Chapter 2
The prime spectrum

Before: affine varieties^{op} $\simeq$ reduced fin.gen. algebras
over $k=\bar{k}$ over $k=\bar{k}$

Now: affine schemes^{op} $\simeq$ rings (assoc. comm. with 1)

This generalization allows to study arithmetic phenomena by geometric methods, by taking rings to be $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathcal{O}_K$ etc.

Recall: X affine k-variety, $k=\bar{k}$ $\Rightarrow$ by Nullstellensatz
points $x \in X$ $\leftrightarrow$ max ideals $m_x \subset k[X]$
$m_x = \{f \in k[X] \mid f(x)=0\}$

def. R ring. Its spectrum is
$\operatorname{Spec} R := \{p \mid p \subset R$ is a prime ideal$\}$.

This way, $x \in \operatorname{Spec} R$ $\leftrightarrow$ $p_x \in R$ (motivation later!)

NB: in general, we cannot think about $f \in R$ as functions with values in a fixed field k.

However, there's a more general notion.

def. Let $x \in \operatorname{Spec} R$ correspond to $\mathfrak{p} \subset R$.
The **residue field** of x (or $\mathfrak{p}$) is

$$\kappa(x) = \kappa(\mathfrak{p}) := R_\mathfrak{p} / \mathfrak{p} \cdot R_\mathfrak{p}$$

– maximal ideal in $R_\mathfrak{p}$

unrelated to any particular k

localization at $\mathfrak{p}$

Every element $f \in R$ has a „value"

$$f(x) := f \bmod \mathfrak{p}_x \in \kappa(x) \quad \forall x \in \operatorname{Spec} R,$$

and the codomain depends on the choice of x.

By definition, $f(x) = 0$ iff $f \in \mathfrak{p}_x$.

Moral: Spec R will be the space on which R is the ring of functions: the affine scheme corresponding to R

def.-Prop. The **Zariski topology** on Spec R is given by the closed subsets

$$Z(\mathfrak{a}) := \{x \in \operatorname{Spec} R \mid f(x) = 0 \ \forall f \in \mathfrak{a}\}$$
$$= \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a}\},$$

$\mathfrak{a} \subseteq R$ any ideal.

Prop. Let $\mathfrak{a}, \mathfrak{b} \subseteq R$ be ideals. Then

1) $Z(\mathfrak{a}) \subseteq Z(\mathfrak{b})$ iff $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.

In particular, $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

2) $Z(\mathfrak{a}) = \emptyset$ iff $\mathfrak{a} = A$

3) $Z(\mathfrak{a}) = \operatorname{Spec} R$ iff $\mathfrak{a} \subseteq \sqrt{(0)} =: \operatorname{Nil} R$

nilradical

4) $Z(\mathfrak{a}) \cup Z(\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b})$; $\bigcap_\lambda Z(\mathfrak{a}_\lambda) = Z(\sum_\lambda \mathfrak{a}_\lambda)$

Proof uses the **Main Fact**: $\sqrt{a} = \bigcap_{p \supseteq a} p$.

In particular, $p \supseteq a$ iff $p \supseteq \sqrt{a}$, so $Z(a) = Z(\sqrt{a})$.

1) $\Rightarrow$ $Z(a) \subseteq Z(b) \Rightarrow \underbrace{\bigcap_{p \in Z(a)} p}_{\sqrt{a}} \supseteq \underbrace{\bigcap_{p \in Z(b)} p}_{\sqrt{b}} \Rightarrow \sqrt{a} \supseteq \sqrt{b}$.

$\Leftarrow$ if $p \in Z(a)$ and $\sqrt{a} \supseteq \sqrt{b}$, then $p \supseteq \sqrt{a} \supseteq \sqrt{b} \supseteq b$ implies $p \in Z(b)$.

2) because $\sqrt{a} = (1)$ iff $a = (1)$, and $p \neq (1)$ $\forall p$, and $\forall a \subset m$ for some m.

3) because $Z(0) = \operatorname{Spec} R$.

Cor. There's an inclusion-reversing bijection

$$\left\{\begin{array}{c}\text{closed subsets} \\ \text{of } \operatorname{Spec} R\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{radical ideals} \\ \text{in } R\end{array}\right\}$$

$$Z(a) \longleftarrow\!\shortmid \; a$$

$$Z \longmapsto I(Z) := \bigcap_{p \in Z} p = \{f \in R \mid f(x) = 0 \;\; \forall x \in Z\}$$

Cor. $\forall$ set $S \subseteq \operatorname{Spec} R$ its closure $\overline{S} = Z(a)$, where $a = \bigcap_{p \in S} p$.

Proof: $\overline{S} = Z(b)$, for some radical ideal b.

$b \subseteq a$: $\forall p \in S$ $b \subseteq p$.

$b \supseteq a$: $Z(a) \supseteq Z(b)$ because $S \subseteq Z(a)$ and $Z(a)$ is closed, so $\overline{S} \subseteq Z(a)$. $\blacksquare$

In particular, for $S = \{p\}$ we get
$\overline{\{p\}} = Z(p) = \{q \in \operatorname{Spec} R \mid q \supseteq p \text{ prime}\}$.

Cor. $x \in \operatorname{Spec} R$ is a closed point iff p_x is a maximal ideal.

In Zariski topology, points don't have to be closed!
(unless R is of Krull dim 0)

Moral: X variety $\Rightarrow X \simeq \underbrace{\operatorname{mSpec}(k[X])}_{\text{maximal ideals}} \subset \underbrace{\operatorname{Spec} k[X]}_{\text{prime ideals}}$

Motivation: why prime ideals instead of maximal?
For varieties over $k = \bar{k}$, Nullstellensatz follows from the Jacobson property of f.g. reduced k-algebras, which tells that

$$\sqrt{I} = \bigcap_{m \supseteq I} m \quad \forall I \subset k[X],$$

and this property leads to the bijection between closed subsets of X and radical ideals in $k[X]$.

For a general R, we must use prime ideals to get such a correspondence:
e.g. R a dvr $\Rightarrow \exists!\ m = (t) \subset R$,
but R has two radical ideals: (0) and (t), so building Spec only out of max ideals wouldn't be enough.

§ Generic points

def. X top. space, $Z \subseteq X$ closed subset.
A generic point of Z (if exists) is
a point $\eta \in Z$ s.t. $\overline{\{\eta\}} = Z$. a dense point!

In our context:
$\forall\, p$ is a generic point of $Z(p) \subseteq \operatorname{Spec} R$.

Main Ex.: R integral domain $\Rightarrow$
$p = (0)$ is the generic point of $\operatorname{Spec} R$,
because $\{q \supseteq (0) \mid q \text{ prime}\} = \operatorname{Spec} R$.

Rem. For general top. spaces, generic points can be non-unique, but for $X = \operatorname{Spec} R$ (and for schemes) we'll see that a generic point is unique $\forall\, Z \subseteq X$.

Ex. 1) $R = K$ a field
$\operatorname{Spec} K = \{(0)\}$ — single point

2) $R = K[t]/(t^n)$ ← thickening
$\operatorname{Spec} R = \{(t)\}$ — „thick" point
Note: (0) is not prime, because $t \cdot t^{n-1} \in (0)$
1) vs 2): same top. spaces, different algebraic structures

3) R Artinian ring

Spec R is a finite set;

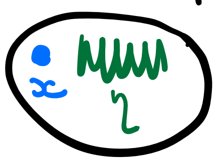
R local Artinian $\Rightarrow$ Spec R is a point

4) R dvr, e.g. $R = \mathbb{Z}_p$

Spec $R = \{x, \eta\}$ with $x \leftrightarrow m$ and $\eta \leftrightarrow (0)$.

Since x is a closed point,

the generic point $\eta = R - \{x\}$ is an open dense point!



Spec R

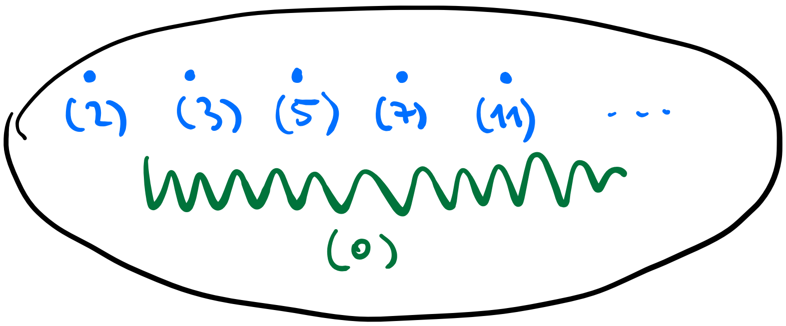
5) $R = \mathbb{Z}$

$$\operatorname{Spec} R \ni (0)$$
$$\quad\ \ \ni (p) \ \forall \text{ prime number } p$$

- $p\mathbb{Z} \subset \mathbb{Z}$ is maximal $\Rightarrow \{(p)\}$ is closed $\forall p$,
  and $k(p) = \mathbb{Z}_{(p)} / p \cdot \mathbb{Z}_{(p)} = \mathbb{F}_p$
- $\mathbb{Z}(0) = \mathbb{Z} \Rightarrow \{(0)\}$ is the generic pt of Spec $\mathbb{Z}$,
  and $k(0) = \mathbb{Z}_{(0)} = \mathbb{Q}$

So, every element $f \in \mathbb{Z}$ gives a "regular" function with values in various fields:

$$f = 17 \in \mathbb{Z} \rightsquigarrow f((0)) = 17 \in \mathbb{Q}$$
$$f((2)) = \bar{1} \in \mathbb{F}_2$$
$$f((3)) = \bar{2} \in \mathbb{F}_3$$
$$f((5)) = \bar{2} \in \mathbb{F}_5$$
$$\ldots$$



$\operatorname{Spec} \mathbb{Z}$

Comments

- Points of Spec R have various residue fields, and this allows us to study simultaneously solutions of equations over different fields (or rings), e.g. $\mathbb{F}_p$ and $\mathbb{Q}$

- When R is a f.g. k-algebra, $k=\bar{k}$, then $\forall$ closed point m we have
$$\kappa(m) = k,$$
since by Weak Nullstellensatz $\kappa(m) \supseteq k$ is a finite field extension.

- For such R, the topology of Spec R is fully detected by the closed pts, that's why we didn't encounter the diversity of residue fields before — but having them lets us put number theory into geometric context and then some of us can prove results like Fermat's Last Thm! :)

def. The affine n-space is

$$\mathbb{A}^n := \operatorname{Spec} \mathbb{Z}[t_1, \ldots, t_n].$$

The affine n-space over R is

$$\mathbb{A}^n_R := \operatorname{Spec} R[t_1, \ldots, t_n].$$

If $k = \bar{k}$, then

$$\mathbb{A}^n_k \supsetneq \mathbb{A}^n(k) \overset{\text{as a set}}{=} k^n$$

points are prime ideals in $k[t_1, .., t_n]$ — points are maximal ideals in $k[t_1, .., t_n]$,

and Zariski topology on $\mathbb{A}^n(k)$ is induced by Zariski topology on $\mathbb{A}^n_k$, but $\mathbb{A}^n_k$ has more points, e.g. (0).