Lecture 1: Why schemes?

§ Summary of affine varieties

k - alg closed field

Main idea:

$$\left\{\begin{array}{c}\text{subsets of } k^n \\ \text{cut out by} \\ \text{polynomial equations}\end{array}\right\}\Big/\cong \quad \longleftrightarrow \quad \left\{\begin{array}{c}\text{finitely generated} \\ k\text{-algebras} \\ \text{without nilpotent} \\ \text{elements}\end{array}\right\}\Big/\cong$$

geometry algebra

- $I \subseteq k[x_1, \dots, x_n]$ ideal

  $X := Z(I) = \{a \in k^n \mid f(a) = 0 \ \forall f \in I\}$ affine variety
  (or $V(I)$) zero locus of all functions in I

- $\mathbb{A}^n(k) =: \mathbb{A}^n$ - n-dimensional affine space
  as a set it's k^n, but it has
  Zariski topology: closed subsets are $Z(I)$.
  Basis of distinguished open sets:
  $D(f) = \{a \in k^n \mid f(a) \neq 0\}$, $f \in k[x_1, \dots, x_n]$.

  Any $X \subseteq \mathbb{A}^n$ has subspace topology.

- $I(X) := \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \ \forall x \in X\}$
  $k[X] := k[x_1, \dots, x_n]/I(X)$ - coordinate ring of X

- $k[X]$ parametrizes functions on X:

$x \in X \rightsquigarrow m_x := \ker(ev_x : k[X] \to k)$ function that vanish at x

and $\forall f \in k[X]$ gives

$$f: X \to \mathbb{A}^1 = k$$
$$x \mapsto f(x) = \overline{f} \text{ in } k[X]/m_x$$

Hilbert's weak Nullstellensatz:

$$\left\{\begin{matrix}\text{points of} \\ X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{maximal ideals} \\ \text{of } k[X]\end{matrix}\right\}$$
$$(a_1, \dots, a_n) \longleftrightarrow (\overline{x_1 - a_1, \dots, x_n - a_n})$$

Hilbert's Nullstellensatz: $I(Z(I)) = \underbrace{\sqrt{I}}_{\text{radical of } I}$

$$\sqrt{I} = \{f \mid \exists N : f^N \in I\}$$

Morphisms: given X and $Y \subseteq \mathbb{A}^m$ a morphism is given by

$$\varphi : X \to Y \subseteq \mathbb{A}^m$$
$$\varphi = (f_1, \dots, f_m), \quad f_i \in k[X] \text{ whose image lie in } Y.$$

That's equivalent to a pullback map

$$\varphi^* : k[Y] \to k[X],$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, dashed, "\varphi^* f"'] & Y \arrow[d, "f"] \\ & \mathbb{A}^1 \end{tikzcd}$$

so $\mathrm{Hom}(X, Y) = \mathrm{Hom}_{k\text{-alg}}(k[Y], k[X])$

which gives the equivalence of cats:

$$\mathrm{AffVar}_k \simeq \mathrm{FinGen}\underbrace{\mathrm{Red}}_{\text{reduced}}\, k\text{-}\mathrm{Alg}^{op}$$

reduced = no nilpotents

§ Why varieties are not good enough?

Some reasons in no specific order:

① embedding into $\mathbb{A}^n$ shouldn't really be part of the data, would be nice to have an intrinsic definition, since you can embed the same variety into different spaces

② for non alg closed fields, Nullstellensatz doesn't work:

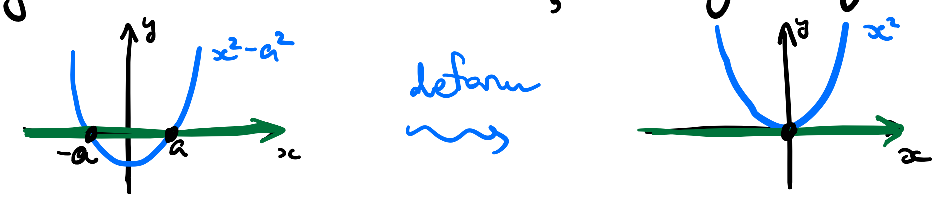
$I = (x^2+y^2+1) \subsetneq \mathbb{R}[x,y]$ is a prime $\Rightarrow$ radical ideal,

but $Z(I) = \emptyset$, so $I(Z(I)) = \mathbb{R}[x,y]$.

③ we can ask, on which top. space is $\mathbb{R}[x,y]/(x^2+y^2+1)$ naturally the space of functions? Or $\mathbb{R}[x]$? Or $\mathbb{Z}[x]$? Or $\mathbb{Z}$?

$\rightsquigarrow$ why don't we consider ALL rings?..

④ nilpotents arise naturally, when you deform varieties, so ignoring them is not good:



$X = Z(y, y - x^2 - a^2)$

$k[X] \simeq k[x]/_{(x-a)} \oplus k[x]/_{(x+a)} \simeq k^2$

k^2 parametrizes values at two points: $\{a\}$ and $\{-a\}$

$X = Z(y, y - x^2)$

$k[X] \simeq k[x]/\sqrt{(x^2)} \simeq k$

we lost info because we don't distinguish x and x^2

We want to have „functions" on the right to be $k[x]/_{x^2}$, not k.

Intuition: Intersections of varieties often don't want to be varieties:

it's a „double" point at 0, not just a point.

Historical motivation (non-examinable):

Weil conjectures (1948)

f homogeneous polynomial in $\mathbb{Z}[x_0, \dots, x_n]$.

$X = Z(f) \subset \mathbb{P}^n$ projective hypersurface

$X(\mathbb{C})$ – compact top. space $\rightsquigarrow b_0(X), \dots, b_{2n}(X)$

Betti numbers of X

$(b_i = \dim H^{2i}(X(\mathbb{C}); \mathbb{Z}))$

$|X(\mathbb{F}_{p^m})| =: N_m$ – number of solutions of mod p^m reduction $\rightsquigarrow \zeta(X, t) := \exp\left(\sum \frac{N_m}{m} t^m\right)$

Weil zeta function

(SGA: Grothendieck, Serre, Artin, Deligne, ...)

Thm. X smooth over $\mathbb{C}$ and over $\overline{\mathbb{F}_p}$ (i.e. $\frac{\partial f}{\partial x_i}$ don't vanish simultaneously $\forall x \in X$), then $\zeta(X;t)$ is a rational function:

$$\zeta(X;t) = \frac{p_1(t)\cdot p_3(t)\cdot\ldots\cdot p_{2n-1}(t)}{p_0(t)\cdot p_2(t)\cdot\ldots\cdot p_{2n}(t)} \quad \text{— polynomials,}$$

and $\underbrace{\deg p_i(t)}_{\text{arithmetic data}} = \underbrace{b_i}_{\text{topological data}}$! miracle :)

Proving Weil conjectures required building the theory of schemes + their cohomology, because to compare data over $\mathbb{C}$ and $\overline{\mathbb{F}_p}$, you need to deal with data over $\underbrace{\mathbb{Z}}_{\text{not a field}}$, and have the notion of „cohomology" for alg varieties and schemes, with properties similar to singular cohomology...

Extra reading (non-examinable): on Weil conjectures

- my lecture (separate pdf)
- Hartshorne App. C
- Milne „Lectures on étale cohomology" Chapter II