# String Theory 1

Lecture #4

# Chapter 1 Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime $M$

1.1 Classical relativistic point particle

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

1.3.2 Conserved charges associated to the symmetries of the action

1.3.3 Solutions of EOM + bound. cond.

1.3.4 Satisfying the constraints

1.3.5 The Witt-algebra & conformal symmetries

## 1.3 Classical solutions continued

**Summary**: starting from the gauge fixed Polyakov action $(\gamma_{ab} = \eta_{ab})$ for $M$ = D dim Minkowski space

Solve

$\xi^{\pm} = \tau \pm \sigma$

- $\partial_a \partial^a X^\mu = 0$ ie $\partial_+ \partial_- X^\mu = 0$ in light cone coords

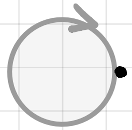
  general soln : $X^\mu(\xi^{\pm}) = X^\mu_R(\xi^-) + X^\mu_L(\xi^+)$

- Impose boundary conditions
  - $\delta X^\mu(\tau_i, \sigma) = 0$ & $\delta X^\mu(\tau_f, \sigma) = 0$ string kept fixed at $\tau_i$ & $\tau_f$
  - $0 = \int_{\tau_i}^{\tau_f} d\tau \; \partial_\sigma X \cdot \delta X \Big|_{\sigma=0}^{\sigma=\ell}$

- Impose constraints comming from $T_{ab} = 0$
  tracelessness, $\eta^{ab} \partial_a T_{bc} = 0$, $T_{ab} = 0$

## 1.3.1 Boundary conditions (continued)

**closed strings** periodicity conditions



$\sigma \sim \sigma + \ell$

Require

$$X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$$

$$\partial_\sigma X^\mu(\tau,\sigma) = \partial_\sigma X^\mu(\tau,\sigma+\ell)$$

ie solutions of the EOM which are periodic in $\sigma$ with period $\ell$

## open strings

boundary conditions on the string endpoints

$$0 = -T \int_{\tau_i}^{\tau_f} d\tau \left( \partial_\sigma X \cdot \delta X \right)\Big|_{\sigma=0}^{\sigma=\ell} \Rightarrow \partial_\sigma X_\mu \, \delta X^\mu = 0 \quad \text{at} \quad \sigma = 0, \ell$$

- **Neumann (NN)** $\partial_\sigma X^\mu(\tau, \ell) = 0$ & $\partial_\sigma X^\mu(\tau, 0) = 0$

  so endpoints move freely in $\mathcal{M}$ (no constraints on $\delta X^\mu$ at $\sigma = 0, \ell$)

  & "no momentum flowing off the string"

ends of string fixed in $\mathcal{M}$

Dirichlet (DD) $\qquad \delta X^\mu = 0 \quad$ at $\quad \sigma = 0, \ell$

ie $\qquad X^\mu(\tau, \ell) = x^\mu_\ell(\tau), \qquad X^\mu(\tau, 0) = x^\mu_0(\tau)$

This involves a choice of spacetime vectors $\Rightarrow$ break Poincaré invariance

One can have mixed boundary conditions: for example

Neumann (NN) on $D-(p+1)$ words

Dirichlet (DD) on $p+1$ words

$\to$ The ends of the string are fixed on a subspace $\mathcal{D} \subset \mathcal{M}$ of $\dim \mathcal{D} = p+1$.

This subspace is called a Dp-brane with $x^\mu_0$ & $x^\mu_\ell$ interpreted as the position of the brane. (Very important: need for internal consistency of the non-perturbative theory; see later)

One can also have (ND) boundary conditions.

## 1.3.2 Conserved charges

Recall Noether's theorem: for each symmetry in the action there is a corresponding conserved current. We also have Noether charges, spatial integral of the $\tau$-component of each current

- Spacetime Poincaré invariance: $\partial_a \left( \frac{\delta \mathcal{L}}{\partial \partial_a X^\mu} \right) = 0$

• translations $\quad X^\mu(\xi) \mapsto X^\mu(\xi) + V^\mu$

current $\quad q^a_\mu = -T\sqrt{-\gamma}\,\gamma^{ab}\partial_b X_\mu = -T\eta^{ab}\partial_b X_\mu$

conservation $\quad \nabla_a q^a_\mu = 0 \;:\; \partial_a q^a_\mu = 0$

conservation of the energy momentum current

charges: $\frac{1}{\ell}\int_0^\ell (q^\mu)^\tau \, d\sigma = \frac{T}{\ell}\int_0^\ell \partial_\tau X^\mu$ $\quad$ total spacetime momentum

- **Lorentz transformations**: $X^\mu \longmapsto \Lambda^\mu{}_\nu X^\nu$

current $J^a_{\mu\nu} = -T\eta^{ab}(X_\mu \partial_b X_\nu - X_\nu \partial_b X_\mu) = X_\mu q^a_\nu - X_\nu q^a_\mu$

conservation $\partial_a J^a_{\mu\nu} = 0$

conservation of angular momentum current

charges: $\frac{1}{\ell}\int_0^\ell (J^{\mu\nu})^\tau d\sigma$

- **WS - symmetries**: WS diffeomorphisms

$$\left.\begin{aligned} \xi^\pm &\longmapsto \tilde{\xi}^\pm(\xi) = \xi^\pm + \epsilon^\pm(\xi) \\ \gamma_{ab} &\longmapsto \gamma_{ab} + \nabla_a \epsilon_b + \nabla_b \epsilon_a \end{aligned}\right\} \Longrightarrow \partial^a T_{ab} = 0$$

conserved current $T_{ab}$, $\eta^{ab}\partial_a T_{bc} = 0$

Tracelessness of $T_{ab}$ is a consequence of Weyl inv.

## 1.3.3 Solutions of EOM + bound. cond.

general solution of the wave eq $X^\mu(\tau,\sigma) = X_R^\mu(\xi^-) + X_L^\mu(\xi^+)$

**Closed strings** : $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\ell)$, $\partial_\sigma X^\mu(\tau,\sigma) = \partial_\sigma X^\mu(\tau,\sigma+\ell)$

Expand in Fourier modes: (periodicity $\xi^\mp \to \xi^\pm \pm \ell$)

separately periodic up to a two mode

$$X_L^\mu(\xi^+) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$X_R^\mu(\xi^-) = \frac{1}{2}x^\mu + \frac{\pi}{\ell}\alpha' p^\mu \xi^- + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi^-}$$

where $x^\mu$, $p^\mu$, $\tilde{\alpha}_n^\mu$ and $\alpha_n^\mu$ are the Fourier coeffts.

$X^\mu$ is real-valued : $x^\mu \in \mathbb{R}$, $p^\mu \in \mathbb{R}$, $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$, $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$

$$\partial_+ X^\mu(\xi^+) = \partial_+ X_L^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_+}$$

$$\partial_- X^\mu(\xi^-) = \partial_- X_R^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu \, e^{-\frac{2\pi i}{\ell} n \xi_-}$$

where $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$

Prefactors for convenient physical interpretations as we will see below

**Open strings** with Neumann (NN) boundary conditions

$$\partial_\sigma X^\mu(\tau, \ell) = 0 \qquad \partial_\sigma X^\mu(\tau, 0) = 0$$

Due to the boundary conditions $X^\mu_L$ & $X^\mu_R$ are no longer independent $(\tilde{\alpha}^\mu_n = \alpha^\mu_n)$

$$X^\mu(\tau,\sigma) = \underbrace{x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau}_{\substack{\text{average position} \\ x^\mu(\tau) = \frac{1}{\ell}\int_0^\ell d\sigma\, X^\mu(\tau,\sigma)}} + i\sqrt{2\alpha'} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha^\mu_n e^{-\frac{i\pi n}{\ell}\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right)$$

$X^\mu$ real-valued : $\alpha^\mu_{-n} = (\alpha^\mu_n)^*$

$$\partial_\pm X^\mu = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_n \alpha^\mu_n e^{-\frac{i\pi n}{\ell}\xi^\pm}, \qquad \alpha^\mu_0 = \sqrt{2\alpha'}\, p^\mu$$

See lecture notes for open strings with DD & ND boundary conds

# Interpretation of the coefficients

## translations

$$\int_0^\ell d\sigma\, (\mathcal{P}^\mu)^\tau = T \int_0^\ell \partial_\tau X^\mu = \begin{cases} p^\mu & \text{closed string \& open string with (NN) bound. conds} \\ 0 & \text{open string with (DD) bound. cond} \end{cases}$$

$\partial_\tau X^\mu = \frac{2\pi\alpha'}{\ell} p^\mu$ + terms that vanish upon $\sigma$-integration

$p^\mu$ : center of mass momentum of the string

## Lorentz transformations

$$M^{\mu\nu} = \int_0^\ell d\sigma \, (J^{\mu\nu})^\tau = \begin{cases} \ell^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} & \text{closed strings} \\ \ell^{\mu\nu} + E^{\mu\nu} & \text{(NN) open string} \end{cases}$$

$$\ell^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$$

centre of mass

two mode of spacetime angular momentum

$$E^{\mu\nu} = -i \sum_{n\neq 0} \frac{1}{n} \left( \alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)$$

$$\tilde{E}^{\mu\nu} = -i \sum_{n\neq 0} \frac{1}{n} \left( \tilde{\alpha}^\mu_{-n} \tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n} \tilde{\alpha}^\mu_n \right)$$

contribution of L & R moving waves to the spacetime angular momentum

## reparametrisation invariance : diffeomorphism inv on $\Sigma$

current $T_{ab}$ conservation $\nabla^a T_{ab} = 0$

$\nabla$ = LC-connection associated to w/ metric $\gamma_{ab}$

(true on shell, i.e. when EoM for $X^\mu$ are satisfied)

## 1.3.4 Satisfying the constraints

Recall that we need to impose constraints from the stress tensor. In the light-cone coordinates

- tracelessness $\eta^{ab} T_{ab} = 0 \Rightarrow T_{+-} + T_{-+} = 0$

  symmetry $T_{ab} = T_{ba}$ $\quad T_{+-} = T_{-+}$

$$\left.\begin{array}{l} T_{+-} + T_{-+} = 0 \\ T_{+-} = T_{-+} \end{array}\right\} \underline{T_{+-} = 0}$$

- conservation on $T_{ab}$ $\quad \eta^{ab} \partial_a T_{bc} = 0 \Longrightarrow$

$$\partial_+ T_{--} + \cancel{\partial_- T_{+-}} = 0$$
$$\partial_- T_{++} + \cancel{\partial_+ T_{-+}} = 0$$

$$\Rightarrow \quad \underline{\partial_+ T_{--} = 0 \qquad \partial_- T_{++} = 0}$$

These are extremely powerful! 2dim Lorentzian version of holomorphicity (antiholomorphicity). These give us an $\boxed{\text{infinite}}$ set of conserved charges! *

- Finally enforce $T_{++} = 0 \qquad T_{--} = 0$

# Closed strings

let $f(\xi^-)$ be an arbitrary function and consider

$$Q_f = \int_0^\ell d\sigma \; f(\xi^-)\, T_{--}(\xi^-) \qquad \partial_+ T_{--} = 0$$

$$\partial_\pm = \tfrac{1}{2}(\partial_\tau \pm \partial_\sigma)$$

$$\text{so } \partial_\tau = 2\partial_+ - \partial_\sigma$$

$$\Rightarrow \frac{\partial}{\partial\tau} Q_f = \int_0^\ell d\sigma \, (\cancel{2\partial_+} - \partial_\sigma)(f(\xi^-)\, T_{--}(\xi^-))$$

$$= -\int_0^\ell d\sigma \, \partial_\sigma (f(\xi^-)\, T_{--}(\xi^-)) = -(f(\xi^-)\, T_{--}(\xi^-))\Big|_{\sigma=0,\ \tau \text{ fixed}}^{\sigma=\ell,\ \tau \text{ fixed}}$$

$= 0$ if $f(\xi^-)$ is periodic

That is: the current $f\, T_{--}$ is also conserved!

Since $f$ is arbitrary $\Rightarrow$ there is an infinite set of conserved currents

Similarly: $T_{++}$ is conserved and so is $g\, T_{++}$, $g = g(\xi^+)$ periodic

A complete set of periodic functions in $\sigma$ is given by

$$f_m(\xi^-) = e^{\frac{2\pi i}{\ell} m \xi^-}, \quad m \in \mathbb{Z}.$$

Define then an infinite set of charges:

$$L_m = \frac{T\ell}{2\pi} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} T_{--}(\xi^-) = \frac{T}{2\ell} \int_0^\ell d\sigma \, e^{\frac{2\pi i}{\ell} m \xi^-} \partial_- X_R \, \partial_- X_R$$

$(T_{--} = \partial_- X \cdot \partial_- X = \partial_- X_R \cdot \partial_- X_R)$

Using the mode expansion for $X^\mu$ $\left(\partial_- X_R^\mu = \frac{2\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-\frac{2\pi i}{\ell} n \xi^-}\right)$

take $\tau = 0$ WLOG as $L_m$ conserved charges

$$L_m = \frac{1}{2} \sum_{n\in\mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad \text{with} \quad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

Note that $L_{-m} = (L_m)^*$ because $T_{--}$ is real

similarly: for $T_{++}(\xi^+)$, $g_m(\xi^+) = e^{\frac{2\pi i}{\ell} m \xi^+}$

$$\tilde{L}_m = \frac{T\ell}{2\pi} \int_0^\ell d\sigma \, e^{\frac{4\pi i}{\ell} m \xi^+} T_{++}(\xi^+) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n, \quad \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu$$

$$\boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n \qquad \tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n}$$

Notice that $L_m$ & $\tilde{L}_m$ are the Fourier components of $T_{--}$ & $T_{++}$ respectively. Then setting

$$\boxed{L_m = 0 \quad , \quad \tilde{L}_m = 0} \qquad \forall \; m\in\mathbb{Z}$$

imposes the constraints $T_{++} = 0$ & $T_{--} = 0$

The vanishing of these charges is equivalent to quadratic constraints on the oscillators $\alpha_n^\mu$ & $\tilde{\alpha}_n^\mu$.

Consider these constraints for $L_0$ & $\tilde{L}_0$ (particularly interesting)

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{n=1} \alpha_{-n}\cdot\alpha_n \qquad \alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu$$

$$(p^2 = p\cdot p = p^\mu p_\mu)$$

$$= \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = 0$$

$$\tilde{L}_0 = \frac{\alpha'}{4}p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = 0$$

Recall $p^\mu$ = spacetime center of mass momentum

we have a mass shell condition : $M^2 = -p^2$

$$-p^2 = M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)$$

contributions of osc. modes to the effective mass of the string in spacetime

rest mass of a string in a given state of oscillation

Moreover: $$\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = -\frac{\alpha'}{4}p^2$$

level matching condition

(relates L/R modes)

(do not forget we still need $L_m = 0$, $\tilde{L}_m = 0$ $\forall\, m \neq 0$ )

Open strings with (NN) boundary conditions on all $X^\mu$

Let $f(\xi^+)$ & $g(\xi^-)$ be arbitrary functions. Then

$$\partial_+\left(g(\xi^-)\,T_{--}(\xi^-)\right) = 0 \quad \& \quad \partial_-\left(f(\xi^+)\,T_{++}(\xi^+)\right) = 0$$

Let

$$Q_{f,g} = \int_0^\ell d\sigma\,\left(f(\xi^+)\,T_{++} + g(\xi^-)\,T_{--}\right)$$

and seek conditions on $f$ & $g$ st $Q$ is conserved. Then

$$\partial_\tau Q_{f,g} = \int_0^\ell d\sigma\,\left(\partial_\sigma\left(f(\xi^+)\,T_{++}\right) - \partial_\sigma\left(g(\xi^-)\,T_{--}\right)\right)$$

$\partial_\tau = 2\partial_- + \partial_\sigma$ $\quad$ $\partial_\tau = 2\partial_+ - \partial_\sigma$

$$= \left(f(\xi^+)\,T_{++} - g(\xi^-)\,T_{--}\right)\Big|_0^\ell$$

At $\sigma = 0, \ell$ : $\partial_+ X^\mu = \partial_- X^\nu$ so $T_{++} = T_{--}$

$$\partial_\pm X^\mu = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n^\mu\, e^{-\frac{i\pi n}{\ell}\xi^\pm}$$

$\Rightarrow$ $Q$ is conserved if $f(\xi^+) = g(\xi^-)$ at $\sigma = 0, \ell$

(i) $\sigma = 0 \Rightarrow f(\tau) = g(\tau)$ so $f$ & $g$ are the same fun

(ii) $\sigma = \ell \Rightarrow f(\tau+\ell) = g(\tau-\ell) = f(\tau-\ell)$

$\therefore f(\sigma) = f(\tau + 2\ell)$

$f$ periodic function with period $2\ell$

Let $f(\xi^+) = e^{\pi i m \xi^+/\ell}$ $\quad g(\xi^-) = e^{\pi i m \xi^-/\ell}$

and define

$$L_m = \frac{T\pi}{\ell}\int_0^\ell d\sigma \left(e^{\pi i m \xi^+/\ell} T_{++} + e^{\pi i m \xi^-/\ell} T_{--}\right) \overset{\tau=0}{=} \frac{T\pi}{\ell}\int_0^\ell d\sigma \left(e^{\pi i m \sigma/\ell} T_{++}(\sigma) + e^{-\pi i m \sigma/\ell} T_{--}(\sigma)\right)$$

$$\Rightarrow \boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n}$$ with $\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu$

$L_0$ gives open-string mass-shell condition

$$L_0 = 0 : \quad M^2 = -p^2 = \frac{1}{\alpha'} \sum_{n=1} \alpha_{-n} \cdot \alpha_n$$

rest mass of a string due to the oscillators

end of lecture #4

Next )

# 1.3.5 The Witt-algebra & conformal symmetries

We have constructed explicitly the space of solutions of the eqs of motion ie, the phase space.

This is an infinite dimensional affine space with coordinates

$$\{ x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu \}$$

for the closed string; for the open string neglect $\tilde{\alpha}, \tilde{L}$

subject to quadratic constraints

$$\{ L_n = 0, \tilde{L}_n = 0, \forall n \in \mathbb{Z} \}$$

where $L_n$ (& $\tilde{L}_n$) constitute an infinite set of conserved charges corresponding to the Fourier modes of $T_{ab}$

$$L_n = \frac{T}{2}\frac{l}{\pi}\int_0^l d\sigma\, e^{2imn\sigma^-}\, T_{--}(\sigma^-) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n \,, \qquad \tilde{L}_n = \cdots$$