B4.3 Distribution Theory

Sheet 4 — MT23/HT24 Week 1

Localization and convolution of distributions. Hypoelliptic differential operators and elliptic regularity

Only work on the questions from Section B should be handed in.

Section A

1. Let Ω be a non-empty open subset of $\mathbb{R}^n$ and $p \in [1, \infty]$.

   (a) What does it mean to say that a distribution on Ω

   (i) has order $k \in \mathbb{N}_0$ on the open subset ω of Ω?

   (ii) is a regular distribution on the open subset ω of Ω?

   (iii) is L^p on the open subset ω of Ω?

   Assume that $u \in \mathscr{D}'(\Omega)$ is a regular distribution on the open subset $\omega \subset \Omega$. Show that u must have order 0 on ω. Next, let (ϕ_j) be a sequence in $\mathscr{D}(\Omega)$ such that for some compact set $K \subset \omega$ and constant $c \geq 0$ we have $\mathrm{supp}(\phi_j) \subseteq K$ and $\sup_{x \in \Omega} |\phi_j(x)| \leq c$ for all $j \in \mathbb{N}$. Prove that if $\phi_j(x) \to 0$ pointwise in almost all $x \in \Omega$ as $j \to \infty$, then $\langle u, \phi_j \rangle \to 0$ as $j \to \infty$.

   (b) Let δ_{x_0} denote Dirac's delta function concentrated at the point $x_0 \in \Omega$. Prove that δ_{x_0} is a distribution on Ω of order 0 that is a regular distribution on $\Omega \setminus \{x_0\}$, but not on Ω.

   (c) Let μ be a locally finite Borel measure on Ω and assume there exists a compact set $N \subset \Omega$ with $\mathscr{L}^n(N) = 0$ and $\mu(N) > 0$ (*the measure μ is then said to have a singular part with respect to $\mathscr{L}^n$*). Consider the corresponding distribution

   $$\langle \mu, \phi \rangle \overset{\mathrm{def}}{=} \int_\Omega \phi \, \mathrm{d}\mu, \quad \phi \in \mathscr{D}(\Omega).$$

   Show it has order 0 and that it is not a regular distribution on Ω.

   (d) Identifying $z \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ in the usual way and denoting by $B_1(0)$ the open unit disc in $\mathbb{R}^2$, show that

   $$u = \frac{1}{\pi z} \mathbf{1}_{B_1(0)}$$

   is a regular distribution on $\mathbb{R}^2$ and calculate its distributional Wirtinger derivative $\partial u / \partial \bar{z}$. Is it a regular distribution on $\mathbb{R}^2$? Find the supports and the singular supports of u and $\partial u / \partial \bar{z}$.

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Solution: (a): (i) means that the restriction $u|_\omega$ is a distribution on ω of order k. The restriction $u|_\omega$ was defined in lectures by the rule $\langle u|_\omega, \phi\rangle = \langle u, \phi^e\rangle$, $\phi \in \mathscr{D}(\omega)$, where ϕ^e denotes

$$\phi^e(x) \stackrel{\text{def}}{=} \begin{cases} \phi(x) & \text{if } x \in \omega, \\ 0 & \text{if } x \in \Omega \setminus \omega. \end{cases}$$

It is clear that $\phi \mapsto \phi^e$ is a linear and $\mathscr{D}$-continuous map of $\mathscr{D}(\omega)$ into $\mathscr{D}(\Omega)$, so that the above definition makes sense. Note that strictly speaking the restriction $u|_\omega$ is nothing but $u|_{\mathscr{D}(\omega)^e}$, but we prefer the former notation as it emphasizes that we think of distributions as generalized functions defined on Ω. (ii) means that there exists $f \in \mathrm{L}^1_{\mathrm{loc}}(\omega)$ such that $\langle u, \phi\rangle = \int_\omega f\phi \,\mathrm{d}x$ for $\phi \in \mathscr{D}(\Omega)$ with support in ω. This is often written simply as $u|_\omega \in \mathrm{L}^1_{\mathrm{loc}}(\omega)$. (iii) means that the f in (ii) belongs to $\mathrm{L}^p(\omega)$ (and we then often write $u|_\omega \in \mathrm{L}^p(\omega)$).

Assume $u|_\omega \in \mathrm{L}^1_{\mathrm{loc}}(\omega)$. Let $L \subset \omega$ be compact. Then denoting $u|_\omega = f$ we have that f is integrable over L and so for $\phi \in \mathscr{D}(\omega)$ with support contained in L it follows that $f\phi \in \mathrm{L}^1(\omega)$ and

$$\bigl|\langle u|_\omega, \phi\rangle\bigr| = \left|\int_\omega f\phi \,\mathrm{d}x\right| \leq c_L \sup_\omega |\phi|,$$

where $c_L = \int_L |f| \,\mathrm{d}x$. Therefore $u|_\omega$ has order 0.

The continuity property of the regular distribution $u|_\omega = f$ follows by use of Lebesgue's DCT: Under the stated assumptions we get $|f\phi_j| \leq c|f|\mathbf{1}_K$ a.e. on ω for all $j \in \mathbb{N}$, and $c|f|\mathbf{1}_K$ is integrable over ω. The pointwise convergence a.e. implies that also $f\phi_j \to 0$ a.e. on ω, hence $\langle u|_\omega, \phi_j\rangle = \int_K f\phi_j \,\mathrm{d}x \to 0$, as required.

(b): It is clear that δ_{x_0} is a distribution of order 0 on Ω. In fact, for any $\phi \in \mathscr{D}(\Omega)$ we have that $\bigl|\langle \delta_{x_0}, \phi\rangle\bigr| = |\phi(x_0)| \leq \sup_\Omega |\phi|$ (note that we do not even need to restrict the test functions to have support contained in compact subsets $K \subset \Omega$ here as the constants c_K can all be taken as 1). Because $\mathrm{supp}(\delta_{x_0}) = \{x_0\}$ we have $u|_{\Omega\setminus\{x_0\}} = 0$, which in particular is a regular distribution on $\Omega \setminus \{x_0\}$. To see that δ_{x_0} is not a regular distribution on Ω we can show that it does not have the continuity property established for regular distributions in (a) above. We use this approach in (c) below. Instead we proceed here by contradiction: assume that we could find $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ such that

$$\phi(x_0) = \int_\Omega f\phi \,\mathrm{d}x \quad \text{ for all } \phi \in \mathscr{D}(\Omega).$$

If we consider only ϕ that are supported in $\Omega \setminus \{x_0\}$, then we get from the fundamental lemma of the calculus of variations that $f = 0$ a.e. in $\Omega \setminus \{x_0\}$. But as $\{x_0\}$ is a null set we can strengthen this to $f = 0$ a.e. in Ω. But $\delta_{x_0} \neq 0$ so this is a contradiction proving that δ_{x_0} cannot be a regular distribution on Ω.

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(c): To see that μ has order 0 we fix a compact set $K \subset \Omega$. Then for $\phi \in \mathscr{D}(\Omega)$ with support in K we have

$$\bigl|\langle \mu, \phi \rangle\bigr| \le \int_K |\phi| \, \mathrm{d}\mu \le \mu(K) \sup_{\Omega} |\phi|,$$

and because μ is locally finite we have $\mu(K) < +\infty$, so the order is 0, as required.

Next, we show that μ cannot be a regular distribution on Ω by showing that it does not have the necessary continuity property established in (a) above. Put $\phi^{\varepsilon} = \rho_{\varepsilon} * \mathbf{1}_{B_{\varepsilon}(N)}$ for $\varepsilon \in (0, \operatorname{dist}(N, \partial\Omega)/3)$. Then $\phi^{\varepsilon} \in \mathscr{D}(\Omega)$ is supported in the compact subset $K = \overline{B_{2d}(N)}$ of Ω when $d = \operatorname{dist}(N, \partial\Omega)/3$. We also record that $0 \le \phi^{\varepsilon}(x) \le 1$ for all $x \in \Omega$, $\varepsilon \in (0, d)$ and that $\phi^{\varepsilon}(x) \to 0$ pointwise in $x \in \Omega \setminus N$ as $\varepsilon \searrow 0$. However since $\phi^{\varepsilon} = 1$ on N,

$$\bigl|\langle \mu, \phi^{\varepsilon} \rangle\bigr| = \int_K \phi^{\varepsilon} \, \mathrm{d}\mu \ge \mu(N) > 0 \quad \text{for all } \varepsilon \in (0, d),$$

it follows that $\langle \mu, \phi^{\varepsilon} \rangle$ does not converge to 0 as $\varepsilon \searrow 0$, and so that μ cannot be a regular distribution on Ω.

(d): u is a regular distribution since (integrating in polar coordinates)

$$\int_{\mathbb{R}^2} \left| \frac{1}{\pi z} \mathbf{1}_{B_1(0)} \right| \, \mathrm{d}(x, y) = 2 < +\infty,$$

so that we even have $u \in \mathrm{L}^1(\mathbb{R}^2)$. In order to calculate the Wirtinger derivative we use that u locally is the product of a C^{∞} function and a distribution so that we can employ the Leibniz rule:

$$\frac{\partial u}{\partial \bar{z}} = \delta_0 \mathbf{1}_{B_1(0)} + \frac{1}{\pi z} \frac{\partial}{\partial \bar{z}} \Bigl(\mathbf{1}_{B_1(0)} \Bigr) = \delta_0 + \frac{1}{\pi z} \frac{\partial}{\partial \bar{z}} \Bigl(\mathbf{1}_{B_1(0)} \Bigr),$$

where we used a result from Problem Sheet 3. In order to calculate the last term we let $\phi \in \mathscr{D}(\mathbb{R}^2)$ and use the divergence theorem:

$$\begin{aligned}
\left\langle \frac{\partial}{\partial \bar{z}} \Bigl(\mathbf{1}_{B_1(0)} \Bigr), \phi \right\rangle &= -\int_{B_1(0)} \frac{\partial \phi}{\partial \bar{z}} \, \mathrm{d}(x, y) \\
&= \int_{\partial B_1(0)} \frac{1}{2} \bigl(x + \mathrm{i}y \bigr) \phi \, \mathrm{d}S_{(x,y)} \\
&= \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta}}{2} \phi\bigl(\cos\theta, \sin\theta \bigr) \, \mathrm{d}\theta,
\end{aligned}$$

where in the last line we wrote out the curve integral. The Wirtinger derivative is therefore

$$\frac{\partial u}{\partial \bar{z}} = \delta_0 + \frac{1}{\pi z} \frac{z}{2} \mathrm{d}S = \delta_0 + \frac{1}{2\pi} \mathrm{d}S,$$

where $\mathrm{d}S$ denotes integration over the unit circle $\partial B_1(0)$ in $\mathbb{R}^2$. With the normalization, $\frac{1}{2\pi}\mathrm{d}S$, is a probability measure μ on $\mathbb{R}^2$: if A is a Borel subset of $\mathbb{R}^2$, then

$$\mu(A) = \frac{1}{2\pi} \int_{\{\theta \in [0, 2\pi] : \, \mathrm{e}^{\mathrm{i}\theta} \in A\}} \mathrm{d}S.$$

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Using (c) above we see that the Wirtinger derivative of u therefore is a distribution of order 0 that is not a regular distribution on $\mathbb{R}^2$.

Finally, we see by inspection that $\operatorname{supp}(u) = \overline{B_1(0)}$, $\operatorname{supp}(\partial u/\partial \bar{z}) = \{0\} \cup \partial B_1(0)$ and $\operatorname{sing.supp}(u) = \{0\} \cup \partial B_1(0)$, $\operatorname{sing.supp}(\partial u/\partial \bar{z}) = \{0\} \cup \partial B_1(0)$. As a check we note that $\operatorname{sing.supp}(u) = \operatorname{sing.supp}(\partial u/\partial \bar{z})$, as it should be according to the elliptic regularity theorem since the Wirtinger differential operator $\partial/\partial \bar{z}$ is hypoelliptic.

2. (a) Let $\theta \in \mathscr{D}(\mathbb{R}^n)$ and denote as usual its L^1 dilation by factor $\varepsilon > 0$ as

$$\theta_\varepsilon(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon^n}\theta\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^n.$$

   (i) Prove that θ_ε has a limit in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

   (ii) Prove that for each $\phi \in \mathscr{D}(\mathbb{R}^n)$ we have $\theta_\varepsilon * \phi \to \int_{\mathbb{R}^n} \theta \,\mathrm{d}x\, \phi$ in $\mathscr{D}(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

   (iii) Prove that for each $\psi \in \mathrm{C}(\mathbb{R}^n)$ we have $\theta_\varepsilon * \psi \to \int_{\mathbb{R}^n} \theta \,\mathrm{d}x\, \psi$ locally uniformly on $\mathbb{R}^n$ as $\varepsilon \searrow 0$. What can you say about the partial derivatives $\partial^\alpha\big(\theta_\varepsilon * \psi\big)$ as $\varepsilon \searrow 0$?

   (b) Let $f\colon \mathbb{R} \to \mathbb{C}$ be locally integrable and T-periodic (that is, $f(x+T) = f(x)$ for all $x \in \mathbb{R}$). For a non-empty open interval $(a,b) \subseteq \mathbb{R}$ and natural numbers $j \in \mathbb{N}$ define $f_j(x) = f(jx)$, $x \in (a,b)$. Prove that

$$f_j \to \frac{1}{T}\int_0^T f \,\mathrm{d}x \quad \text{in } \mathscr{D}'(a,b) \text{ as } j \to \infty.$$

   [*This result is sometimes called the generalized Riemann-Lebesgue lemma.*]

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Solution: (a)(i): If $\phi \in \mathscr{D}(\mathbb{R}^n)$, then by the distributional definition of the L^1 dilation (so by change of variables) and continuity of ϕ at 0 we have as $\varepsilon \searrow 0$:

$$\langle \theta_\varepsilon, \phi \rangle = \int_{\mathbb{R}^n} \theta(x)\phi(\varepsilon x)\,\mathrm{d}x \to \int_{\mathbb{R}^n} \theta(x)\,\mathrm{d}x\,\phi(0).$$

Consequently, $\theta_\varepsilon \to \int_{\mathbb{R}^n} \theta\,\mathrm{d}x\,\delta_0$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \searrow 0$.

(ii): Let $\phi \in \mathscr{D}(\mathbb{R}^n)$. Then by the support rule $\mathrm{supp}(\theta_\varepsilon * \phi) \subseteq \mathrm{supp}(\theta_\varepsilon) + \mathrm{supp}(\phi) = \varepsilon\mathrm{supp}(\theta) + \mathrm{supp}(\phi)$. Take $r > 0$ such that $\mathrm{supp}(\theta) \subset B_r(0)$ and note that for $\varepsilon \in (0,1)$ we have $\mathrm{supp}(\theta_\varepsilon * \phi) \subseteq B_r(0) + \mathrm{supp}(\phi) \subset K$, where $K = \overline{B_r(\mathrm{supp}(\phi))}$ is a fixed compact set. By the differentiation rule we have for $\alpha \in \mathbb{N}_0^n$ that $\partial^\alpha\bigl(\theta_\varepsilon * \phi\bigr) = \theta_\varepsilon * (\partial^\alpha \phi)$ and since $\partial^\alpha \phi$ is uniformly continuous employing the argument of (i) above this converges uniformly to $\int_{\mathbb{R}^n} \theta\,\mathrm{d}x\,\partial^\alpha \phi$ on $\mathbb{R}^n$ as $\varepsilon \searrow 0$. But this is what we were required to prove.

(iii): Because ψ is continuous, and hence is uniformly continuous on compact sets, we have as $\varepsilon \searrow 0$,

$$\bigl(\theta_\varepsilon * \psi\bigr)(x) = \int_{\mathbb{R}^n} \theta(y)\psi(x - \varepsilon y)\,\mathrm{d}y \to \int_{\mathbb{R}^n} \theta\,\mathrm{d}y\,\psi(x)$$

locally uniformly in $x \in \mathbb{R}^n$. The derivatives $\partial^\alpha \psi$ must be understood as distributions here and so it follows from the differentiation rule and (i) above that, as $\varepsilon \searrow 0$,

$$\partial^\alpha\bigl(\theta_\varepsilon * \psi\bigr) = \theta_\varepsilon * \partial^\alpha \psi \to \int_{\mathbb{R}^n} \theta\,\mathrm{d}x\,\partial^\alpha \psi \quad \text{in } \mathscr{D}'(\mathbb{R}^n).$$

(b): Put $A = \frac{1}{T}\int_0^T f(t)\,\mathrm{d}t$ and $F(x) = \int_0^x \bigl(f(t) - A\bigr)\,\mathrm{d}t$, $x \in \mathbb{R}$. Then by results from lectures F is locally absolutely continuous with $F' = f - A$ in $\mathscr{D}'(\mathbb{R})$. Since $F(x+T) - F(x) = \int_x^{x+T}\bigl(f(t) - A\bigr)\,\mathrm{d}t = 0$ for all $x \in \mathbb{R}$, where the last equality follows because f is T-periodic, also F is T-periodic. In particular, F is then a bounded function on $\mathbb{R}$. Define $F_j(x) = F(jx)$, $x \in (a,b)$. Then F_j is absolutely continuous and by inspection, $F_j' = j(f_j - A)$ in $\mathscr{D}'(a,b)$. Now for $\phi \in \mathscr{D}(a,b)$ we get

$$\langle f_j - A, \phi \rangle = \left\langle \frac{1}{j}F_j', \phi \right\rangle = -\frac{1}{j}\langle F_j, \phi' \rangle = -\frac{1}{j}\int_a^b F(jx)\phi'(x)\,\mathrm{d}x,$$

and since

$$\frac{1}{j}\left|\int_a^b F(jx)\phi'(x)\,\mathrm{d}x\right| \leq \frac{\sup_{t\in\mathbb{R}}|F(t)|}{j}\int_a^b |\phi'(x)|\,\mathrm{d}x \to 0 \text{ as } j \to \infty,$$

we are done.

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Section B

3. *Distributions defined by finite parts.*

On Problem Sheet 2 it was shown that the distributional derivative of $\log|x|$ is the distribution $\mathrm{pv}\left(\frac{1}{x}\right)$ defined by the principal value integral

$$\left\langle \mathrm{pv}\left(\tfrac{1}{x}\right), \phi \right\rangle \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\phi(x)}{x} \, \mathrm{d}x \,, \quad \phi \in \mathscr{D}(\mathbb{R}).$$

In order to represent the higher order derivatives one can use finite parts: Let $n \in \mathbb{N}$ with $n > 1$. We then define $\mathrm{fp}\left(\frac{1}{x^n}\right)$ for each $\phi \in \mathscr{D}(\mathbb{R})$ by the *finite part integral*

$$\left\langle \mathrm{fp}\left(\tfrac{1}{x^n}\right), \phi \right\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\phi(x) - \sum_{j=0}^{n-2} \frac{\phi^{(j)}(0)}{j!} x^j - \frac{\phi^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)}{x^n} \, \mathrm{d}x.$$

(a) Check that hereby $\mathrm{fp}\left(\frac{1}{x^n}\right)$ is a well-defined distribution on $\mathbb{R}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\left(\tfrac{1}{x}\right) = -\mathrm{fp}\left(\tfrac{1}{x^2}\right) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fp}\left(\tfrac{1}{x^n}\right) = -n\mathrm{fp}\left(\tfrac{1}{x^{n+1}}\right)$$

for all $n > 1$. Is $\mathrm{fp}\left(\frac{1}{x^n}\right)$ homogeneous? (See Problem Sheet 2 for the definition of homogeneity.) Determine the order, the support and the singular support of $\mathrm{fp}\left(\frac{1}{x^n}\right)$.

(b) Show that for $n > 1$ we have $x^n \mathrm{fp}\left(\frac{1}{x^n}\right) = 1$ and find the general solution to the equation $x^n u = 1$ in $\mathscr{D}'(\mathbb{R})$. What is the general solution to the equation $(x-a)^n v = 1$ in $\mathscr{D}'(\mathbb{R})$ when $a \in \mathbb{R} \setminus \{0\}$?

(c) Let $p(x) \in \mathbb{C}[x] \setminus \{0\}$ be a polynomial. Describe the general solution $w \in \mathscr{D}'(\mathbb{R})$ to the equation

$$p(x)w = 1 \text{ in } \mathscr{D}'(\mathbb{R}).$$

Solution: (a): Fix $\phi \in \mathscr{D}(\mathbb{R})$. To see that the definition yields a distribution note first that the function

$$\frac{\phi(x) - \sum_{j=0}^{n-2} \frac{\phi^{(j)}(0)}{j!} x^j}{x^n}$$

is continuous for $|x| \geq 1$ and that it is $\mathrm{O}(x^{-2})$ as $|x| \to +\infty$. It is therefore integrable over $\mathbb{R} \setminus (-1,1)$, and we record the bound

$$\left| \int_{\mathbb{R}\setminus(-1,1)} \frac{\phi(x) - \sum_{j=0}^{n-2} \frac{\phi^{(j)}(0)}{j!} x^j}{x^n} \, \mathrm{d}x \right| \leq c_1 \max\left\{ |\phi^{(j)}(t)| : t \in \mathbb{R}, \, 0 \leq j \leq n \right\}, \qquad (1)$$

where (for instance) $c_1 = 10$ will do. Next, for $|x| < 1$ we have by Taylor's formula with Lagrange remainder term that for some $\xi = \xi_x$ between 0 and x that

$$\frac{\phi(x) - \sum_{j=0}^{n-1} \frac{\phi^{(j)}(0)}{j!} x^j}{x^n} = \frac{\phi^{(n)}(\xi)}{n!}.$$

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Therefore the function on the left-hand side is uniformly continuous on $(-1,0)\cup(0,1)$, and so integrable over $(-1,1)$. We record a corresponding bound:

$$\left|\int_{(-1,1)} \frac{\phi(x)-\sum_{j=0}^{n-1}\frac{\phi^{(j)}(0)}{j!}x^j}{x^n}\,\mathrm{d}x\right| \leq \max|\phi^{(n)}|. \tag{2}$$

Consequently, by linearity of the integral, $\mathrm{fp}\big(\frac{1}{x^n}\big)\colon \mathscr{D}(\mathbb{R})\to\mathbb{C}$ is a well-defined linear functional. Combining (1) and (2) we get the bound

$$\left|\left\langle \mathrm{fp}\big(\frac{1}{x^n}\big),\phi\right\rangle\right| \leq c\max\big\{|\phi^{(j)}(t)|\,:\,t\in\mathbb{R},\,0\leq j\leq n\big\} \tag{3}$$

valid for all $\phi\in\mathscr{D}(\mathbb{R})$, where (for instance) $c=1+c_1$ will do. Note that (3) in particular implies the $\mathscr{D}$-continuity of $\mathrm{fp}\big(\frac{1}{x^n}\big)$ proving that it is a distribution on $\mathbb{R}$.

Before we calculate the distributional derivative it is convenient to denote the k-th Taylor polynomial for $\phi\in\mathscr{D}(\mathbb{R})$ about 0 as

$$T_\phi^k(x)\equiv\sum_{j=0}^{k}\frac{\phi^{(j)}(0)}{j!}x^j$$

and note that because $1/x$ is an odd function,

$$\left\langle \mathrm{fp}\left(\frac{1}{x^n}\right),\phi\right\rangle = \lim_{r\searrow 0}\left(\int_{-\frac{1}{r}}^{-r}+\int_r^{\frac{1}{r}}\right)\big(\phi(x)-T_\phi^{n-1}(x)\big)\,\frac{\mathrm{d}x}{x^n}.$$

This also holds for $n=1$ and so unifies the notation (so for $n=1$ we have pv = fp). Observe $\phi'(x)-T_{\phi'}^{n-1}(x)=\big(\phi(x)-T_\phi^n(x)\big)'$, so

$$\begin{aligned}\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fp}\left(\frac{1}{x^n}\right),\phi\right\rangle &= -\left\langle \mathrm{fp}\left(\frac{1}{x^n}\right),\phi'\right\rangle\\ &= -\lim_{r\searrow 0}\left(\int_{-\frac{1}{r}}^{-r}+\int_r^{\frac{1}{r}}\right)\big(\phi(x)-T_\phi^n(x)\big)'\,\frac{\mathrm{d}x}{x^n}.\end{aligned}$$

Now for $r>0$ so small that ϕ is supported in $(-1/r,1/r)$ we have by partial integration

$$\begin{aligned}\left(\int_{-\frac{1}{r}}^{-r}+\int_r^{\frac{1}{r}}\right)\big(\phi(x)-T_\phi^n(x)\big)'\,\frac{\mathrm{d}x}{x^n} &= \frac{\phi(-r)-T_\phi^n(-r)}{(-r)^n}+T_\phi^n(-\tfrac{1}{r})(-r)^n-T_\phi^n(\tfrac{1}{r})r^n\\ &\quad-\frac{\phi(r)-T_\phi^n(r)}{r^n}\\ &\quad+n\left(\int_{-\frac{1}{r}}^{-r}+\int_r^{\frac{1}{r}}\right)\big(\phi(x)-T_\phi^n(x)\big)\,\frac{\mathrm{d}x}{x^{n+1}},\end{aligned}$$

and therefore

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fp}\left(\frac{1}{x^n}\right),\phi\right\rangle = -n\left\langle \mathrm{fp}\left(\frac{1}{x^{n+1}}\right),\phi\right\rangle,$$

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as required.

The distribution $\mathrm{fp}\big(\frac{1}{x^n}\big)$ is homogeneous of degree $-n$: This follows for instance by direct verification (showing and exploiting that $T^{n-1}_{\phi_r} = (T^{n-1}_{\phi})_r$) or by use of results established on Problem Sheet 2: it was shown there that $\mathrm{pv}\big(\frac{1}{x}\big)$ is homogeneous of degree -1 and that the derivative of a β-homogeneous distribution is $(\beta-1)$-homogeneous. The result then follows from the above identification of derivatives.

Note that $1/x^n$ is a regular distribution on $\mathbb{R}\setminus\{0\}$ and that $\mathrm{fp}\big(\frac{1}{x^n}\big)|_{\mathbb{R}\setminus\{0\}} = \frac{1}{x^n}$. Therefore the support of $\mathrm{fp}\big(\frac{1}{x^n}\big)$ is $\mathbb{R}$ (closure of $\mathbb{R}\setminus\{0\}$) and its singular support must be contained in $\{0\}$. The singular support must be $\{0\}$ because if it was not, it would be empty and the distribution would then have a C^∞ function as representative on $\mathbb{R}$. This function would have to equal $1/x^n$ on $\mathbb{R}\setminus\{0\}$, preventing it from being a C^∞ function on $\mathbb{R}$.

The bound (3) implies that $\mathrm{fp}\big(\frac{1}{x^n}\big)$ has order at most n. To see that the order is n we recall from Problem Sheet 2 that $\mathrm{pv}\big(\frac{1}{x}\big)$ has order 1, and so using the identity $\mathrm{fp}\big(\frac{1}{x^n}\big) = c_n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{pv}\big(\frac{1}{x}\big)$, where $c_n = \frac{(-1)^{n-1}}{(n-1)!}$, we infer that the order is n. (It is acceptable if the students assert this as I have mentioned it, though not proved it, in lectures.)

Lemma: *If $u \in \mathscr{D}'(\mathbb{R})$ has order $k \in \mathbb{N}$, then u' has order $k+1$.* (Note this is false if $k=0$.)

Proof. It is easy to see that u' has order at most $k+1$. To see that the order is $k+1$ assume for a contradiction that the order is at most k. Then for any $R>0$ we find a constant $c = c_R \geq 0$ such that

$$\big|\langle u', \phi\rangle\big| \leq c\sum_{j=0}^{k} \max|\phi^{(j)}| \tag{4}$$

holds for all $\phi \in \mathscr{D}(\mathbb{R})$ with support in $(-R,R)$. Take $\chi \in \mathscr{D}(-R,R)$ with $\int_{\mathbb{R}}\chi\,\mathrm{d}x = 1$. For $\phi \in \mathscr{D}(-R,R)$ put $\varphi = \phi - c\chi$ with $c = \int_{\mathbb{R}}\phi\,\mathrm{d}x$. Then $\varphi \in \mathscr{D}(-R,R)$ and $\int_{\mathbb{R}}\varphi\,\mathrm{d}x = 0$ and so $\psi(x) = \int_{-R}^{x}\varphi(t)\,\mathrm{d}t$, $x \in \mathbb{R}$, belongs to $\mathscr{D}(-R,R)$. Plugging it into (4) yields, after rearranging terms, the bound

$$\big|\langle u, \phi\rangle\big| \leq C\sum_{j=0}^{k-1} \max|\phi^{(j)}|$$

for some constant $C = C_R$. But this is contradicting the assumption that u has order k. □

(b): For $\phi \in \mathscr{D}(\mathbb{R})$ we calculate

$$\begin{aligned}\left\langle x^n \mathrm{fp}\left(\frac{1}{x^n}\right), \phi\right\rangle &= \left\langle \mathrm{fp}\left(\frac{1}{x^n}\right), x^n\phi\right\rangle \\ &= \lim_{r\searrow 0}\left(\int_{-\frac{1}{r}}^{-r} + \int_r^{\frac{1}{r}}\right)\left(x^n\phi(x) - T^{n-1}_{x^n\phi}(x)\right)\frac{\mathrm{d}x}{x^n} \\ &= \lim_{r\searrow 0}\left(\int_{-\frac{1}{r}}^{-r} + \int_r^{\frac{1}{r}}\right)\phi(x)\,\mathrm{d}x = \int_{\mathbb{R}}\phi(x)\,\mathrm{d}x,\end{aligned}$$

as required. This also yields a particular solution to the equation $x^n u = 1$. To find the general solution to the homogeneous equation $x^n u = 0$ we note that any such u must be supported in $\{0\}$, and so, by a result from lectures, be a linear combination of δ_0 and its derivatives: $u = \sum_{j=0}^{J} c_j \delta_0^{(j)}$ for some $J \in \mathbb{N}_0$ and $c_j \in \mathbb{C}$. Plugging in this u we see that when $J = n-1$, then we are free to choose the coefficients $c_j \in \mathbb{C}$. To see that these are all the solutions we use special test functions. Indeed if it is a solution then we get for any $\phi \in \mathscr{D}(\mathbb{R})$ that for $J \geq n$:

$$0 = \sum_{j=0}^{J} c_j \langle \delta_0^{(j)}, x^n\phi\rangle = \sum_{j=0}^{J} c_j \sum_{k=0}^{j}\binom{j}{k}\frac{\mathrm{d}^k}{\mathrm{d}x^k}|_{x=0}\left(x^n\right)\phi^{(j-k)}(0) = \sum_{j=n}^{J} c_j \binom{j}{n} n!\phi^{(j-n)}(0).$$

For $s \in \{0, \ldots, J-n\}$ we select $\phi \in \mathscr{D}(\mathbb{R})$ with $\phi^{(j)}(0) = \delta_{j,s}$ for all $j \in \mathbb{N}_0$ (see Problem Sheet 1 for a possible construction of such ϕ). We now have that the terms in the above sum are 0 unless $j - n = s$ and so we get $0 = c_{n+s}\binom{n+s}{n}n!$, that is, $c_j = 0$ for all $j \geq n$. Therefore GS is $u = \mathrm{fp}\left(\frac{1}{x^n}\right) + \sum_{j=0}^{n-1} c_j \delta_0^{(j)}$, where $c_1, \ldots, c_{n-1} \in \mathbb{C}$. By inspection we see that GS to the equation $(x-a)^n v = 1$ is $v = \mathrm{fp}\left(\frac{1}{(x-a)^n}\right) + \sum_{j=0}^{n-1} c_j \delta_a^{(j)}$, where $c_1, \ldots, c_{n-1} \in \mathbb{C}$.

(c): We find the GS as $w_H + w_{PS}$, where w_H is a general solution to the homogeneous equation and w_{PS} is any solution to the inhomogeneous equation. Suppose $a_1, \ldots, a_n \in \mathbb{R}$ are all the distinct real roots of $p(x)$ and let $m_j \in \mathbb{N}$ be the multiplicity of a_j. Then $p(x) = q(x)\prod_{j=1}^{n}(x-a_j)^{m_j}$, where $q(x)$ is a polynomial without real roots. We first find w_H and note that since $1/q(x) \in \mathrm{C}^\infty(\mathbb{R})$ this is GS to $\left(\prod_{j=1}^{n}(x-a_j)^{m_j}\right)w = 0$. Using localization and a result from (b) we find

$$w_H = \sum_{k=1}^{n}\sum_{j=0}^{m_k-1} c_{k,j}\delta_{a_k}^{(j)},$$

where $c_{k,j} \in \mathbb{C}$ are arbitrary. To find w_{PS} we expand in partial fractions:

$$\frac{1}{p(x)} = \frac{1}{q(x)}\sum_{k=1}^{n}\sum_{j=1}^{m_k}\frac{b_{k,j}}{(x-a_k)^j},$$

and so using localization and a result from (b) we get

$$w_{PS} = \frac{1}{q(x)}\sum_{k=1}^{n}\sum_{j=1}^{m_k}\mathrm{fp}\left(\frac{b_{k,j}}{(x-a_k)^j}\right).$$

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4. (a) Let $n \in \mathbb{N}$. Calculate the limits

$$\big(x+\mathrm{i}0\big)^{-n} \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0}\big(x+\mathrm{i}\varepsilon\big)^{-n} \quad \text{and} \quad \big(x-\mathrm{i}0\big)^{-n} \stackrel{\text{def}}{=} \lim_{\varepsilon \searrow 0}\big(x-\mathrm{i}\varepsilon\big)^{-n}$$

in $\mathscr{D}'(\mathbb{R})$. Prove the *Plemelj-Sokhotsky jump relations*:

$$(x+\mathrm{i}0)^{-n}-(x-\mathrm{i}0)^{-n} = 2\pi\mathrm{i}\frac{(-1)^n}{(n-1)!}\delta_0^{(n-1)},$$

where δ_0 is Dirac's delta-function on $\mathbb{R}$ concentrated at 0.

(b) Show that for $\phi \in \mathscr{D}(\mathbb{R})$ with $\phi^{(j)}(0)=0$ for each $j \in \{0, \ldots, n\}$ we have

$$\big\langle \big(x \pm \mathrm{i}0\big)^{-n}, \phi \big\rangle = \int_{-\infty}^{\infty} \frac{\phi(x)}{x^n}\,\mathrm{d}x.$$

(c) Show that

$$x\big(x \pm \mathrm{i}0\big)^{-1} = 1 \quad \text{in } \mathscr{D}'(\mathbb{R}).$$

Deduce that

$$\big(x \pm \mathrm{i}0\big)^{-1}\big(x\delta_0\big) = 0 \neq \delta_0 = \Big(\big(x \pm \mathrm{i}0\big)^{-1}x\Big)\delta_0.$$

Next, show that

$$\big(x \pm \mathrm{i}0\big)^{-n}x^n = 1 \quad \text{in } \mathscr{D}'(\mathbb{R})$$

holds for each $n \in \mathbb{N}$

(d) Find distributions $u_+,\ u_- \in \mathscr{D}'(\mathbb{R})$ such that their n-th distributional derivatives satisfy

$$u_+^{(n)} = \big(x+\mathrm{i}0\big)^{-n} \quad \text{and} \quad u_-^{(n)} = \big(x-\mathrm{i}0\big)^{-n}.$$

Solution: (a): First note that $\big(x \pm \mathrm{i}\varepsilon\big)^{-n} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, so for $\phi \in \mathscr{D}(\mathbb{R})$ we calculate by $n-1$ successive integrations by parts:

$$\big\langle \big(x \pm \mathrm{i}\varepsilon\big)^{-n}, \phi \big\rangle = \frac{1}{(n-1)!}\int_{\mathbb{R}} \phi^{(n-1)}(x)\frac{\mathrm{d}x}{x \pm \mathrm{i}\varepsilon}.$$

Now a primitive for $1/(x \pm \mathrm{i}\varepsilon)$ is the principal logarithm $\mathrm{Log}(x \pm \mathrm{i}\varepsilon)$, and we record that, as $\varepsilon \searrow 0$,

$$\mathrm{Log}(x \pm \mathrm{i}\varepsilon) = \log(x^2+\varepsilon^2)^{\frac{1}{2}} + \mathrm{i}\mathrm{Arg}(x \pm \mathrm{i}\varepsilon) \to \log|x| \pm \pi\mathrm{i}H(-x)$$

pointwise in $x \in \mathbb{R} \setminus \{0\}$. Here H denotes Heaviside's function. Therefore another integration by parts and then Lebesgue's DCT yields as $\varepsilon \searrow 0$ that

$$\big\langle \big(x \pm \mathrm{i}\varepsilon\big)^{-n}, \phi \big\rangle \to \frac{-1}{(n-1)!}\int_{\mathbb{R}} \phi^{(n)}(x)\big(\log|x| \pm \pi\mathrm{i}H(-x)\big)\,\mathrm{d}x.$$

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It is clear that the right-hand sides define distributions on $\mathbb{R}$. Their difference is

$$\left\langle \bigl(x+\mathrm{i}\varepsilon\bigr)^{-n} - \bigl(x-\mathrm{i}\varepsilon\bigr)^{-n}, \phi \right\rangle = -\frac{2\pi\mathrm{i}}{(n-1)!}\int_{-\infty}^{0} \phi^{(n)}(x)\,\mathrm{d}x = -\frac{2\pi\mathrm{i}}{(n-1)!}\phi^{(n-1)}(0),$$

and since $\phi^{(n-1)}(0) = (-1)^{n-1}\langle \delta_0^{(n-1)}, \phi\rangle$ this is the required jump relation.

(b): If $\phi^{(j)}(0) = 0$ for $0 \leq j \leq n$, then $x^{-n}\phi(x) \to 0$ as $x \to 0$, and so $x^{-n}\phi(x)$ is integrable on $\mathbb{R}$. Because $\left|\bigl(x \pm \mathrm{i}\varepsilon\bigr)^{-n}\phi(x)\right| \leq |x^{-n}\phi(x)|$ for all $x \in \mathbb{R}\setminus\{0\}$ we get by Lebesgue's DCT that

$$\left\langle \bigl(x \pm \mathrm{i}0\bigr)^{-n}, \phi \right\rangle = \int_{\mathbb{R}} x^{-n}\phi(x)\,\mathrm{d}x.$$

(c): First we note that if $\phi \in \mathscr{D}(\mathbb{R})$, then $x\phi(x)$ is a test function that vanishes at 0 and so by (b) we get $\langle x\bigl(x \pm \mathrm{i}0\bigr)^{-1}, \phi\rangle = \langle \bigl(x \pm \mathrm{i}0\bigr)^{-1}, x\phi\rangle = \int_{\mathbb{R}}\phi\,\mathrm{d}x$, thus $x\bigl(x \pm \mathrm{i}0\bigr)^{-1} = 1$. The deduction is then clear.

For $n \in \mathbb{N}$ and $\phi \in \mathscr{D}(\mathbb{R})$ the function $x^n\phi$ is a test function that vanishes to order n at 0, and so we may use (b) as above to see that $x^n\bigl(x \pm \mathrm{i}0\bigr)^{-n} = 1$.

(d): This calculation was essentially done in our solution of (a) above:

$$u_{\pm} = \frac{(-1)^{n-1}}{(n-1)!}\bigl(\log|x| \pm \pi\mathrm{i}H(-x)\bigr)$$

is the limit in $\mathscr{D}'(\mathbb{R})$ of $\frac{(-1)^{n-1}}{(n-1)!}\mathrm{Log}(x \pm \mathrm{i}\varepsilon)$ as $\varepsilon \searrow 0$. The latter is the n-th primitive of $\bigl(x \pm \mathrm{i}\varepsilon\bigr)^{-n}$. By continuity of differentiation in $\mathscr{D}'(\mathbb{R})$ we therefore see that $u_{\pm}^{(n)} = \bigl(x \pm \mathrm{i}0\bigr)^{-n}$.

5. A real-valued distribution u on $\mathbb{R}^2$ is called *subharmonic* provided $\Delta u \geq 0$ in $\mathscr{D}'(\mathbb{R}^2)$. In the following we identify $z \in \mathbb{C}$ in the usual way with $(x,y) \in \mathbb{R}^2$ and we assume $f\colon \mathbb{C} \to \mathbb{C}$ is an entire function that is not identically zero. Define

$$\langle u, \phi\rangle \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2} \phi(x,y)\log|f(z)|\,\mathrm{d}(x,y), \quad \phi \in \mathscr{D}(\mathbb{R}^2).$$

Show that u is a well-defined and subharmonic distribution on $\mathbb{R}^2$.

What happens above if $f\colon \mathbb{C} \to \mathbb{C}\cup\{\infty\}$ is allowed to be meromorphic?

Solution: First we note that the zero set for f, $Z = \bigl\{z \in \mathbb{C} : f(z) = 0\bigr\}$, cannot have limit points in $\mathbb{C}$ since otherwise $f \equiv 0$ by the identity theorem. The set Z is therefore locally finite (so for each $z_0 \in \mathbb{C}$ and $r > 0$ the intersection $B_r(z_0) \cap Z$ is a finite set). It is then in particular at most countable and so can be enumerated, say $Z = \bigl\{z_j : j \in J\bigr\}$, where J is at most countable. We also record that for each $w \in \mathbb{C}$ we can select $r > 0$

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such that $B_{2r}(w)$ contains at most one zero for f. When $B_{2r}(w) \cap Z = \emptyset$ the function $\log|f(z)|$ is C^∞ on $B_{2r}(w)$, and so is in particular integrable on $B_r(w)$. If $w = z_j \in Z$ choose $r > 0$ such that $Z \cap B_{2r}(z_j) = \{z_j\}$ and by applying Taylor's theorem to f about z_j we get $f(z) = \sum_{s=m_j}^{\infty} \frac{f^{(s)}(z_j)}{j!}(z-z_j)^s$ on $\mathbb{C}$. Defining $g(z) = \sum_{s=m_j}^{\infty} \frac{f^{(s)}(z_j)}{j!}(z-z_j)^{s-m_j}$ we have that $g\colon \mathbb{C} \to \mathbb{C}$ is an entire function that has no zeroes on $B_{2r}(z_j)$ and $f(z) = (z-z_j)^{m_j} g(z)$ on $\mathbb{C}$. Thus $\log|f(z)| = m_j \log|z-z_j| + \log|g(z)|$ and the second term is integrable on $B_r(z_j)$. The first term is also seen to be integrable on $B_r(z_j)$ (for instance we may check this by integrating in polar coordinates about z_j). Thus $\log|f(z)|$ is locally integrable, and so defines a regular distribution u on $\mathbb{C}$. By the above we see that the restriction of u to the open set $\mathbb{C} \setminus Z$ can be represented by a C^∞ function, and so we may calculate on $\mathbb{C} \setminus Z$:

$$\begin{aligned}
\Delta u &= 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\log|f(z)|\right) \\
&= 2\frac{\partial}{\partial \bar{z}}\left(\frac{1}{|f(z)|^2}\left(\frac{\partial f(z)}{\partial z}\overline{f(z)} + f(z)\frac{\partial \overline{f(z)}}{\partial z}\right)\right) \\
&= 2\frac{\partial}{\partial \bar{z}}\left(\frac{f'(z)}{f(z)} + 0\right) = 0,
\end{aligned}$$

where we applied the Cauchy-Riemann equations twice. If $w = z_j \in Z$ we select $r > 0$ as above such that $B_{2r}(z_j) \cap Z = \{z_j\}$ and $u = m_j \log|\cdot - z_j| + \log|g|$ on $B_r(z_j)$. Using results from Problem Sheet 3 and lectures we then calculate in the sense of distributions on $B_r(z_j)$:

$$\Delta u = m_j \Delta \log|z-z_j| = 2m_j \frac{\partial}{\partial \bar{z}}\left(\frac{1}{z-z_j}\right) = 2\pi m_j \delta_{z_j}.$$

It follows by localization that

$$\Delta u = \sum_{j \in J} 2\pi m_j \delta_{z_j} \ \text{ in } \ \mathscr{D}'(\mathbb{C}),$$

and since each $m_j \in \mathbb{N}$, $\Delta u \geq 0$, so that u is subharmonic. By *localization* we intend: given $\phi \in \mathscr{D}(\mathbb{C})$ we cover $\mathrm{supp}(\phi)$ by a finite number of balls as above, say $\mathrm{supp}(\phi) \subset \bigcup_{k=1}^{K} B_{r_k}(w_k)$, and then we select a smooth partition of unity η_k, $k \in K$, that is subordinated the open cover $\{B_{r_k}(w_k)\}_{k \in K}$. We now have $\langle \Delta u, \phi \rangle = \sum_{k=1}^{K} \langle \Delta u, \eta_k \phi \rangle$. Here each term in the sum is covered by the above calculation, namely $\langle \Delta u, \eta_k \phi \rangle = 0$ when $B_{2r_k}(w_k) \cap Z = \emptyset$ and $\langle \Delta u, \eta_k \phi \rangle = 2\pi m_j (\eta_k \phi)(z_j)$ when $B_{2r_k}(w_k) \cap Z = \{z_j\}$. The result then follows as asserted.

When $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is meromorphic we allow a set P of isolated poles for f to exist. The set P will then be locally finite, so in particular at most countable and so can be enumerated, say $P = \{p_k : k \in K\}$. Denote the order of the pole p_k by $n_k \in \mathbb{N}$. Clearly $Z \cap P = \emptyset$. As before we have if $w \in \mathbb{C} \setminus (Z \cup P)$ that u is C^∞ near w and if $w \in Z$

we take $r > 0$ such that $B_{2r}(w) \cap (Z \cup P) = \{w\}$ so u is again integrable on $B_r(w)$. If $w = p_k \in P$ we take $r > 0$ such that $B_{2r}(p_k) \cap (P \cup Z) = \{p_k\}$ and have then by Laurent's theorem $f(z) = \sum_{s=-n_k}^{\infty} c_s(z-p_k)^s$ for suitable $c_s \in \mathbb{C}$ and $c_{-n_k} \neq 0$. Write $g(z) = \sum_{s=0}^{\infty} c_{s-n_k}(z-p_k)^s$ and note that $g \colon B_{r_k}(p_k) \to \mathbb{C}$ is nonvanishing, holomorphic and $f(z) = (z-p_k)^{-n_k} g(z)$. We have therefore on $B_{r_k}(p_k)$ that

$$u = -n_k \log|z-p_k| + \log|g(z)|,$$

which shows that u is integrable on $B_{r_k}(p_k)$. Thus u is again a regular distribution on $\mathbb{C}$ and calculating as above we find that

$$\Delta u = \sum_{j\in J} 2\pi m_j \delta_{z_j} - \sum_{k \in K} 2\pi n_k \delta_{p_k}.$$

This is clearly not a positive distribution, so u is not subharmonic in this case.

6. (a) Let $A \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a bijective linear map and define for $\phi \in \mathscr{D}(\mathbb{R}^2)$ the function $A\phi \colon \mathbb{R}^2 \to \mathbb{C}$ by $(A\phi)(x,y) \stackrel{\text{def}}{=} \phi\big(A(x,y)\big)$, $(x,y) \in \mathbb{R}^2$.

   Prove that $A \colon \mathscr{D}(\mathbb{R}^2) \to \mathscr{D}(\mathbb{R}^2)$ is a well-defined, linear and $\mathscr{D}$-continuous map. How should we define Au for a distribution $u \in \mathscr{D}'(\mathbb{R}^2)$? Find a formula for $A^{-1}\Delta A u$, where Δ denotes the Laplacian on $\mathbb{R}^2$.

   (b) Let $p(\partial)$ be a differential operator on $\mathbb{R}^2$ with real coefficients and that is homogeneous of order 2:

   $$p(\partial) = \alpha\partial_x^2 + \beta\partial_x\partial_y + \gamma\partial_y^2,$$

   where α, β, $\gamma \in \mathbb{R}$ are not all zero. Prove that $p(\partial)$ is elliptic if and only if $\beta^2 < 4\alpha\gamma$. Next, show that if $p(\partial)$ is elliptic, then it is also hypoelliptic.

   (c) Let Ω be an open non-empty subset of $\mathbb{R}^2$ and assume that $u \in \mathscr{D}'(\Omega)$ satisfies

   $$\partial_x^2 u + \partial_x\partial_y u + \partial_y^2 u = f \quad \text{in } \mathscr{D}'(\Omega),$$

   where $f \in \mathscr{D}'(\Omega)$. Prove that if f is C^∞ on $B_r(x_0,y_0) \subset \Omega$, then so is u.

   Optional: What can you say about u if f is $\mathrm{L}^2_{\mathrm{loc}}$ on $B_r(x_0,y_0) \subset \Omega$?

Solution: (a): Put $A(x,y) = (ax+by, cx+dy)$, where a, b, c, $d \in \mathbb{R}$ and $ad-bc \neq 0$. Then by iterative use of the chain rule it follows that for $\phi \in \mathscr{D}(\mathbb{R}^2)$ the function $A\phi(x,y) = \phi(ax+by, cx+dy)$ is in $\mathrm{C}^\infty(\mathbb{R}^2)$ and since $\mathrm{supp}(A\phi) = A^{-1}\mathrm{supp}(\phi)$ is compact, we have $A\phi \in \mathscr{D}(\mathbb{R}^2)$. It is then clear that $A \colon \mathscr{D}(\mathbb{R}^2) \to \mathscr{D}(\mathbb{R}^2)$ is a well-defined linear map. To show that it is $\mathscr{D}$-continuous it suffices by linearity to consider a null sequence. Let $\phi_j \in \mathscr{D}(\mathbb{R}^2)$ and suppose that $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^2)$. That is, for some

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compact set $K \subset \mathbb{R}^2$ we have $\mathrm{supp}(\phi_j) \subseteq K$ for all j, and $\partial^\alpha \phi_j \to 0$ uniformly on $\mathbb{R}^2$ for each multi-index α. Then $\mathrm{supp}(A\phi_j) \subseteq A^{-1}K$ for all j, where again $A^{-1}K$ is compact. For the uniform convergence of partial derivatives we use induction on the lenght of the multi-index α. First, we clearly have $A\phi_j \to 0$ uniformly, and for the induction step we use the formulas

$$\partial_x A\psi = aA(\partial_x \psi) + cA(\partial_y \psi) \text{ and } \partial_y A\psi = bA(\partial_x \psi) + dA(\partial_y \psi) \tag{5}$$

that are valid for any $\psi \in \mathrm{C}^1(\mathbb{R}^2)$. Assume we have established the uniform convergence to 0 for all partial derivatives of order less than $n \in \mathbb{N}$. Then for $\alpha \in \mathbb{N}_0^2$ of lenght n we put $\beta = \alpha - (1,0)$ if $\alpha_1 > 0$ (and $\alpha - (0,1)$ if $\alpha_1 = 0$). Assuming the former we then have $\partial^\alpha A\phi_j = \partial^\beta\big(aA(\partial_x\phi_j) + cA(\partial_y\phi_j)\big) \to 0$ uniformly by the induction hypothesis. The $\mathscr{D}$-continuity of A follows.

In order to extend the definition of A to distributions we use the adjoint identity scheme, and for that purpose we derive an adjoint identity. For ϕ, $\psi \in \mathscr{D}(\mathbb{R}^2)$ we calculate by the change-of-variables formula:

$$\int_{\mathbb{R}^2} A\phi\psi \,\mathrm{d}(x,y) = \int_{\mathbb{R}^2} \phi A^{-1}\psi |\det A| \,\mathrm{d}(x,y).$$

Here we note that the map $\mathscr{D}(\mathbb{R}^2) \ni \psi \mapsto A^{-1}\psi|\det A| \in \mathscr{D}(\mathbb{R}^2)$ is linear and $\mathscr{D}$-continuous. We must therefore for $u \in \mathscr{D}'(\mathbb{R}^2)$ define Au by the rule

$$\langle Au, \phi \rangle \stackrel{\text{def}}{=} \langle u, A^{-1}\phi|\det A| \rangle, \quad \phi \in \mathscr{D}(\mathbb{R}^2).$$

The adjoint identity scheme then ensures that hereby $A\colon \mathscr{D}'(\mathbb{R}^2) \to \mathscr{D}'(\mathbb{R}^2)$ is a well-defined, linear and $\mathscr{D}'$-continuous map.

By successive use of the formulas (5) we next calculate for $u \in \mathscr{D}'(\mathbb{R}^2)$ (first assume $u \in \mathrm{C}^\infty(\mathbb{R}^2)$ and then transfer the formula to general u by mollification):

$$\big(A^{-1}\Delta A\big)u = (a^2+b^2)\partial_x^2 u + 2(ac+bd)\partial_x\partial_y u + (c^2+d^2)\partial_y^2 u.$$

(b): $p(\partial)$ is elliptic iff $p(x,y) = \alpha x^2 + \beta xy + \gamma y^2 \neq 0$ for $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Consider the equation $\alpha x^2 + \beta xy + \gamma y^2 = 0$ and assume first that $\alpha \neq 0$. If there is a solution (x,y) with $y = 0$, then it can only be $(0,0)$. If (x,y) is a solution with $y \neq 0$, then we must have $\alpha\left(\frac{x}{y}\right)^2 + \beta\frac{x}{y} + \gamma = 0$. The solutions are real iff $\beta^2 - 4\alpha\gamma \geq 0$. It thus follows that when $\alpha \neq 0$, $p(\partial)$ is elliptic iff $\beta^2 < 4\alpha\gamma$ holds. When $\alpha = 0$ we have that $p(x,0) = 0$ for all $x \in \mathbb{R}$ and so in this case $p(\partial)$ is never elliptic, concluding the proof that we have ellipticity iff $\beta^2 < 4\alpha\gamma$ holds.

Assume that $p(\partial)$ is elliptic. To prove it is hypoelliptic, so that it admits a fundamental solution with singular support $\{(0,0)\}$ we aim to find an automorphism A of $\mathbb{R}^2$ such that $p(\partial) = A^{-1}\Delta A$ and use that $G(z) = \log|z|/2\pi$ is a fundamental solution for Δ.

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By above we must have $\beta^2 < 4\alpha\gamma$, and so in particular that α, γ have the same sign. Because also $-p(\partial)$ is elliptic we can without loss in generality assume that $\alpha > 0$, $\gamma > 0$. For the following calculation it is convenient to introduce $\vec{v} = \sqrt{\alpha}(\cos\theta_1, \sin\theta_1)$ and $\vec{w} = \sqrt{\gamma}(\cos\theta_2, \sin\theta_2)$ for some angles θ_1, $\theta_2 \in (-\pi, \pi]$ still to be determined. Note that $\vec{v}\cdot\vec{w} = \sqrt{\alpha\gamma}\cos(\theta_1 - \theta_2)$ and if $\underline{A}$ denotes the square matrix whose first and second row is $\vec{v}$ and $\vec{w}$, respectively, then $\det\underline{A} = \sqrt{\alpha\gamma}\sin(\theta_1 - \theta_2)$. Thus $\underline{A}$ is regular iff $\theta_1 - \theta_2 \notin \pi\mathbb{Z}$, and so in this case the corresponding linear map A is an automorphism of $\mathbb{R}^2$. We may then write $A^{-1}\Delta A = \alpha\partial_x^2 + 2\sqrt{\alpha\gamma}\cos(\theta_1 - \theta_2)\partial_x\partial_y + \gamma\partial_y^2$. Because $\beta^2 < 4\alpha\gamma$, that is, $|\beta| < 2\sqrt{\alpha\gamma}$, we can solve $\cos(\theta_1 - \theta_2) = \beta/2\sqrt{\alpha\gamma} \in (-1, 1)$ for $\theta_1 - \theta_2 \in \mathbb{R}\setminus\pi\mathbb{Z}$. Take (for instance) $\theta_2 = 0$ and $\theta_1 \in (-\pi, 0)\cup(0, \pi)$ solving the equation and let A denote the corresponding automorphism. Then $p(\partial) = A^{-1}\Delta A$, or $\Delta = Ap(\partial)A^{-1}$, hence we have $\delta_0 = \Delta G = Ap(\partial)A^{-1}G$ and so $p(\partial)A^{-1}G = A^{-1}\delta_0$. Using the definition we see that $A^{-1}\delta_0 = \delta_0|\det A|$ and thus if we define

$$E(x, y) = \frac{1}{2\pi|\det A|}\log|A^{-1}(x, y)|,$$

then E is a regular distribution on $\mathbb{R}^2$ with singular support $\{(0, 0)\}$ and by construction $p(\partial)E = \delta_0$, as required.

(c): This follows easily from (b) and the elliptic regularity theorem from lectures. Indeed the differential operator $\partial_x^2 + \partial_x\partial_y + \partial_y^2$ is elliptic because $\alpha = \beta = \gamma = 1$, so evidently $\beta^2 = 1 < 4 = 4\alpha\gamma$. It is then also hypoelliptic and so if the right-hand side f is C^∞ on the ball $B \subset \Omega$, then so is the solution u.

Optional part: Assume that $f|_{B_r(x_0,y_0)} \in \mathrm{L}^2_{\mathrm{loc}}(B_r(x_0, y_0))$. Fix a ball $B \Subset B_r(x_0, y_0)$ and denote

$$f_B = \begin{cases} f & \text{in } B, \\ 0 & \text{in } \mathbb{R}^2\setminus B. \end{cases}$$

Then $f_B \in \mathrm{L}^2(\mathbb{R}^2)$ has compact support and we may define $v = E * f_B$. By the differentiation rule for convolutions, $p(\partial)v = \big(p(\partial)E\big) * f_B = f_B$, and so $p(\partial)(u - v) = 0$ on B, so that by hypoellipticity, $u - v$ is C^∞ on B, and so u is locally as regular as v is on B. Because the coefficients are real we may assume that f is real-valued as otherwise we can consider separately the PDEs with the real and imaginary parts of f on the right-hand side. First, we check that $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$: for $R > 0$ and writing $B_R = B_R(0)$, $z = x + \mathrm{i}y$ and $w = w_1 + \mathrm{i}w_2$ we have

$$\begin{aligned}\int_{B_R} v^2\,\mathrm{d}(x, y) &\overset{\text{Cauchy-Schwarz}}{\leq} \int_{B_R}\int_B E(z - w)^2\,\mathrm{d}(w_1, w_2)\|f_B\|_2^2\,\mathrm{d}(x, y) \\ &\leq \pi R^2\int_{B_R - B} E(w)^2\,\mathrm{d}(w_1, w_2)\,\|f_B\|_2^2 < +\infty\end{aligned}$$

For the higher order regularity it is convenient to transform the equation back to the Laplacian (though it is not necessary). We have $f_B = p(\partial)v = A^{-1}\Delta Av$ on $\mathbb{R}^2$, and so

$g = \Delta w$ on $\mathbb{R}^2$, where $g = Af_B$, $w = Av$. Clearly the regularity of w and v are the same on corresponding sets and $g \in \mathrm{L}^2(\mathbb{R}^2)$. We have established that $w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$. We assert that $w \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^2)$, and consequently that u is $\mathrm{W}^{2,2}_{\mathrm{loc}}$ on $B_r(x_0, y_0)$. We of course have the formula $w = G * g$, but it seems easier to use $w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\Delta w = g$.

Put $w_\varepsilon = \rho_\varepsilon * w$, $g_\varepsilon = \rho_\varepsilon * g$ and note that $\Delta w_\varepsilon = g_\varepsilon$. For $\phi \in \mathscr{D}(\mathbb{R}^2)$ we have $\int_{\mathbb{R}^2} g_\varepsilon \phi = \int_{\mathbb{R}^2} \Delta w_\varepsilon \phi = -\int_{\mathbb{R}^2} \nabla w_\varepsilon \cdot \nabla \phi$. Fix two concentric balls $B' \Subset B''$ and take a cut-off function $\eta \in \mathscr{D}(\mathbb{R}^2)$ satisfying $\mathbf{1}_{B'} \leq \eta \leq \mathbf{1}_{B''}$. Test $\Delta W = G$ with $\phi = \eta^2 W$, where $W = w_{\varepsilon_1} - w_{\varepsilon_2}$ and $G = g_{\varepsilon_1} - g_{\varepsilon_2}$ to get

$$\int_{\mathbb{R}^2} G\eta^2 W = -\int_{\mathbb{R}^2} \nabla W \cdot \nabla(\eta^2 W) = -\int_{\mathbb{R}^2} \bigl(\eta^2 |\nabla W|^2 + 2\eta W \nabla W \cdot \nabla \eta\bigr),$$

and hence (using $2st \leq s^2/2 + 2t^2$ and $2st \leq s^2 + t^2$)

$$\begin{aligned}\int_{\mathbb{R}^2} \eta^2 |\nabla W|^2 &= \int_{\mathbb{R}^2} \bigl(2\eta W \nabla W \cdot \nabla \eta - \eta^2 W G\bigr)\\ &\leq \int_{\mathbb{R}^2} \bigl(\frac{1}{2}\eta^2 |\nabla W|^2 + 2W^2 |\nabla \eta|^2 + \frac{1}{2}\eta^2 W^2 + \frac{1}{2}\eta^2 G^2\bigr),\end{aligned}$$

and thus

$$\int_{B'} |\nabla W|^2 \leq c \int_{B''} \bigl(W^2 + G^2\bigr),$$

where $c = 1 + 4\max|\nabla \eta|^2$ will do. Thus we have shown

$$\int_{B'} |\nabla w_{\varepsilon_1} - \nabla w_{\varepsilon_2}|^2 \leq c \int_{B''} \bigl(|w_{\varepsilon_1} - w_{\varepsilon_2}|^2 + |g_{\varepsilon_1} - g_{\varepsilon_2}|^2\bigr)$$

and since w, $g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$ it follows that the sequence of restrictions, $((\nabla w_\varepsilon)|_{B'})$ is Cauchy in $\mathrm{L}^2(B', \mathbb{R}^2)$ as $\varepsilon \searrow 0$, and so by completeness we find $\chi \in \mathrm{L}^2(B', \mathbb{R}^2)$ such that $(\nabla w_\varepsilon)|_{B'} \to \chi$ in $\mathrm{L}^2(B', \mathbb{R}^2)$. (Note: $\mathrm{L}^2_{\mathrm{loc}}(B', \mathbb{R}^2)$ denotes the space of $\mathbb{R}^2$-valued $\mathrm{L}^2_{\mathrm{loc}}$ functions on B' normed by using the usual norm on $\mathbb{R}^2$.) But we also have that $\nabla w_\varepsilon \to \nabla w$ in $\mathscr{D}'(\mathbb{R}^2, \mathbb{R}^2)$ and so we conclude that $(\nabla w)|_{B'} = \chi \in \mathrm{L}^2(B', \mathbb{R}^2)$. Because B' was an arbitrary ball we conclude that $\nabla w \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. Finally, for the second order derivatives we retain the above notation and calculate

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla^2(\eta W)|^2 &= \int_{\mathbb{R}^2} \Bigl(\Delta(\eta W)^2 + 2\bigl((\eta W)^2_{xy} - (\eta W)_{xx}(\eta W)_{yy}\bigr)\Bigr)\\ &= \int_{\mathbb{R}^2} \bigl(\eta \Delta W + 2\nabla \eta \cdot \nabla W + W \Delta \eta\bigr)^2\\ &\quad +2\int_{\mathbb{R}^2} \Bigl(\bigl((\eta W)_x (\eta W)_{xy}\bigr)_y - \bigl((\eta W)_x (\eta W)_{yy}\bigr)_x\Bigr).\end{aligned}$$

Here the last integral is 0 and so as $\Delta W = G$ we get after routine estimations (as above) that

$$\int_{B'} |\nabla^2 W|^2 \leq c \int_{B''} \bigl(G^2 + |\nabla W|^2 + W^2\bigr),$$

where $c = 3(1 + \max |\nabla \eta|^2 + \max(\Delta \eta)^2)$ will do. Thus we can conclude exactly as above that also the Hessian matrix $\nabla^2 w$ is locally square integrable on $\mathbb{R}^2$ and so $w \in \mathrm{W}^{2,2}_{\mathrm{loc}}(\mathbb{R}^2)$, as asserted.

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Section C

7. Let $p(x) \in \mathbb{C}[x]$ be a complex polynomial of degree $d \in \mathbb{N}$ in n indeterminates and let

$$p(\partial) = \sum_{|\alpha| \leq d} c_\alpha \partial^\alpha$$

be the corresponding differential operator. Prove that

$$\text{sing.supp}\big(p(\partial)u\big) \subseteq \text{sing.supp}(u)$$

holds for all distributions u on an open non-empty subset Ω of $\mathbb{R}^n$. Give an example where the inclusion is strict. Does such an example exist when the dimension $n = 1$?

Solution: Recall that

$$\Omega \setminus \text{sing.supp}(u) = \Omega \setminus \big\{x \in \Omega : \, u|_{\Omega \cap B_r(x)} \in \mathrm{C}^\infty(\Omega \cap B_r(x)) \text{ for some } r > 0\big\}.$$

Fix x in this set. We may then find $r \in (0, \text{dist}(x, \partial\Omega))$ such that u is C^∞ on $B_r(x)$. But then also $p(\partial)u$ is C^∞ on $B_r(x)$, and therefore $x \in \Omega \setminus \text{sing.supp}\big(p(\partial)u\big)$ so that we have proved the required inclusion. An example with strict inclusion can for instance be obtained using the wave differential operator $\partial_t^2 - k^2\partial_x^2$ on $\mathbb{R}^2$ and the solutions mentioned in Question 2 on Problem Sheet 2. However, more dramatic examples can also be constructed: For instance consider the differential operator $p(\partial) = \partial_y$ on $\mathbb{R}^2$. Let $\{q_n\}_{n \in \mathbb{N}}$ be an enumeration of the rational numbers $\mathbb{Q}$ and define

$$u(x, y) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} 2^{-n} \mathbf{1}_{(-\infty, q_n)}(x), \quad (x, y) \in \mathbb{R}^2.$$

Then $u \colon \mathbb{R}^2 \to \mathbb{R}$ is a well-defined function that is independent of y and is increasing and right-continuous in $x \in \mathbb{R}$. It is therefore in particular locally integrable on $\mathbb{R}^2$ and so defines a regular distribution on $\mathbb{R}^2$. Since u is discontinuous at each point of the set $\mathbb{Q} \times \mathbb{R}$, which is dense in $\mathbb{R}^2$, it follows that $\text{sing.supp}(u) = \mathbb{R}^2$. However, since $\partial_y u = 0$ in $\mathscr{D}'(\mathbb{R}^2)$ we have $\text{sing.supp}(\partial_y u) = \emptyset$.

No such example can exist when $n = 1$. We skip the details here and merely indicate a possible proof of the following result: if $f \in \mathrm{C}^\infty(a, b)$ and $u \in \mathscr{D}'(a, b)$ satisfies $p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) u = \sum_{j=0}^{n} c_j u^{(j)} = f$ in $\mathscr{D}'(a, b)$, then $u \in \mathrm{C}^\infty(a, b)$ too. In order to show this, one can use the factorization of the polynomial $p(x)$ to factorize the differential operator $p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$, then use the fundamental theorem and the constancy theorem iteratively as was done in an example with a second order operator on Problem Sheet 3.

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8. A function $f\colon \mathbb{R} \to \mathbb{C}$ is Lipschitz continuous if there exists a constant $L \geq 0$ such that $\bigl|f(x) - f(y)\bigr| \leq L|x-y|$ holds for all x, $y \in \mathbb{R}$.

   Prove that $u \in \mathscr{D}'(\mathbb{R})$ has a Lipschitz continuous representative if and only if $u' \in \mathrm{L}^\infty(\mathbb{R})$. What does this mean for elements of the Sobolev space $\mathrm{W}^{1,\infty}(\mathbb{R})$?

Solution: If $u' \in \mathrm{L}^\infty(\mathbb{R})$, then by results from lectures $f(x) = c + \int_{x_0}^x u'(t)\,\mathrm{d}t$, $x \in \mathbb{R}$, is for suitable $c \in \mathbb{C}$ and $x_0 \in \mathbb{R}$, an absolutely continuous representative of u. Because

$$\bigl|f(x) - f(y)\bigr| = \left|\int_y^x u'(t)\,\mathrm{d}t\right| \leq \|u'\|_\infty |x-y|$$

for all x, $y \in \mathbb{R}$, f is also Lipschitz continuous. Conversely, assume that u has a Lipschitz continuous representative and denote it by u again, so that $u\colon \mathbb{R} \to \mathbb{C}$ satisfies $|u(x) - u(y)| \leq L|x-y|$ for all x, $y \in \mathbb{R}$, where $L \geq 0$ is a constant. In an example from lectures we saw that the difference quotients of a distribution converge in the sense of distributions to its distributional derivative, so

$$\frac{\Delta_h u}{h} = \frac{\tau_h u - u}{h} \to u' \text{ in } \mathscr{D}'(\mathbb{R}) \text{ as } h \to 0.$$

Here it follows from the Lipschitz condition that $\|\Delta_h u/h\|_\infty \leq L$ for all real $h \neq 0$. It follows then by general results of Functional Analysis that $u' \in \mathrm{L}^\infty(\mathbb{R})$ (more precisely one can use that $\mathrm{L}^\infty(\mathbb{R})$ is the dual space of the separable space $\mathrm{L}^1(\mathbb{R})$), but as we have not developed that here we use a mollification argument instead. Put $u_\varepsilon = \rho_\varepsilon * u$. Then $u_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$ and by inspection $|u_\varepsilon(x) - u_\varepsilon(y)| \leq L|x-y|$ holds for all x, $y \in \mathbb{R}$ and $\varepsilon > 0$. We start by showing that u' is a regular distribution on $\mathbb{R}$ and to that end fix $a > 0$. Now for ε_1, $\varepsilon_2 > 0$ we estimate

$$\begin{aligned}
\bigl\|u'_{\varepsilon_1} - u'_{\varepsilon_2}\bigr\|_{\mathrm{L}^1(-a,a)} &= \int_{-a}^a \left|\int_{-1}^1 \rho'(y)\bigl(u(x-\varepsilon_1 y) - u(x-\varepsilon_2 y)\bigr)\,\mathrm{d}y\right|\,\mathrm{d}x \\
&\leq \int_{-a}^a \int_{-1}^1 \|\rho'\|_\infty L|\varepsilon_1 - \varepsilon_2||y|\,\mathrm{d}y\,\mathrm{d}x \\
&\leq 4a\|\rho'\|_\infty L|\varepsilon_1 - \varepsilon_2|,
\end{aligned}$$

showing that $(u'_\varepsilon|_{(-a,a)})$ is Cauchy in $\mathrm{L}^1(-a,a)$ as $\varepsilon \searrow 0$. By completeness of $\mathrm{L}^1(-a,a)$ we find $v_a \in \mathrm{L}^1(-a,a)$ such that $u'_\varepsilon|_{(-a,a)} \to v_a$ in $\mathrm{L}^1(-a,a)$ as $\varepsilon \searrow 0$. Consequently, $\langle u', \phi\rangle = \int_{-a}^a v_a \phi\,\mathrm{d}x$ for all $\phi \in \mathscr{D}(\mathbb{R})$ supported in $(-a,a)$, and since $a > 0$ was arbitrary here we infer that u' is a regular distribution on $\mathbb{R}$. To conclude we fix $a > 0$ again, take a null sequence $\varepsilon_j \searrow 0$ such that $u'_{\varepsilon_j}(x) \to u'(x)$ pointwise in a.e. $x \in (-a,a)$. Using that the u_{ε_j} satisfy an L-Lipschitz condition we get $|u'_{\varepsilon_j}(x)| \leq L$ for all $x \in \mathbb{R}$, hence $|u'(x)| = \lim_{j\to\infty} |u'_{\varepsilon_j}(x)| \leq L$ holds for a.e. $x \in (-a,a)$, and since $a > 0$ is arbitrary we are done.

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Recall that we defined $\mathrm{W}^{1,\infty}(\mathbb{R})$ to consist of all $u \in \mathscr{D}'(\mathbb{R})$ for which u, $u' \in \mathrm{L}^{\infty}(\mathbb{R})$. It therefore follows that $\mathrm{W}^{1,\infty}(\mathbb{R})$ consists of those distributions on $\mathbb{R}$ that admit a bounded Lipschitz function as representative.

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