B8.3: Mathematical Modelling of Financial Derivatives

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Option pricing: binomial model

Overview

- Arbitrage pricing
- Binomial trees
- Risk-neutral valuation

Financial Options

Definition

A **European call (put) option** gives the right to the holder of the option to purchase (sell) the underlying, for example a stock S, at a pre-specified time, called the expiration date T, for a pre-specified amount known as the strike price K.

Definition

An **American call/put option** is like a European option with the difference that it can be exercised (i.e., buy or sell the underlying) at any time up until, and including, expiration T.

Pricing

Simple example of binomial tree setup to price a call option

- Assume that there are two possible states.
  - A stock is trading at 100 and tomorrow it will either
    - go up to 101 or
    - go down to 99.
- What is the value of a European call option with strike price $K = 100$?

Pricing

Simple example of binomial tree setup to price a call option

- Assume that there are two possible states.
  - A stock is trading at 100 and tomorrow it will either
    - go up to 101 or
    - go down to 99.
- What is the value of a European call option with strike price $K = 100$?
- What happens if the probability of landing in the down state is $q = \{.25\,,0.5\,,0.95\}$?

Simple world

- Two states of nature that occur with probability p and $q = 1 - p$, and two traded assets
- Asset 1 (A_1) pays 1 in state 1 and 1 in state 2, i.e., pays $(1, 1)$
- Asset 2 (A_2) pays 0 in state 1 and 3 in state 2, i.e., pays $(0, 3)$
- Price of A_1 is p_1 and of A_2 is p_2

Now, assume that there is a third asset in this simple economy paying (2,3). What is its initial price p_3?

Pricing asset A_3

- Set up a portfolio $\Pi(t=0)$ consisting of a units of A_1 and b units of A_2. Find a and b such that $\Pi(t=1) = A_3(1)$.

$$\Pi_u(1) = a \times 1 + b \times 0$$

and

$$\Pi_d(1) = a \times 1 + b \times 3\,.$$

- We require that $\Pi_u(1) = 2$ and $\Pi_d(1) = 3$, i.e., we replicate A_3's payoff.
- Therefore, $a = 2$ and $b = 1/3$ and at time $t = 0$,

$$p_3 = 2\,p_1 + \frac{1}{3}\,p_2\,.$$

Pricing a Call option in a Binomial model

- Two states of the world, up and down, with probabilities p and $q = 1 - p$, respectively.
- Starting value of stock is S.
- In the 'up' state, with probability p, asset becomes $u\,S$ where u is a constant.
- In the 'down' state asset becomes $d\,S$ where d is a constant.
- There is a risk-free bond that pays a constant interest rate r.
- In the up state the payoff of the call is

$$C_u^E = \max(u\,S - K, 0)\,.$$

- In the down state the payoff of the call is

$$C_d^E = \max(d\,S - K, 0)\,.$$

Pricing the call at $t = 0$

- As above, set up a portfolio with B cash in a bond and Δ amount of the stock to replicate the payoff of option:

$$\Pi(0) = B + \Delta\, S\,.$$

- Choose Δ such that

$$\Delta\, u\, S + R\, B = C_u^E\,,$$

and

$$\Delta\, d\, S + R\, B = C_d^E\,,$$

where the gross risk free rate is $R = 1 + r$.

In matrix form, we solve the system of equations

$$\begin{bmatrix} u\,S & R \\ d\,S & R \end{bmatrix} \begin{bmatrix} \Delta \\ B \end{bmatrix} = \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

therefore

$$\begin{bmatrix} \Delta \\ B \end{bmatrix} = \frac{1}{R\,(u\,S - d\,S)} \begin{bmatrix} R & -R \\ -d\,S & u\,S \end{bmatrix} \begin{bmatrix} C_u^E \\ C_d^E \end{bmatrix},$$

so

$$\Delta = \frac{C_u^E - C_d^E}{u\,S - d\,S} \qquad \text{and} \qquad B = \frac{-d\,C_u^E + u\,C_d^E}{R\,(u - d)}\,.$$

Hence, the value of the portfolio at time $t = 0$ is, by **no arbitrage**, the same value as that of the call, i.e., $\Pi(0) = C^E(S, t = 0; K, 1)$.

$$\begin{aligned} C^E(S, t=0; K, 1) &= \Delta S + B \\ &= \frac{C_u^E - C_d^E}{u\,S - d\,S}\,S + \frac{-d\,C_u^E + u\,C_d^E}{R\,(u-d)} \\ &= \frac{1}{R}\left[\frac{R-d}{u-d}\,C_u^E + \frac{u-R}{u-d}\,C_d^E\right]. \end{aligned}$$

Risk-neutral valuation

The value of the call can be seen as the discounted weighted average of the payoff at expiry, with weights

$$p^\star = \frac{R-d}{u-d} \qquad \text{and} \qquad q^\star = \frac{u-R}{u-d}\,,$$

and write the price of the call as the expectation (under the new measure) as

$$C^E(S, t=0; K, 1) = \frac{1}{R}\,\left[p^\star C_u^E + q^\star C_d^E\right].$$

Can we, in this risk-neutral world, calculate the discounted expected value of the stock price $R^{-1}\,\mathbb{E}^\star[S_1]$?

- First, note that

$$C^E(S, t=0; K=0,1) = S\,.$$

- Then

$$\begin{aligned}
C^E(S, t=0; K=0,1) &= \frac{1}{R}\left[p^\star\, C^E_u + q^\star\, C^E_d\right] \\
S &= \frac{1}{R}\,[p^\star\, u\, S + q^\star\, d\, S] \\
&= \frac{1}{R}\,[p^\star\, u\, S + (1-p^\star)\, d\, S] \\
&= \frac{1}{R}\,[p^\star\, u\, S + (1-p^\star)\, d\, S] \\
&= \frac{1}{R}\,\mathbb{E}^\star[S_1]\,.
\end{aligned}$$

Model independent properties

Call prices satisfy the following inequalities

1. $$C^A(S,t;K,T) \geq C^E(S,t;K,T)\,,$$

2. $$C^A(S,t;K_1,T) \leq C^A(S,t;K_2,T)\,, \qquad \text{if } K_1 \geq K_2\,,$$

3. $$C^A(S,t;K,T_1) \geq C^A(S,t;K,T_2)\,, \qquad \text{if } T_1 \geq T_2\,,$$

4. $$C^A(S,t;K,T) \leq S\,,$$

5. $$C^A(0,t;K,T) = C^E(0,t;K,T) = 0\,.$$

Early Exercise

Proposition

Let S be an underlying security that pays no dividends. Then an American call written on S is **never** *exercised early.*

First we establish the inequality

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)}\,.$$

Consider the portfolio $C^A(S,t;K,T) - S + K\,e^{-r\,(T-t)}$. If the American call is exercised early we obtain

$$S - K - S + K\,e^{-r\,(T-t)} = K\,(e^{-r\,(T-t)} - 1) < 0\,.$$

If we wait until T we exercise if $S \geq K$ and obtain 0 profit; if $S < K$ we do not exercise the option and obtain $K - S > 0$.

Therefore we are better off waiting until T, hence we have shown

$$C^A(S,t;K,T) \geq S - K\,e^{-r\,(T-t)}\,.$$

To show that an American call written on a stock that pays no dividend is never exercised we observe that a call yields $S-K$ if exercised but

$$S-K \leq S - K\,e^{-r\,(T-t)} \leq C^A(S,t;K,T)\,.$$

QED

Proposition

Put-call-parity *for European options:*

$$C^E(S,t;K,T) - P^E(S,t;K,T) = S - K\,e^{-r\,(T-t)}\,.$$

Brownian Motion, Stochastic Integrals, Ito's Lemma

Overview

- Brownian Motion, Wiener Process
- Stochastic Integrals
- Itô's Lemma
- Modelling returns

Wiener process, Brownian motion

Definition

A stochastic process W is called a Wiener process or Brownian motion if the following conditions hold.

1. $W_0 = 0$.
2. The process W has independent increments, i.e., if $r < s \leq t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables.
3. For $s < t$ the distribution of the stochastic variable $W_t - W_s$ is $N(0, t - s)$.
4. W has continuous trajectories (almost surely, i.e., with probability one).

Note: It is not immediately obvious that we can rigorously construct a process W which satisfies these four properties, but it can be done.

Elementary properties of Brownian motion

Proposition

Let W_t be a Brownian motion and let $u > 0$, then

$$W_u \sim N(0, u) \tag{1}$$

and therefore

$$\mathbb{E}[W_u] = 0 \qquad \textit{and} \qquad \mathrm{Var}(W_u) = u\,.$$

Proof.

The result in (1) is a consequence of the third property for $t = u$ and $s = 0$ together with the property that $W_0 = 0$. □

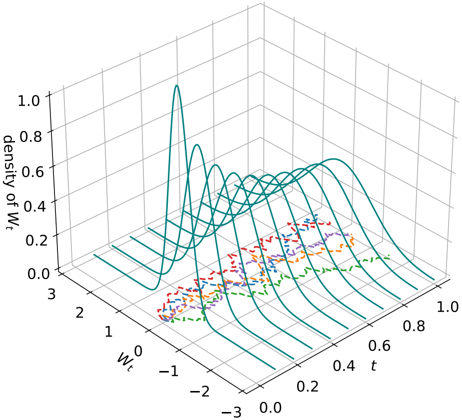

Figure: Five paths of Brownian motion and its density at various points in time.

Proposition

Let W_t be a Brownian motion. Given that $W_t \sim N(0,t)$ we have

$$\mathbb{P}(W_t > x) = \Phi^c\left(\frac{x}{\sqrt{t}}\right) \qquad \textit{and} \qquad \mathbb{P}(W_t \le x) = \Phi\left(\frac{x}{\sqrt{t}}\right). \tag{2}$$

Proof.

We have that

$$\mathbb{P}(W_t > x) = \mathbb{P}\left(\frac{W_t - 0}{\sqrt{t}} > \frac{x-0}{\sqrt{t}}\right) = \mathbb{P}\left(Z > \frac{x}{\sqrt{t}}\right) = \Phi^c\left(\frac{x}{\sqrt{t}}\right) \tag{3}$$

where $\Phi^c(x) = \int_x^\infty \frac{1}{\sqrt{2\,\pi}}\, e^{-\,z^2}\, dz$, and we are using that $Z = (W_t - 0)/\sqrt{t}$ is a standard Normal random variable. The second equality follows from the identity $\mathbb{P}(A) = 1 - \mathbb{P}(A^c)$. □

Proposition

Let W_t be a Brownian motion, then

$$\mathbb{E}[W_s\, W_t] = \min(s,t)\,. \tag{4}$$

Proof.

Let $0 \leq s \leq t$. Then

$$\mathbb{E}[W_s W_t] \;=\; \mathbb{E}[W_s(W_s + W_t - W_s)] \;=\; \mathbb{E}[W_s^2] \;=\; \mathrm{Var}(W_s) \;=\; s$$

because
$\mathbb{E}[W_s\,(W_t - W_s)] = \mathbb{E}[(W_s - W_0)\,(W_t - W_s)] = \mathbb{E}[W_s - W_0]\,\mathbb{E}[W_t - W_s] = 0$, (last step is because of independent increments). This means that in general, for $s, t \geq 0$

$$R(s,t) \;:=\; \mathbb{E}[W_s\, W_t] \;=\; \min(s,t)\,. \tag{5}$$

This is known as the **covariance function** of Brownian motion. □

Proposition

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$\mathbb{P}\left(W_t \leq x | W_s = y\right) = \Phi\left(\frac{x-y}{\sqrt{t-s}}\right) . \tag{6}$$

Proof.

$$\begin{aligned}
\mathbb{P}\left(W_t \leq x | W_s = y\right) &= \mathbb{P}\left(W_t - W_s + W_s \leq x | W_s = y\right) && (7)\\
&= \mathbb{P}\left(W_t - W_s + y \leq x | W_s = y\right) && (8)\\
&= \mathbb{P}\left(W_t - W_s \leq x - y | W_s = y\right) && (9)\\
&= \mathbb{P}\left(W_t - W_s \leq x - y\right) , && (10)
\end{aligned}$$

because $W_t - W_s$ is independent from W_s. Lastly,

$$\begin{aligned}
\mathbb{P}\left(W_t \leq x | W_s = y\right) &= \mathbb{P}\left(\frac{W_t - W_s}{\sqrt{t-s}} \leq \frac{x-y}{\sqrt{t-s}}\right) && (11)\\
&= \Phi\left(\frac{x-y}{\sqrt{t-s}}\right) . && (12)
\end{aligned}$$

Corollary

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$\mathbb{E}\,[W_t \mid W_s = y] = y\,. \tag{13}$$

Corollary

Let W_t be a Brownian motion, and let $0 < s < t$, then

$$f_{W_t \mid W_s = y}(x) = \frac{1}{\sqrt{2\,\pi(t-s)}}\, e^{-\frac{(x-y)^2}{2\,(t-s)}}\,. \tag{14}$$

Corollary

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion then $(-W_t)_{t\geq 0}$ is also a standard Brownian motion.

Quadratic Variation

Partitions and QV

A partition of the time interval $[0, t]$ is a set of the form $\Pi = t_0 = 0 < t_1 < ... < t_n = t$. The size of the partition is

$$\|\Pi\| = \max_{0 \le i \le n-1} (t_{i+1} - t_i) ,$$

i.e., equal to the largest interval of the partition. The **quadratic variation** (QV) of a random process X over a fixed time interval $[0, t]$ is

$$[X, X]_t = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})^2$$

if this limit exists and does not depend on the choice of the sequence of partitions Π.[1]

[1] I follow closely the material in "Stochastic Calculus for Finance. II Continuous-time models", by S. Shreve

QV of deterministic function

Let $f(t)$ be a continuous function defined on $0 \leq t \leq T$. The QV of f up to T is

$$[f, f]_T^n = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} [f(t_{i+1}) - f(t_i)]^2 , \tag{15}$$

where the partition Π is $\{t_0, t_1, \cdots, t_n\}$, $0 = t_0 < t_1 < \cdots < t_n = T$, and $n = n(\Pi)$ denotes the number of partition points in Π.

Next, we show that the QV of the function f is zero.

QV of deterministic function

$$\begin{aligned}
\sum_{i=0}^{n-1}[f(t_{i+1}) - f(t_i)]^2 &= \sum_{i=0}^{n-1} f'(t_i^*)^2 \, (t_{i+1} - t_i)^2 \leq \|\Pi\| \sum_{i=0}^{n-1} |f'(t_i^*)|^2 \, (t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \sum_{i=0}^{n-1} |f'(t_i^*)|^2 \, (t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \lim_{\|\Pi\|\to 0} \sum_{i=0}^{n-1} |f'(t_i^*)|^2 \, (t_{i+1} - t_i) \\
&= \lim_{\|\Pi\|\to 0} \|\Pi\| \lim_{\|\Pi\|\to 0} \int_0^T |f'(t)|^2 \, dt = 0 \, .
\end{aligned}$$

In last step $\int_0^T |f'(t)|^2 \, dt$ is finite because f is continuous.

Sampled QV of Brownian motion

Let W denote a standard Brownian motion. We define

$$[W,W]_t^n = \sum_{i=0}^{n-1} (W_{t_{i+1}} - W_{t_i})^2 \tag{16}$$

to be the sampled quadratic variation for a single partition Π.

Proposition

The following holds true

$$\begin{aligned} \mathbb{E}[[W,W]_t^n] &= t\,, \\ \mathrm{Var}([W,W]_t^n) = \mathbb{E}[([W,W]_t^n - t)^2] &\to 0\,, \end{aligned}$$

as $n \to \infty$.

proof We first note that

$$\begin{aligned}
\mathbb{E}\left[[W,W]_t^n\right] &= \mathbb{E}\left[\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2\right] \\
&= \sum_{i=0}^{n-1}\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] \\
&= \sum_{i=0}^{n-1}(t_{i+1} - t_i) \\
&= t\,,
\end{aligned}$$

because $W_{t_{i+1}} - W_{t_i} \sim N(0, t_{i+1} - t_i)$. Thus the expected sampled quadratic variation is independent of the partition, and trivially $\lim_{n\to\infty}\mathbb{E}[[W,W]_t^n] = t$, because the expectation here does not depend on n. Next,

$$
\begin{aligned}
\mathrm{Var}([W,W]_t^n) &= \mathbb{E}\left[([W,W]_t^n - t)^2\right] = \mathbb{E}\left[\left(\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 - t\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=0}^{n-1}\left[(W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right]\right)^2\right]
\end{aligned}
$$

(all cross products when squaring the above have expectation zero)

$$
\begin{aligned}
&= \sum_{i=0}^{n-1}\mathbb{E}\left[\left((W_{t_{i+1}} - W_{t_i})^2 - (t_{i+1} - t_i)\right)^2\right] \\
&= \sum_{i=0}^{n-1}\mathbb{E}\left[(W_{t_{i+1}} - W_{t_i})^4 - 2\,(t_{i+1} - t_i)\,(W_{t_{i+1}} - W_{t_i})^2 + (t_{i+1} - t_i)^2\right] \\
&= \sum_{i=0}^{n-1} 3\,(t_{i+1} - t_i)^2 - 2\,(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2
\end{aligned}
$$

(use $W_{t_{i+1}} - W_{t_i} \sim \sqrt{t_{i+1} - t_i}\,Z$ and $\mathbb{E}[Z^4] = 3$ where $Z \sim N(0,1)$)

$$
\begin{aligned}
&= 2\sum_{i=0}^{n-1}(t_{i+1} - t_i)^2 \le 2\,\|\Pi\|\sum_{i=0}^{n-1}(t_{i+1} - t_i) \\
&= 2\,\|\Pi\|\,t \qquad \text{which tends to zero if } \|\Pi\| \to 0.
\end{aligned}
$$

QV of Brownian motion

We have proved the following theorem.

Theorem

Let W denote a Brownian motion. Then $[W, W]_T = T$ for all $T \geq 0$ almost surely.

- We proved convergence in mean square, also called L^2 convergence.
- In general the quadratic variation $[X, X]_t$ of a process X is a random process, but for Brownian motion W, $[W, W]_t = t$ **almost surely** (a.s.).
  - Almost surely means that there are some paths of the Brownian motion for which $[W, W]_t = t$ is not true.
  - The probability of the set of paths for which $[W, W]_t = t$ is not true is zero.

To bear in mind

Above we used

$$\mathbb{E}[(W_{t_i+1} - W_{t_i})^2] = t_{i+1} - t_i \qquad \text{and} \qquad \mathrm{Var}[(W_{t_i+1} - W_{t_i})2] = t_{i+1} - t_i \,.$$

Intuitively, one would like to claim that

$$(W_{t_i+1} - W_{t_i})^2 \sim t_{i+1} - t_i \,,$$

which makes sense because for a small time increment both sides are very small. However, best to think about this as the square of a Normal r.v.

$$Y_{i+1} = \frac{W_{t_i+1} - W_{t_i}}{\sqrt{t_{i+1} - t_i}} \,; \tag{17}$$

the distribution of both sides is the same regardless of the time interval.

Now, take time interval $t_{i+1} - t_i = T/n$ and write

$$T\,\frac{Y_{i+1}^2}{n} = (W_{t_i+1} - W_{t_i})^2\,. \tag{18}$$

- By the LLN

$$\frac{1}{n}\sum_{i=0}^{n-1} Y_{i+1}^2 \to \mathbb{E}[Y_{i+1}^2] = 1 \qquad \text{as} \qquad n \to \infty\,.$$

- Thus,

$$\sum_{i=0}^{n-1}(W_{t_i+1} - W_{t_i})^2 \to T\,. \tag{19}$$

- Each term of the sum above can be different from its mean

$$t_{i+1} - t_i = T/n\,,$$

but when we sum many of them the differences average out to zero.