B8.3: Mathematical Modelling of Financial Derivatives
—Exercises—

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Exercise Sheet 1

The exercises in all sheets draw from previous years (Sam Cohen) and from "The Mathematics of Financial Derivatives —A Student Introduction", by Paul Wilmott, Sam Howison and Jeff Dewynne.

Part A

1. If $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$, where S_t denotes the time t price of the stock, dW_t denotes the increments of a standard Wiener process, and μ, σ, A and n are constants, find the stochastic equations satisfied by
(a) $f(S) = A\, S$,
(b) $f(S) = S^n$,
(c) Show that $\mathbb{E}[S_T] = S_t\, e^{\mu\,(T-t)}$.

Part B

1. An Ornstein–Uhlenbeck process X satisfies the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

where $\kappa > 0$ and $\theta \in \mathbb{R}$.

(a) By using Itô's lemma applied to $e^{\kappa t} X_t$, show that for a given initial value X_0, the value of X_t is given by

$$X_t = \theta + \Big((X_0 - \theta) + \sigma \int_0^t e^{\kappa s} dW_s\Big) e^{\kappa t}$$

(b) Show that this implies that, for any deterministic initial value X_0, X_t has a Gaussian distribution, with mean and variance you should determine.

(c) Calculate $f(x,t) = \mathbb{E}[X_T^2 | X_t = x]$, and check explicitly that this is a solution to the corresponding PDE:

$$\frac{\partial f}{\partial t} + \kappa(\theta - x)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} = 0.$$

2. Let the price S_t of an asset satisfy

$$dS_t = \alpha\ (\mu - \ln S_t)\ S_t\, dt + \sigma\, S_t\, dW_t\,, \qquad (1)$$

where α and σ are non-negative constants, μ is a constant, and W a standard Brownian motion.

(a) Show that

$$x_T = x_t\, e^{-b\,(T-t)} + \frac{a}{b}\,\left(1 - e^{-b\,(T-t)}\right) + \sigma\, e^{-b\,T} \int_t^T e^{b\,s}\, dW_u\,,$$

where $x_t = \ln S_t$, $a = \alpha\,\hat{\mu}$ (for a choice of $\hat{\mu}$, and $b = \alpha$.

(b) Use the above to calculate $\mathbb{E}_t[S_T]$.

3. In the following $(W_t)_{t\geq 0}$ denotes a standard Brownian motion and $t \geq 0$ denotes time. A partition Π of the interval $[0,t]$ is a sequence of points $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ and $|\Pi| = \max_k(t_{k+1} - t_k)$. On a given partition $W_k \equiv W_{t_k}$, $\delta W_k \equiv W_{k+1} - W_k$, $\delta t_k \equiv t_{k+1} - t_k$ and if f is a function on $[0,t]$, $f_k \equiv f(t_k)$ and $\delta f_k \equiv f_{k+1} - f_k$.

   (a) Show that if $t, s \geq 0$ then $\mathbb{E}[W_s\, W_t] = \min(s,t)$.

   (b) Assuming that both the integral and its variance exist, show that

   $$\mathrm{Var}\Big[\int_0^t f(W_s, s)\, dW_s\Big] = \int_0^t \mathbb{E}\, f(W_s, s)^2\, ds\,.$$

   Is it generally the case that $\int_0^t f(W_s, s) dW_s$ has a Gaussian distribution?

   [Note: if the integral and its variance exist then it is legitimate to interchange the order of expectation and dt-integration.]

4. Use the differential version of Itô's lemma to show that

   (a) $\displaystyle\int_0^t W_s\, ds = t\, W_t - \int_0^t s\, dW_s = \int_0^t (t-s)\, dW_s,$

   (b) $\displaystyle\int_0^t W_s^2\, dW_s = \tfrac{1}{3} W_t^3 - \int_0^t W_s\, ds,$

Part C

1. Define X_t to be the 'area under a Brownian motion', $X_0 = 0$ and $X_t = \int_0^t W_u\, du$ for $t > 0$. Show that X_t is normally distributed with

   $$\mathbb{E}\big[X_t\big] = 0, \quad \mathbb{E}\big[X_t^2\big] = \tfrac{1}{3}\, t^3.$$

   Now define Y_t as the 'average area under a Brownian motion',

   $$Y_t = \begin{cases} 0 & \text{if } t = 0, \\ X_t/t & \text{if } t > 0. \end{cases}$$

   Show that Y_t has $\mathbb{E}[Y_t] = 0$, $\mathbb{E}\big[Y_t^2\big] = t/3$ and that Y_t is continuous for all $t \geq 0$. Is $\sqrt{3}\, Y_t$ a Brownian motion? Give reasons for your answer.

2. Consider the general stochastic differential equation

$$dG = A(G,t)\,dt + B(G,t)\,dW_t\,,$$

where G and B are functions and W is Brownian motion. Use Itô's Lemma to show that it is theoretically possible to find a function $f(G,t)$ which itself follows a random walk with zero drift.