

B5.6 Nonlinear Dynamics, Bifurcations and Chaos

Sheet 1 — HT 2024

Solutions to all problems in Sections A and C

Section A: Problems 1, 2 and 3

1. Find the stable, unstable and center subspaces E^s , E^u and E^c of the linear system

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}$$

with matrix $M \in \mathbb{R}^{4 \times 4}$ given by

(a)

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b)

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

(c)

$$M = \begin{pmatrix} -11 & 0 & 9 & -2 \\ -5 & -12 & 7 & 6 \\ -19 & 0 & 17 & -2 \\ -17 & -8 & 19 & 2 \end{pmatrix}$$

Solution: We denote the eigenvalues and generalized eigenvectors of M by

$$\lambda_j = a_j + i b_j \quad \text{and} \quad \mathbf{w}_j = \mathbf{u}_j + i \mathbf{v}_j,$$

where $a_j, b_j \in \mathbb{R}$, $\mathbf{u}_j, \mathbf{v}_j \in \mathbb{R}^4$, for $j = 1, 2, 3, 4$.

(a) The matrix M is diagonalizable (semi-simple). It has four different eigenvalues and eigenvectors given by

$$a_1 = a_2 = -\frac{1}{2}, \quad a_3 = a_4 = \frac{1}{2}, \quad b_1 = b_3 = \frac{\sqrt{3}}{2}, \quad b_2 = b_4 = -\frac{\sqrt{3}}{2},$$

$$\mathbf{u}_1 = \mathbf{u}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_1 = -\mathbf{v}_2 = \sqrt{3} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\mathbf{u}_3 = \mathbf{u}_4 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_3 = -\mathbf{v}_4 = \sqrt{3} \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have

$$E^s = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad E^u = \text{span} \left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad E^c = \emptyset.$$

(b) The matrix M is diagonalizable (semi-simple). It has four different eigenvalues and eigenvectors given by

$$\lambda_1 = -2, \quad \lambda_2 = 2, \quad \lambda_3 = i, \quad \lambda_4 = -i,$$

$$\mathbf{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{u}_3 = \mathbf{u}_4 = \begin{pmatrix} -5 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = -\mathbf{v}_4 = \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix}.$$

Consequently, we have

$$E^s = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad E^u = \text{span} \left\{ \begin{pmatrix} 0 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}, \quad E^c = \text{span} \left\{ \begin{pmatrix} -5 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(c) The characteristic polynomial of matrix M is $(\lambda - 8)(\lambda + 4)^3$. Consequently, matrix M has two eigenvalues: $\lambda_1 = 8$ (with multiplicity 1) and $\lambda_2 = -4$ (with algebraic multiplicity 3 and geometric multiplicity 1). The corresponding eigenvectors are

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (1)$$

The generalized eigenvectors corresponding to $\lambda_2 = -4$ are

$$\mathbf{w}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{w}_4 = \begin{pmatrix} 4 \\ 1 \\ 4 \\ 0 \end{pmatrix}. \quad (2)$$

They satisfy the equations $(M - \lambda_2 I)\mathbf{w}_3 = 2\mathbf{w}_2$ and $(M - \lambda_2 I)\mathbf{w}_4 = 8\mathbf{w}_3$, which implies $(M - \lambda_2 I)^2\mathbf{w}_3 = \mathbf{0}$ and $(M - \lambda_2 I)^3\mathbf{w}_4 = \mathbf{0}$. Consequently, we have

$$E^s = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 4 \\ 0 \end{pmatrix} \right\}, \quad E^u = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}, \quad E^c = \emptyset.$$

2. Consider the linear system

$$\frac{d\mathbf{x}}{dt} = M\mathbf{x}$$

with matrix $M \in \mathbb{R}^{4 \times 4}$ given as in Question 1(c), i.e.

$$M = \begin{pmatrix} -11 & 0 & 9 & -2 \\ -5 & -12 & 7 & 6 \\ -19 & 0 & 17 & -2 \\ -17 & -8 & 19 & 2 \end{pmatrix}$$

and the initial condition

$$\mathbf{x}(0) = \mathbf{x}_0.$$

- (a) Assume that $\mathbf{x}_0 \in E^u$ where E^u is the unstable subspace calculated in part 1(c). Assume that $\mathbf{x}_0 \neq \mathbf{0}$. Show that

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \mathbf{x}(t) = \mathbf{0}$$

- (b) Assume that $\mathbf{x}_0 \in E^s$ where E^s is the stable subspace calculated in part 1(c). Assume that $\mathbf{x}_0 \neq \mathbf{0}$. Show that

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0} \quad \text{and} \quad \lim_{t \rightarrow -\infty} \|\mathbf{x}(t)\| = \infty$$

Solution:

- (a) If $\mathbf{x}_0 \in E^u$ and $\mathbf{x}_0 \neq \mathbf{0}$, then there exists a constant $\alpha_1 \neq 0$ such that $\mathbf{x}_0 = \alpha_1 \mathbf{w}_1$, where \mathbf{w}_1 is given by (1), and the solution of our initial value problem is

$$\mathbf{x}(t) = \alpha_1 e^{\lambda_1 t} \mathbf{w}_1.$$

Since $\lambda_1 = 8 > 0$ and $\alpha_1 \neq 0$, we have

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = \infty \quad \text{and} \quad \lim_{t \rightarrow -\infty} \mathbf{x}(t) = \mathbf{0}.$$

- (b) If $\mathbf{x}_0 \in E^s$, then there exist constants α_2, α_3 and α_4 such that

$$\mathbf{x}_0 = \alpha_2 \mathbf{w}_2 + \alpha_3 \mathbf{w}_3 + \alpha_4 \mathbf{w}_4, \quad (3)$$

where $\mathbf{w}_2, \mathbf{w}_3$ and \mathbf{w}_4 are given by (1)–(2), and the solution of our initial value problem is

$$\mathbf{x}(t) = (\alpha_2 + 2\alpha_3 t + 8\alpha_4 t^2) e^{\lambda_2 t} \mathbf{w}_2 + (\alpha_3 + 8\alpha_4 t) e^{\lambda_2 t} \mathbf{w}_3 + \alpha_4 e^{\lambda_2 t} \mathbf{w}_4. \quad (4)$$

Indeed, substituting $t = 0$ into (4) and using (3), we confirm that the solution (4) satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Moreover, multiplying (4) by M and using $M\mathbf{w}_2 = \lambda_2 \mathbf{w}_2$, $M\mathbf{w}_3 = \lambda_2 \mathbf{w}_3 + 2\mathbf{w}_2$ and $M\mathbf{w}_4 = \lambda_2 \mathbf{w}_4 + 8\mathbf{w}_3$, we get

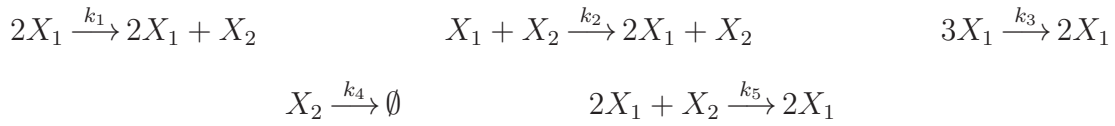
$$M\mathbf{x}(t) = \lambda_2 \mathbf{x}(t) + 2(\alpha_3 + 8\alpha_4 t) e^{\lambda_2 t} \mathbf{w}_2 + 8\alpha_4 e^{\lambda_2 t} \mathbf{w}_3,$$

which we also obtain by differentiating (4) as $d\mathbf{x}/dt$. In particular, we have confirmed that formula (4) is the solution of our initial value problem with \mathbf{x}_0 given by (3). Since $\lambda_2 = -4$ and $\mathbf{x}_0 \neq \mathbf{0}$, we conclude that at least one α_j , $j = 2, 3, 4$, is nonzero and equation (4) implies that

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0} \quad \text{and} \quad \lim_{t \rightarrow -\infty} \|\mathbf{x}(t)\| = \infty.$$

B5.6 Nonlinear Dynamics, Bifurcations and Chaos: Sheet 1 — HT 2024

3. Consider the system of $n = 2$ chemical species X_1 and X_2 which are subject to the following $\ell = 5$ chemical reactions:



Let $x_1(t)$ and $x_2(t)$ be the concentrations of the chemical species X_1 and X_2 , respectively.

- (a) Assuming mass action kinetics, write a system of ODEs (reaction rate equations) describing the time evolution of $x_1(t)$ and $x_2(t)$.
- (b) Assume the problem has already been non-dimensionalized and choose the values of dimensionless rate constants as

$$k_1 = \mu, \quad k_2 = 1, \quad k_3 = 4, \quad k_4 = 1 \quad \text{and} \quad k_5 = 1,$$

where $\mu > 0$ is a single parameter that we will vary.

Use an analysis of the dynamics on the center manifold to show that

- (i) The origin $[x_1, x_2] = [0, 0]$ is an asymptotically stable critical point if $\mu \leq 4$.
- (ii) The origin $[x_1, x_2] = [0, 0]$ is an asymptotically unstable critical point if $\mu > 4$.
- (c) Find and classify all critical points and sketch the phase plane in the nonnegative quadrant $\{x_1 \geq 0, x_2 \geq 0\}$ for: (i) $\mu \in (0, 4)$; and (ii) $\mu > 4$.

Solution:

- (a) Using the definition of mass action kinetics (covered in Lecture 1), we have:

$$\begin{aligned}
 \frac{dx_1}{dt} &= k_2 x_1 x_2 - k_3 x_1^3 \\
 \frac{dx_2}{dt} &= k_1 x_1^2 - k_4 x_2 - k_5 x_1^2 x_2
 \end{aligned}$$

- (b) Using our values of parameters $k_1 = \mu, k_2 = 1, k_3 = 4, k_4 = k_5 = 1$, we have

$$\frac{dx_1}{dt} = x_1 x_2 - 4 x_1^3 \tag{5}$$

$$\frac{dx_2}{dt} = \mu x_1^2 - x_2 - x_1^2 x_2 \tag{6}$$

The origin $[x_1, x_2] = [0, 0]$ is a critical point and we have

$$D\mathbf{f}(0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, the linearization has eigenvalues -1 and 0 and the center subspace is spanned by the eigenvector $(1, 0)^T$ which corresponds to the 0 eigenvalue. The center manifold is tangent to the center subspace, so it can be locally written as

$$x_2 = c_2 x_1^2 + c_3 x_1^3 + c_4 x_1^4 + \mathcal{O}(x_1^5). \quad (7)$$

Differentiating with respect of t and substituting (5) and (6), we get

$$(\mu - c_2) x_1^2 - c_3 x_1^3 + (c_2(7 - 2c_2) - c_4) x_1^4 + \mathcal{O}(x_1^5) = 0.$$

Consequently, $c_2 = \mu$, $c_3 = 0$ and $c_4 = \mu(7 - 2\mu)$. Substituting into equation (7), we get

$$x_2 = \mu x_1^2 + \mu(7 - 2\mu) x_1^4 + \mathcal{O}(x_1^5).$$

Substituting into equation (5), we get the dynamics on the center manifold as

$$\frac{dx_1}{dt} = (\mu - 4) x_1^3 + \mu(7 - 2\mu) x_1^5 + \mathcal{O}(x_1^6).$$

Consequently, we obtain that the origin is asymptotically stable for $\mu < 4$ and unstable for $\mu > 4$. If $\mu = 4$, we have

$$\frac{dx_1}{dt} = -4x_1^5 + \mathcal{O}(x_1^6).$$

Thus we conclude that the origin is asymptotically stable for $\mu \leq 4$.

(c) The critical points are given as solutions of the system :

$$\begin{aligned} 0 &= x_1 x_2 - 4x_1^3 \\ 0 &= \mu x_1^2 - x_2 - x_1^2 x_2 \end{aligned}$$

Consequently, the first equation implies that we either have $x_1 = 0$ or $x_2 = 4x_1^2$. Substituting into the second equation, we deduce:

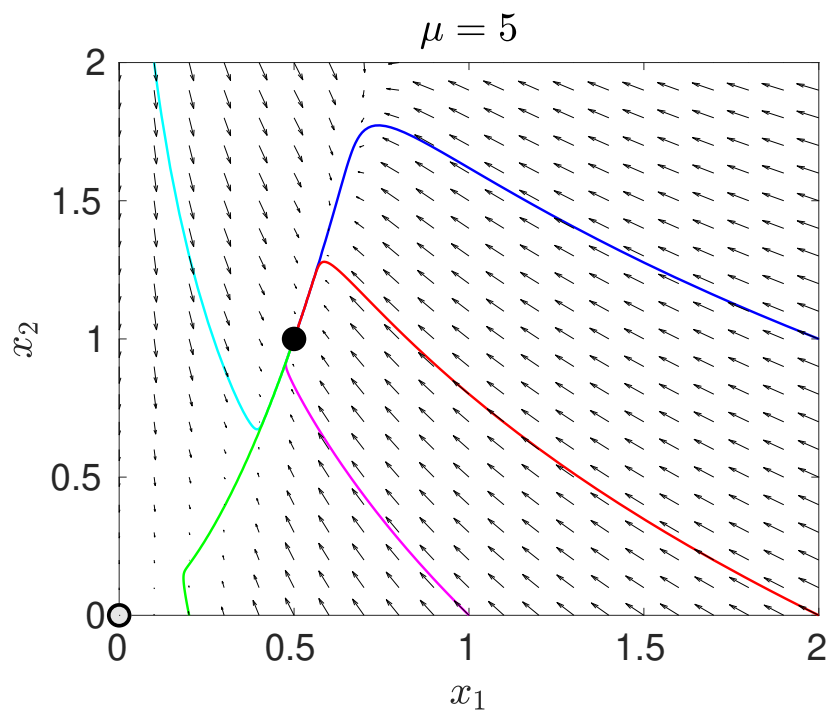
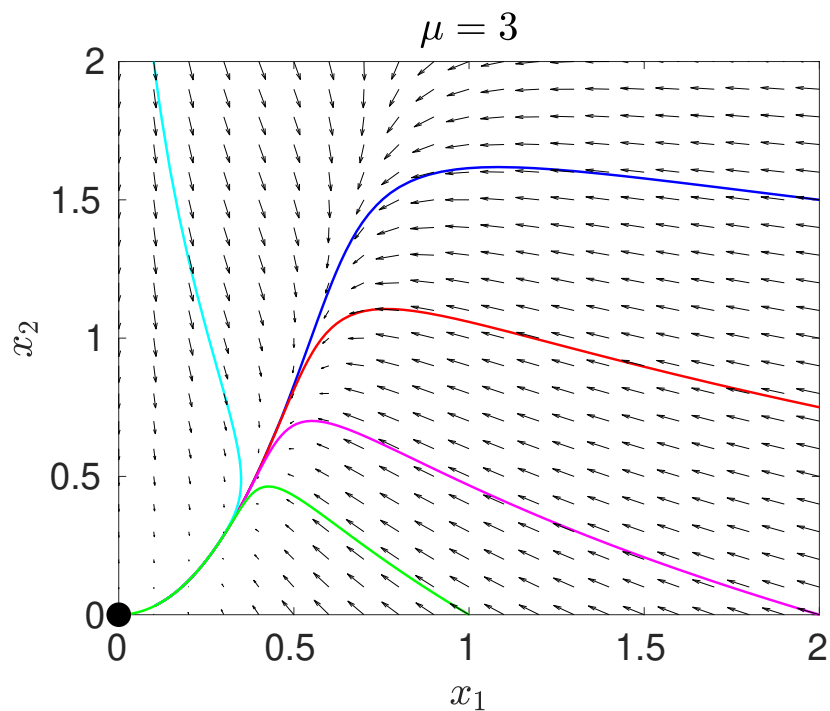
- (i) If $\mu \in (0, 4)$, then the origin is the only critical point which is asymptotically stable.
- (ii) If $\mu > 4$, then there are three critical points given by

$$[0, 0], \quad \left[\frac{\sqrt{\mu - 4}}{2}, \mu - 4 \right], \quad \left[-\frac{\sqrt{\mu - 4}}{2}, \mu - 4 \right].$$

The first two critical points are in the nonnegative quadrant $\{x_1 \geq 0, x_2 \geq 0\}$. The origin $[x_1, x_2] = [0, 0]$ is asymptotically unstable and $\left[\frac{\sqrt{\mu - 4}}{2}, \mu - 4 \right]$ is a stable node for $\mu > 4$.

B5.6 Nonlinear Dynamics, Bifurcations and Chaos: Sheet 1 — HT 2024

The illustrative phase planes in domain $[0, 2] \times [0, 2]$ are plotted below for $\mu = 3$ and $\mu = 5$. The black dots denote the critical points (filled-in dots are stable and empty dots are unstable). Five illustrative trajectories starting at the boundary of the box are plotted using different colours. They converge to the origin $[0, 0]$ for $\mu = 3$, and to the critical point $[1/2, 1]$ for $\mu = 5$.



Section C: Problem 7

7. Let $g \in C^1(\mathbb{R})$ be a given function satisfying $g(x) \geq 1$ for all $x \in \mathbb{R}$. Consider the initial value problem

$$\frac{dx}{dt} = x^2 \quad \text{with} \quad x(0) = x_0 \quad (\star)$$

and the initial value problem

$$\frac{dx}{d\tau} = \frac{x^2}{g(x)} \quad \text{with} \quad x(0) = x_0. \quad (\blacktriangle)$$

(a) Find the solution of the initial value problem (\star) and the maximum interval $I^\star(x_0)$ where the solution is defined for each initial condition $x_0 \in \mathbb{R}$.

(b) Let x_0 be given and denote the orbits corresponding to systems (\star) and (\blacktriangle) by $\Gamma_{x_0}^\star$ and $\Gamma_{x_0}^\blacktriangle$, respectively. Show that

$$\Gamma_{x_0}^\star = \Gamma_{x_0}^\blacktriangle, \quad \text{for all } x_0 \in \mathbb{R},$$

i.e. the ODEs (\star) and (\blacktriangle) have the same phase portrait.

(c) Find $g(x)$ such that the initial value problem (\blacktriangle) has its unique solution on the maximum interval $I^\blacktriangle(x_0) = \mathbb{R}$ for each initial condition $x_0 \in \mathbb{R}$.

Solution:

(a) Given the initial condition $x(0) = x_0 \in \mathbb{R}$, the solution of ODE (\star) is

$$x(t) = \frac{x_0}{1 - t x_0} \quad \text{for } t \in I^\star(x_0), \quad (8)$$

where the maximal interval of existence $I^\star(x_0)$ is

$$I^\star(x_0) = \begin{cases} \left(-\infty, \frac{1}{x_0}\right) & \text{for } x_0 > 0, \\ \mathbb{R} & \text{for } x_0 = 0, \\ \left(\frac{1}{x_0}, \infty\right) & \text{for } x_0 < 0. \end{cases}$$

(b) We define the new time τ by

$$\tau \equiv \tau(t) = \int_0^t g(x(s; x_0)) \, ds. \quad (9)$$

Since $g(x) \geq 1$ for all $x \in \mathbb{R}$, we conclude that $\tau(t)$ is a strictly increasing function of t . In particular, its inverse $t(\tau)$ exists and we can use the chain rule to deduce

$$\frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = \frac{x^2}{g(x)}, \quad (10)$$

i.e. $x(t(\tau))$ solves the initial value problem (\blacktriangle) . Using (8), the orbit $\Gamma_{x_0}^\star \subset \mathbb{R}$ based on $x_0 \in \mathbb{R}$ is given by

$$\Gamma_{x_0}^\star = \left\{ x(t; x_0) \mid t \in I^\star(x_0) \right\} = \left\{ \frac{x_0}{1 - t x_0} \mid t \in I^\star(x_0) \right\}.$$

Using (10), this can be rewritten as

$$\Gamma_{x_0}^\star = \{x(t; x_0) \mid t \in I^\star(x_0)\} = \{x(t(\tau); x_0) \mid \tau \in I^\blacktriangle(x_0)\} = \Gamma_{x_0}^\blacktriangle.$$

(c) Let $g(x) = 1 + x^2$. Then the initial value problem (\blacktriangle) is given as

$$\frac{dx}{d\tau} = \frac{x^2}{1 + x^2} \quad \text{with} \quad x(0) = x_0.$$

It has the unique solution $x(\tau) \equiv 0$ for $x_0 = 0$ and

$$x(\tau) = \frac{\tau x_0 + x_0^2 - 1 + \sqrt{(\tau x_0 + x_0^2 - 1)^2 + 4 x_0^2}}{2x_0} \quad \text{for } x_0 \neq 0, \quad (11)$$

which is defined on the maximum interval of existence $I^\blacktriangle(x_0) = \mathbb{R}$ for each initial condition $x_0 \in \mathbb{R}$. Substituting $g(x) = 1 + x^2$ and (8) into our rescaling of time equation (9), we get

$$\tau(t) = t \left(1 + \frac{x_0^2}{1 - t x_0} \right). \quad (12)$$

Substituting equation (12) into the solution formula (11), we obtain the solution formula (8). In particular, we confirm that the ODEs (\star) and (\blacktriangle) have the same phase portrait and the trajectories of (\blacktriangle) are defined for all $\tau \in \mathbb{R}$.