$ .
- It just remains to construct a model for the mush.



- The first observation is that, for both phases to coexist, the temperature must everywhere be close to the melting temperature, so we may set  $T = T_m$  in the mush.
- The energy source does not heat up the mixture but supplies the latent heat needed to melt the solid.
- The energy equation in the mush is  $\rho L \frac{\partial \theta}{\partial t} = \frac{J^2}{\sigma}$  (\*)

where  $\theta$  is the liquid fraction in the mixture:  $\theta = 0$  in a pure solid and  $\theta = 1$  in pure liquid.

- We have two free boundaries. The equation for conservation of energy at each of these is

$$\left[ \rho L \theta \frac{ds_j}{dt} + k \frac{\partial T}{\partial x} \right]_{-}^{+} = 0 \quad \text{at } s = s_j(t) \quad j=1,2$$

- Finally, we require one initial condition for the PDE (\*), namely that  $\theta$  is continuous across the advancing melting boundary  $x = s_1(t)$  (so that the liquid fraction is zero when the mush first forms).

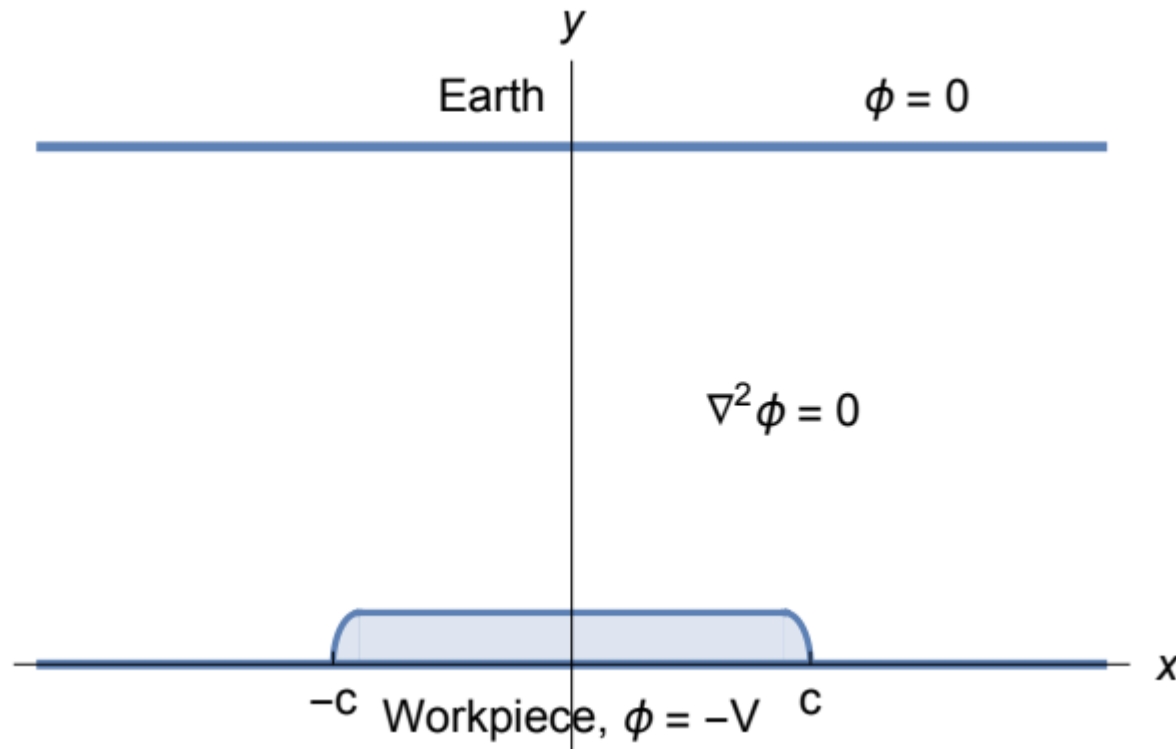
- The dimensionless problem we must work with is thus:

$$\begin{aligned}
 \text{St} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + q && 0 < x < 1, \quad t > 0, \\
 \frac{\partial u}{\partial x} &= 0 && x = 0, \quad t > 0, \\
 u &= -1 && x = 1, \quad t > 0, \\
 u &= -1 && 0 < x < 1, \quad t = 0, \\
 u = 0, \quad \frac{\partial \theta}{\partial t} &= q && s_2(t) < x < s_1(t), \quad t > 0, \\
 u = 0, \quad \left[ \theta \frac{ds_j}{dt} + k \frac{\partial u}{\partial x} \right]_{-}^{+} &= 0 && \text{at } s = s_j(t), \quad t > 0
 \end{aligned}$$

- This modified free boundary problem is **well posed**.
- In general, **numerical solution** is required. A very useful approach is to use the **enthalpy**, which measures the total internal energy in the material, including thermal energy and latent heat.
- However, the initial stages at least may be analysed analytically, and in the quasi-steady limit  $St \rightarrow 0$  the problem may be solved completely.

## Co-dimension two free boundary problems

- In [electrochemical painting](#), metal objects are coated using electrochemistry.



- The object (“workpiece”) is immersed in an electrolyte solution, across which a potential difference  $V$  is imposed.
- This drives a current  $j$  of ions which attach themselves as a layer of paint on the workpiece.
- A model is needed to determine when (and if) the entire surface of the workpiece is covered in paint, and the thickness of the paint layer achieved.

## Modelling assumptions

- The concentration of ions in solution is sufficiently large that their depletion during the process is negligible. We may then treat the concentration as a constant.
- The flux of ions in the solution due to the imposed electric field is then given by

$$j = -\sigma \nabla \phi$$

where  $\phi$  is the electric potential and  $\sigma$  is the conductivity.

- Conservation of charge then gives

$$\nabla \cdot j = 0$$

- We apply boundary conditions  $\phi = -V$  on the workpiece and  $\phi = 0$  on the outer surface.

- Let  $h(x,t)$  denote the thickness of the paint layer (where it exists).
- Assume the paint acts as a resistive layer with conductivity  $\bar{\sigma}$  much lower than that of the solution. If the layer is thin, then it will act as a classical resistor with surface conductivity  $\bar{\sigma}/h$ .
- We can relate the potential difference across the layer to the current across it:

$$\sigma \frac{\partial \phi}{\partial y} = \frac{\bar{\sigma}}{h} (V + \phi) \quad \text{at } y = h(x, t)$$

- It is observed experimentally that, when the current is switched off, the paint dissolves back into solution at some rate  $d$ . So,

$$\frac{\partial h}{\partial t} = -d + \alpha \sigma \frac{\partial \phi}{\partial y} \quad \text{at } y = h(x, t)$$

where  $\alpha = V_M/F$ , where  $V_M$  is the molar volume of solid paint and  $F$  is Faraday's constant.

- We non-dimensionalize via:

$$(x, y) = L(\tilde{x}, \tilde{y}), \quad \phi = -V + V\tilde{\phi} \quad h = \epsilon L\tilde{h}, \quad t = \frac{\epsilon L^2}{\alpha\sigma V}\tilde{t}, \quad \epsilon = \frac{\bar{\sigma}}{\sigma} \ll 1$$

- The boundary conditions then read

$$\frac{\partial\phi}{\partial y} = \frac{\phi}{h}, \quad \frac{\partial h}{\partial t} = \frac{\partial\phi}{\partial y} - \delta \quad \text{at } y = \epsilon h(x, t) \quad \delta = \frac{Ld}{\alpha\sigma V}$$

- We exploit the smallness of  $\epsilon$  and expand:

$$\phi(x, \epsilon h, t) = \phi(x, 0, t) + \epsilon h \frac{\partial\phi}{\partial y}(x, 0, t) + O(\epsilon^2)$$

- We can then replace the boundary condition above to apply on  $y=0$ .



- In places where there is no paint ( $h=0$ ) we apply

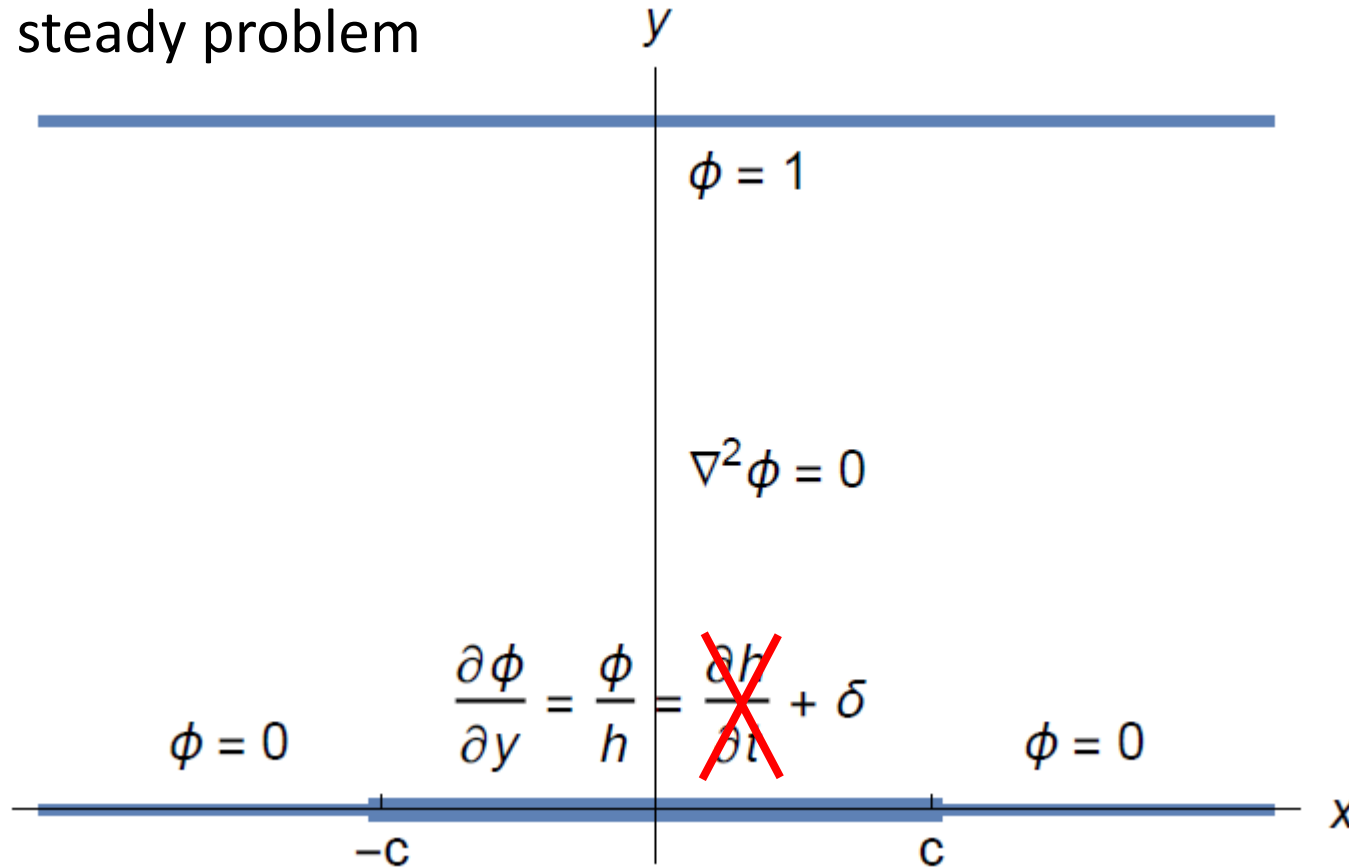
$$\phi = 0 \quad \text{at } y = 0$$

## Summary of lecture 7

- Solid melting is stable – a well-posed problem.
- Liquid freezing is unstable – an ill-posed problem.
- To model freezing we introduce a mushy region composed of liquid fraction  $\theta$  and solid fraction  $1-\theta$ .
- We solve for the temperature  $T$  in the solid and liquid regions and for  $\theta$  in the mushy region.

- The geometry to solve Laplace's equation is specified in advance.
- But the points  $x = \pm c$  where the boundary conditions switch from Dirichlet to Neumann are not known in advance and must be found as part of the solution.
- These *points* are the free boundaries.
- This is called a codimension two free boundary problem because two is the codimension between the space in which the problem is posed (two dimensions) and the free boundaries (points, i.e., zero dimensions).

- Consider the steady problem



- We can solve for  $\phi$  and then determine  $h$  after using the remaining boundary condition

$$h(x) = \frac{\phi(x, 0)}{\delta}$$

- Note that since  $h \geq 0$  this means that  $\phi(x, 0) \geq 0$ .

- Suppose that the earth is a single point at  $(0,1)$ . Then the potential near this point is a point singularity with unit strength:

$$\phi \sim -\frac{1}{2\pi} \log|(x, y) - (0, 1)| = -\frac{1}{4\pi} \log(x^2 + (y - 1)^2) \quad \text{as } (x, y) \rightarrow (0, 1)$$

- We subtract off the singularity, an image at (0,-1) and  $\delta y$  to satisfy the boundary condition  $\partial\phi/\partial y=\delta$  on  $y=h$ :

$$\phi(x, y) = \delta y - \frac{1}{4\pi} \log(x^2 + (y - 1)^2) + \frac{1}{4\pi} \log(x^2 + (y + 1)^2) + \Phi(x, y)$$

- The problem then reads:

$$\begin{array}{ll} \nabla^2 \Phi = 0 & y > 0, \\ \frac{\partial \Phi}{\partial y} \rightarrow -\delta & x^2 + y^2 \rightarrow \infty, \\ \Phi = 0, \quad \frac{\partial \Phi}{\partial y} \leq -f(x) & y = 0, \quad |x| > c, \\ \Phi \geq 0, \quad \frac{\partial \Phi}{\partial y} = -f(x) & y = 0, \quad |x| < c, \end{array}$$

where  $f(x) = \frac{1}{\pi(1+x^2)}$ .

- This is a mixed boundary value problem. We expect the solution to have singularities at  $(\pm c, 0)$  where the boundary conditions switch from Dirichlet to Neumann.
- This problem can be solved by taking a Fourier transform in  $x$ , in terms of the function  $\Phi$  at the surface of the workpiece,  $\Phi_0(x) = \Phi(x, 0)$ . Doing this, we find that

$$\frac{\partial \Phi}{\partial y} = \frac{1}{\pi} \int_{-c}^c \frac{(x-s)\Phi_0'(s)}{(x-s)^2 + y^2} ds$$



- Carefully taking the limit  $y \rightarrow 0$  and applying the boundary condition

$$\frac{\partial \Phi}{\partial y} = -f(x) \quad y = 0, \quad |x| < c.$$

we find that  $\Phi_0$  satisfies the singular integral equation:

$$-f(x) = \frac{1}{\pi} \int_{-c}^c \frac{\Phi_0'(s)}{x-s} ds$$

- Here we take the principal value of the integral:

$$\int_{-c}^c \frac{\Phi_0(s)}{x-s} ds = \lim_{\epsilon \rightarrow 0} \left( \int_{-c}^{x-\epsilon} + \int_{x+\epsilon}^c \right) \frac{\Phi_0(s)}{x-s} ds$$

- We solve for  $\Phi_0(x)$ , and find the value of  $c$  such that is differentiable at  $(\pm c, 0)$ .

- The thickness of the painted layer is then recovered from  $h(x) = \frac{\Phi_0(x)}{\delta}$ .

## Solution of problem

- The most elegant way to solve this problem is using complex variable theory.

- We write  $\Phi(x, y) + i\Psi(x, y) = w(z)$

where  $w(z)$  is holomorphic in  $\text{Im}(z) > 0$ .

(The imaginary part  $\Psi$  is the harmonic conjugate to  $\Phi$  – they are related through the Cauchy–Riemann equations.)

- We differentiate with respect to  $z$ :

$$\frac{\partial \Phi}{\partial x} - i \frac{\partial \Phi}{\partial y} = w'(z)$$

- Our task is then to find a  $w'(z)$  that is holomorphic on the upper half plane and satisfies the conditions

$$\begin{array}{ll} w'(z) \rightarrow \delta i & z \rightarrow \infty, \\ \operatorname{Re} [w'(z)] = 0, & y = 0, |x| > c, \\ \operatorname{Im} [w'(z)] = f(x) & y = 0, |x| < c. \end{array}$$

- The mixed character of the problem is reflected in the fact that we specify the real part of  $w'$  in one place and the imaginary part in another.

- But we can use a trick to transform this into a problem for just the real part by letting

$$w'(z) = \sqrt{z^2 - c^2} W(z)$$

where  $\sqrt{z^2 - c^2} = \sqrt{z - c} \sqrt{z + c} = \sqrt{r_1 r_2} e^{i(\theta_1 + \theta_2)/2}$

- The key point is that  $\sqrt{z^2 - c^2}$  is real when  $x > c$  and imaginary when  $x < c$  as  $y \rightarrow 0$  so  $W(z)$  stays real everywhere.

- The new function  $W$  satisfies a classical Dirichlet problem, satisfying Laplace's equation with its real part specified everywhere on the boundary

$$W(z) \sim \frac{i\delta}{z} \quad z \rightarrow \infty, \quad (\text{A})$$

$$\operatorname{Re} [W(z)] = \begin{cases} 0 & |x| > c, \\ \frac{f(x)}{\sqrt{c^2 - x^2}} & |x| < c \end{cases} \quad y = 0. \quad (\text{B})$$

- The solution is given using Poisson's formula:

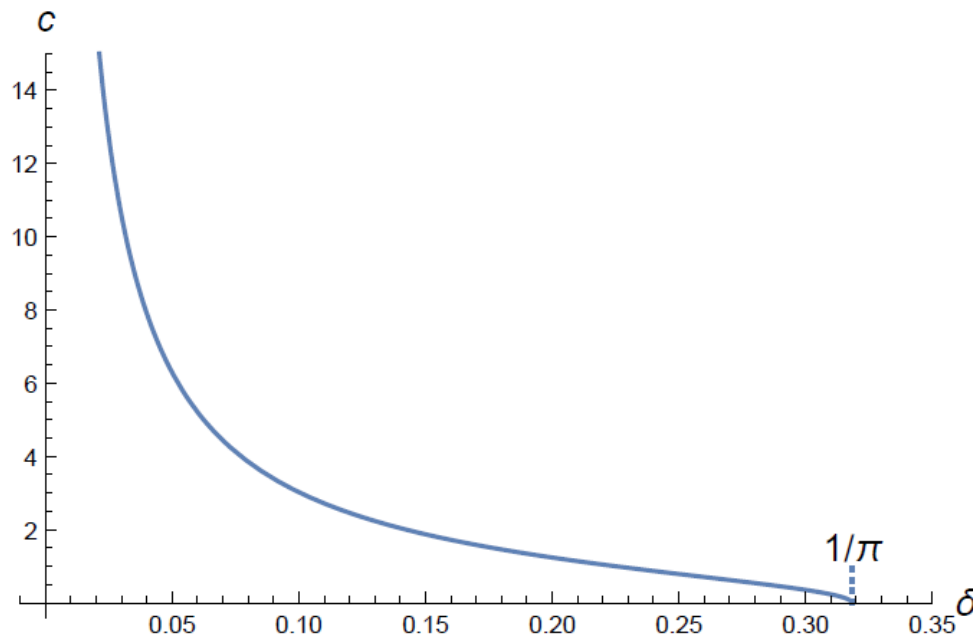
$$W(z) = \frac{1}{\pi i} \int_{-c}^c \frac{f(t) dt}{(t - z)\sqrt{c^2 - t^2}}$$

- Comparing (\*) with (A) gives a relation between  $\delta$  and  $c$ .

- Taking the limit of our integral (\*) as  $z \rightarrow \infty$  and comparing with (A) gives

$$\delta = \frac{1}{\pi\sqrt{1+c^2}} \quad \Longrightarrow \quad c = \sqrt{\frac{1}{\delta^2\pi^2} - 1}.$$

- This tells us how the size of the painted region,  $2c$ , depends on the control parameter  $\delta$

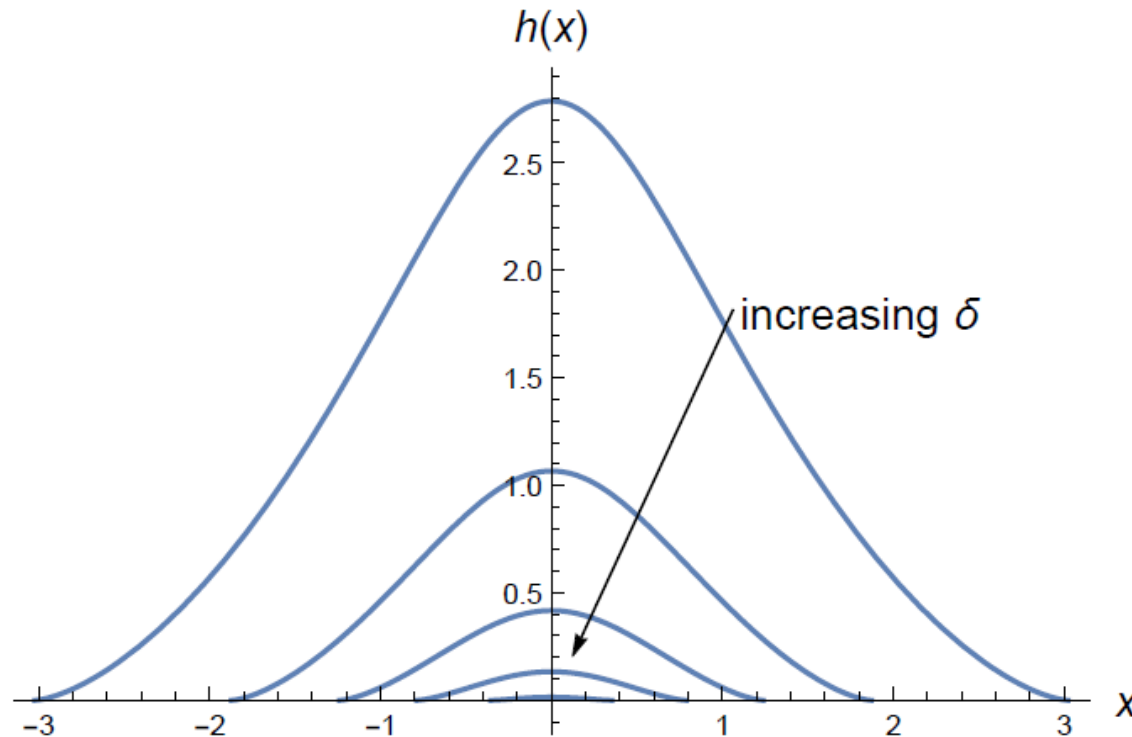


$$\delta = \frac{Ld}{\alpha\sigma V}$$

- Using our result for  $W$  and the definition of  $W$ , we can relate back to  $\Phi_0$  to get

$$\Phi_0(x) = \frac{1}{\pi} \left[ \tanh^{-1} \left( \sqrt{\frac{c^2 - x^2}{1 + c^2}} \right) - \sqrt{\frac{c^2 - x^2}{1 + c^2}} \right]$$

- The thickness profile is then given by  $h(x) = \Phi_0(x)/\delta$



## Useful texts for more reading and examples

- 1) I. Barenblatt, 1979, *Similarity, Self-similarity and Intermediate Asymptotics*, Consultants Bureau.
- 2) A.B. Tayler, 2001, *Mathematical Models in Applied Mechanics*. Oxford University Press.
- 3) S.C. Gupta, 2017, *The Classical Stefan Problem: Basic Concepts, Modelling and Analysis with Quasi-analytical Solutions and Methods*. Elsevier.



