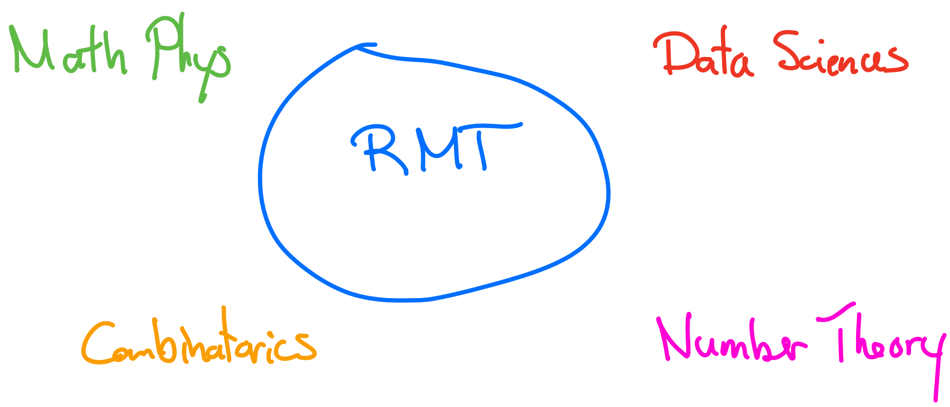
Welcome to C7.7 RMT.

Prof. Louis-Pierre Arguin

A bit about me:
- Math physics
- Stat Mech
- Number theory

Expect that students have diverse backgrounds.



Some technicalities about the class

- 4 PS. (Part B sheet 1, 3 graded)
- 4 classes: Week 3, 5, 7 and 1 of TT
- Exam June.
- Following Prof. Keating's Lecture Notes
  BUT will bring a different perspective

Why random matrices?

A matrix M represents a linear operator

$$M: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$x \longmapsto Mx \qquad \text{s.t. } M(ax+by) = aMx+bMy$$

So in effect we are studying random linear operators between finite-dim. vector spaces

Such operators are ubiquitous:

(1) $Ax = b$ Linear eqns

(2) Quantum Mechanics

$$\underset{\text{Hamiltonian}}{H}\psi \qquad \psi \in L^2(\mathbb{R}^d)$$

(3) Dynamics $\frac{dx(t)}{dt} = Mx(t)$

(4) Principal Component Analysis Machine learning Data sciences

(5) Random networks

(6) Number theory

Why random? For large systems, we might expect the operator to be "typical"

Example

(1)

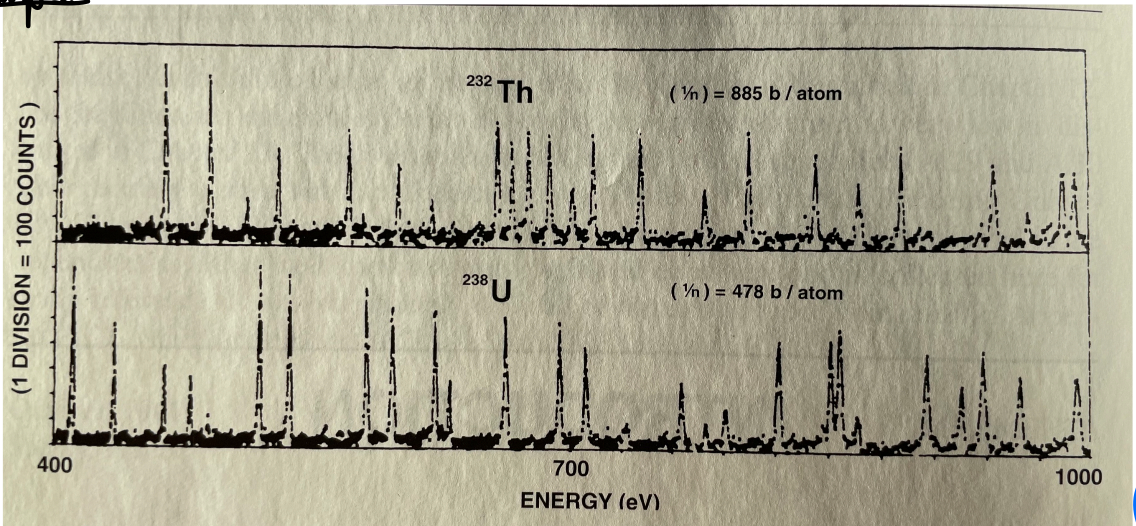


Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei. Reprinted with permission from The American Physical Society, Rahn et al., Neutron resonance spectroscopy, X, *Phys. Rev. C* 6, 1854–1869 (1972).

(Mehta p2)

Excitation energies of a nucleus of U_{238}
(eigenvalues)

Wigner 1950's : · $H = (H_{ij})_{i \le j \le n}$ n large

· H Hermitian so eigenvalues are real

Take · $H_{ij} \sim N(0,1)$ r.v. $i<j$ IID

· $H_{ji} = \overline{H_{ij}}$

· $H_{ii} \sim N(0,2)$ $\left(\frac{M+M^{+}}{2}\right)$

(2) Statistics : Wishart 1928.

Consider $X^{(1)}$ a multivariate Gaussian r.v.

$X^{(1)} = (X_1^{(1)}, \dots, X_p^{(1)})$ mean 0 covariance $N(0, \Sigma)$

Take n IID copies $X^{(1)} X^{(2)} \dots X^{(n)}$

Then construct $X = \begin{pmatrix} X^{(1)} & \dots & X^{(n)} \\ | & & | \end{pmatrix}$ $p \times n$ matrix.

Consider $M = \frac{1}{n} X X^T$ pxp matrix.

This is a sample covariance matrix since.

$$M_{ij} = \frac{1}{n} \sum_k X_{ik} X_{jk}$$

$$X_i^{(k)} X_j^{(k)}$$

Estimator for Σ_{ij}.

Probability Framework

To model a random matrix, we use the standard prob framework

$$(\underset{\text{sample space}}{\Omega}, \underset{\sigma\text{-algebra}}{\mathcal{F}}, \underset{\text{prob}}{\mathbb{P}})$$

(collection of events s.t. $\mathbb{P}(A)$ $A \subseteq \Omega$ is well defined.

A real-valued random variable X is a function

$$\Omega \longrightarrow \mathbb{R}$$

$$\omega \longmapsto X(\omega)$$

s.t. $\{\omega : X(\omega) \in (a,b]\} \in \mathcal{F} \quad \forall (a,b] \subseteq \mathbb{R}$.

The distribution of X is the probability or prob measure on subsets of $\mathbb{R}$

$$\mathbb{P}\big(X \in (a,b]\big) = \mu_X\big((a,b]\big)$$

Ex.: $\cdot$ $X \sim \mathcal{N}(0,1)$

$$\mathbb{P}(X \in B) = \int_B \underbrace{\frac{e^{-y^2/2}}{\sqrt{2\pi}}}_{\text{PDF or density}} dy.$$

$\cdot$

N.B.: $(\mathbb{R}, \underset{\substack{\downarrow \\ \text{smallest } \sigma\text{-algebra containing } (a,b]\text{'s}}}{\mathcal{B}(\mathbb{R})}, \mu)$ is the correct prob. space.

A $m \times n$ random matrix is simply a mble fct or

$$M: \quad \Omega \longrightarrow \mathbb{R}^{m\times n} \text{ or } \mathbb{C}^{m\times n}$$
$$\omega \mapsto (M_{ij}(\omega))$$

Important Remark

RMT is really just multivariate distributions!

A random matrix distribution or ensemble is jus a prob on $\mathbb{R}^{m\times n}$.

Examples of distributions/ensembles

(i) Ginibre: $n\times n$ matrix M

$$M_{ij} \sim \mathcal{N}(0,1) \quad \text{IID}$$

$$M = \begin{pmatrix} M_{11} & & M_{ij} \\ & \ddots & \\ & & M_{nn} \end{pmatrix}$$

So the PDF is

$$\mathbb{P}(M \in B) = \int_B \prod_{i,j} \frac{e^{-x_{ij}^2/2}}{\sqrt{2\pi}} \, dx_{ij}$$

Abs cont. wrt Lebesgue measure on $\mathbb{R}^{n\times n}$.

(ii) GOE (Gaussian Orthogonal Ensemble)

- $M^T = M$ $\quad$ M is symmetric!

- $M_{ij} \sim \mathcal{N}(0,1)$ $i<j$

  $M_{ij} \sim \mathcal{N}(0,2)$ $i=j$ $\frac{M+M^T}{\sqrt{2}}$. symmetrize.

Note that there are now $n + \frac{n(n-1)}{2}$ parameters

Density $f(m) = \prod_{i<j} \frac{e^{-m_{ij}^2/2}}{\sqrt{2\pi}} \prod_{i} \frac{e^{-m_{ii}^2/4}}{\sqrt{2\pi}}$

- Distribution is supported on symmetric matrices.
- singular wrt $\mathbb{R}^{n \times n}$
- abs cont wrt $\mathbb{R}^{n(n+1)/2}$

<u>Important Observation</u>

$$f(M) = \frac{\exp\left(-\frac{1}{4}\operatorname{Tr}(M^2)\right)}{(2\pi)^{n/2}(2\pi)^{n(n-1)/2}} \quad \text{check!}$$

Why is it called GOE? The distribution is invariant under orthogonal transf.

<u>Lemma</u>: Let $O \in \mathcal{O}(n)$, $O^T O = I$ and M a GOE matrix.

Then OMO^T has same distribution ~~as~~ M.

$$P\left(OMO^T \in B\right)$$

$$= \int_{O^T B O} \frac{e^{-\frac{1}{4}\operatorname{Tr} X^2}}{\text{Norm}}\, dX$$

Change of variable $Y = O^T X O$ Linear

$$= \int_B e \qquad dY$$

- Bound
- Jacobian
- Density

Bound $\quad B$

Density: $\mathrm{Tr}\left((OXO^T)^2 \right) = \mathrm{Tr}\, X^2$ since $\mathrm{Tr}\, AB = \mathrm{Tr}\, BA$

Jacobian: we will have to do this later but here just observe that.

$$\mathrm{Tr}\, M^2 = \sum_{i,k} M_{ik}^2 = \|M\|^2$$

- Hilbert-Schmidt norm!
- Or Euclidean norm.

$$(M^2)_{ii} = \sum_k M_{ik} M_{ki} = \sum_k M_{ik}^2$$

But $\|OMO^T\|^2 = \|M\|^2$ isometry so the volume-element is preserved

$\blacksquare$

(iii) GUE

This is similar to GOE but for Hermitian matrices

Take M s.t.
- $M^\dagger = \overline{M}^T = M$
- $M_{ij} \sim N_{\mathbb{C}}(0,1) = \frac{1}{\sqrt{2}} Z_1 + \frac{i Z_2}{\sqrt{2}}$ $\quad$ IID $N(0,1)$
- $M_{ii} \sim N(0,1)$ $\qquad \frac{M + M^\dagger}{\sqrt{2}}$

Now the distribution of M has

$$n + 2\left(n \frac{(n-1)}{2}\right) = n^2 \text{ free real parameters.}$$

PDF: $\displaystyle \prod_i \frac{e^{-x_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i<j} \frac{e^{-x_{ij}^2} e^{-y_{ij}^2}}{(\pi)^{1/2}} = \exp\left(-\frac{\mathrm{Tr}\, M^2}{2}\right)$

Now $\operatorname{Tr}(M^2) = \sum_i \sum_k M_{ik} M_{ki} = \sum_i \sum_k M_{ik} \overline{M_{ik}} = \sum_{i,k} |M_{ik}|^2 = \sum_i M_{ii}^2 + 2\sum_{k<j} \operatorname{Re} M_{ij}^2 + \operatorname{Im} M_{ij}^2$

As before, the distribution is invariant under unitary transformation

<u>Lemma</u>: $UMU^\dagger$ has same distribution as M Check

$\boxed{\text{Gaussian Ensembles}}$

PDF

- GOE $\propto e^{-\operatorname{Tr} M^2/4}$
- GUE $\propto e^{-\operatorname{Tr} M^2/2}$
- GSE $\propto e^{-\operatorname{Tr} M^2}$ M_{ij} takes value in Quaternions $\mathbb{H}$
- GβE $\propto e^{-\beta \operatorname{Tr} M^2/4}$

(iv) <u>Wishart Matrices</u> (Sample Covariance Matrix)

$X^{(1)} \sim N(0,\Sigma)$. as above.

- $$P(X^{(1)} \in B) = \int_B \frac{e^{-x^T \Sigma^{-1} x/2}}{(2\pi)^{n/2} \det \Sigma}\, dx$$

A Wishart matrix is a matrix of the form

$$M = \frac{1}{n} X X^T \quad \text{where } X \text{ is as above}$$

Note that this is a non-linear transformation of the X_{ij}.

But we can compute the PDF

$$P(M \in B) = \int_B f(M)\, dM$$

<u>Example:</u> $p=1$ $n=1$ $M = X^2$ $X \sim \mathcal{N}(0,1)$ say

$$P(M \le x) = P(X^2 \le x) = P(-\sqrt{x} \le X \le \sqrt{x})$$

$$= 2\int_0^{\sqrt{x}} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

so PDF is $\frac{2}{2} \frac{1}{x^{1/2}} \frac{e^{-x/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}\, x^{1/2}} e^{-x/2}$

$$f(M) = \frac{1}{2^{np/2}\Gamma_p(\frac{n}{2})(\det \Sigma)^{n/2}} (\det M)^{(n-p-1)/2} \exp\left[-\frac{1}{2}\mathrm{Tr}(\Sigma^{-1}M)\right]$$

;o Lebesgue measure on the cone of symmetric positive definite matrices.

$$\Gamma_p(\tfrac{n}{2}) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \Gamma\left(\frac{n}{2} - \frac{j-1}{2}\right).$$

(v) <u>Random Rotations</u>

$$M = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ pick θ unif on $[0, 2\pi]$.

This is like sampling from $SO(2)$ random rotations.

More generally we can sample unif. from $O(n)$, $U(n)$ any compact Lie group.

HAAR MEASURE

CUE $U(n)$

COE $O(n)$

CBE ?