Chapter 3
Sheaves

Main idea: a scheme is a space that locally looks like Spec R.
To make sense of it, we will need to use sheaves of functions.

§ Sheaves

invented by Jean Leray in prison during WWII!

def. A presheaf of sets (groups, rings, spaces, ...) on a category $\mathcal{C}$ is a functor

$$F: \mathcal{C}^{op} \to \text{Set} / \text{Grp} / \text{Ring} / \text{Top} / \dots$$

A presheaf on a top. space X is a presheaf on the category Open(X):

Obj = open subsets U of X

Mor = inclusions $V \subseteq U$.

This means that a presheaf R on X consists of the following data:

$\forall\, U \subseteq X$ open $\rightsquigarrow R(U)$ (set / group / ring / ...)

$\forall\, V \subseteq U$ opens $\rightsquigarrow R(U) \xrightarrow{\rho_{UV}} R(V)$ (map / hom / ...)

restriction map

s.t. $\rho_{UU} = id_{R(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for $U \supseteq V \supseteq W$.

Elements of $R(U)$ are called sections, elements of $R(X)$ are global sections and $\rho_{UV}(f)$ is denoted $f|_V$.

Ex. 1) constant presheaf A_X (on X)

pick $A \in$ Set/Ring/Top/...

$A_X(U) = A \quad \forall U$ open in X

$\rho_{UV} = id_A \quad \forall V \subseteq U$

2) presheaf of C^∞-functions

X smooth manifold,

$R(U) := C^\infty(U, \mathbb{R})$

ρ_{UV} = usual restriction of functions.

Want: glue values on X from local data

def. A sheaf R on X is a presheaf on X s.t.:

1) $\forall\ U = \bigcup_i U_i \subseteq X$ open cover; $s, t \in R(U)$

$s|_{U_i} = t|_{U_i} \ \forall i \Rightarrow s = t$

"sections agree locally $\Rightarrow$ agree globally"

2) $U = \bigcup U_i$ open cover, $s_i \in R(U_i)$

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i,j \Rightarrow \exists s \in R(U):$

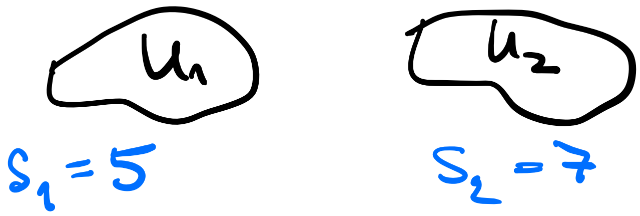
$s|_{U_i} = s_i \quad \forall i.$

"sections agreeing on overlaps can be glued"

Rem. $F(\emptyset) = *$ (take empty cover)

<u>Ex.</u> 1) the presheaf of C^∞ functions on a smooth manifold is a sheaf

2) constant presheaf is NOT a sheaf: say, $X = U_1 \amalg U_2$, $A = \mathbb{Z}$



$s_i \in A_X(U_i)$, $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2}$ because $U_1 \cap U_2 = \emptyset$ (assuming $A_X(\emptyset) = 0$)

but $\nexists\, s \in A_X(X) = \mathbb{Z}$: $s|_{U_i} = s_i$, because restriction maps are identities.

To fix this, one defines <u>constant sheaf</u>:

$$A_X(U) := \{\text{locally constant } U \to A\} \overset{U \text{ loc-connected}}{=\!=} \prod_{\pi_0 U} A$$

$\pi_0 U$ = connected components

This is an example of a more general, canonical procedure „<u>sheafification</u>":
presheaf $F \rightsquigarrow$ sheaf F^+ in a universal way
(see lecture notes by Ritter)

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \text{ - sheaf} \\ \mathcal{F}^+ \arrow[ur, dashed, "\exists!"'] \end{tikzcd}$$

<u>def.</u> A <u>morphism</u> of (pre)sheaves $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is a nat. transformation of functors.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r, "\varphi_V"'] & \mathcal{G}(V) \end{tikzcd}$$

A <u>sub(pre)sheaf</u> $\mathcal{F} \subseteq \mathcal{G}$: $\mathcal{F}(U) \subseteq \mathcal{G}(U)$, compatible with ρ_{UV}

§ Stalks

def. Let $\mathcal{F}$ be a (pre)sheaf on X, $x \in X$.
The stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \operatorname*{colim}_{x \in U \subseteq X \text{ open}} \mathcal{F}(U)$$

direct limit over restriction maps

Explicitly: any element of $\mathcal{F}_x$ is determined by some $f \in \mathcal{F}(U)$ for $U \ni x$ open, and $(f, U) \sim (f', U')$ if $\exists\, U_1 \cap U_2 \supseteq W \ni x$ open s.t. $f|_W = f'|_W$.
The class of equivalence of (f, U) in $\mathcal{F}_x$ is called the germ f_x of f at x.

Rem. 0) $\mathcal{F}_x$ has the same alg structure as $\mathcal{F}$ (e.g. $\mathcal{F}$ sheaf of abelian groups $\Rightarrow \mathcal{F}_x \in Ab$)

1) stalks encode local data: $x \in U \subseteq X$ open $\Rightarrow$ $\mathcal{F}_x$ is the same for $(\mathcal{F}, X)$ and $(\mathcal{F}|_U, U)$.

2) $\forall x \in X \quad \mathcal{F}_x = \mathcal{F}^+_x$

3) A morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ induces maps $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x \quad \forall x \in X$.

Exercise. (stalks are powerful!)
Let $\mathcal{F} \to \mathcal{G} \in Ab(X)$ – sheaves of abelian groups on X.
Then $\mathcal{F} \to \mathcal{G}$ is an isom in $Ab(X)$ iff $\forall x \in X$ $\mathcal{F}_x \to \mathcal{G}_x$ isom.

§ Kernels & cokernels

<u>def.</u> Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves.
The presheaf kernel/image/cokernel is the presheaf Ker / ... is
$$U \rightsquigarrow \underset{\dots}{\mathrm{Ker}}(\mathcal{F}(U) \to \mathcal{G}(U))$$

Exercise. φ map of sheaves $\Rightarrow$ Ker is a sheaf.

Ex: NOT true for cokernels!
Take $X = \mathbb{C}$, $\mathcal{F}_X = $ (holomorphic functions, $+$),
$\mathcal{F}_X^* := $ (non-zero holom functions, $\times$).

Consider exponent map
$$\exp : \mathcal{F}_X \to \mathcal{F}_X^*,$$
then $\mathrm{Ker}(\exp) = $ constant sheaf $2\pi i \cdot \mathbb{Z}$.
But Coker is NOT a sheaf!

Take $U_1 = \mathbb{C} \setminus [0, \infty)$,
$U_2 = \mathbb{C} \setminus [0, -\infty)$,
$U = U_1 \cup U_2 = \mathbb{C} \setminus 0$.

Take $f = z$ in $\mathcal{F}_X(U)$. Then
$\bar{f} \in \mathrm{Coker}(\exp)(U)$, but $\forall i$:
$\mathrm{Coker}(\exp)(U_i) = 0$ because log exists,
so exp is surjective on U_i.

def. Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then the sheaf cokernel / image is the sheafification of the presheaf cokernel / image.

A morphism φ is injective / surjective if $\operatorname{Ker} \varphi = 0$ / $\underbrace{\operatorname{Im}}_{\text{sheaf!}} \varphi = \mathcal{G}$.

Ex. The sequence

$$0 \to 2\pi i \mathbb{Z} \to \mathcal{F}_X \to \mathcal{F}_X^{\times} \to 0$$

is exact as a sequence of sheaves, $\forall$ cpx manifold X.

def. $\mathcal{F} \subseteq \mathcal{G}$ a subsheaf, i.e. $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ $\forall U$, compatible with restrictions.

The quotient sheaf is the sheafification of

$$U \rightsquigarrow \mathcal{G}(U) / \mathcal{F}(U).$$

Exercise. 1) stalks of Ker and Im are the kernel and image of the stalk maps

2) Injectivity and surjectivity are stalk-local properties (can be checked on stalks), but the maps φ_U don't have to be surjective!

§ Moving between spaces

Say, we're given $f: X \to Y$ map of top. spaces with $\mathcal{F}$ sheaf on X; $\mathcal{G}$ sheaf on Y.

also called direct image

def. The pushforward $f_*\mathcal{F}$ on Y is the presheaf: $\underset{Y}{\overset{\cap}{U}} \mapsto \mathcal{F}(\underset{\text{open}}{f^{-1}U})$

~~Prop~~ Exercise: $f_*\mathcal{F}$ is a sheaf.

def. The inverse image presheaf is

$$f^{-1}\mathcal{G}^{pre}(\underset{X}{\overset{\cap}{V}}) := \operatorname*{colim}_{U \supseteq f(V)} \mathcal{G}(U) = \{(s_u, u) \mid f(V) \subseteq U \underset{\text{open}}{\subseteq} Y,\ s_u \in \mathcal{G}(U)\} / \sim$$

where $\sim$ identifies pairs that agree in an open nbhd of $f(V)$.

The inverse image $f^{-1}\mathcal{G}$ is its sheafification.

Rem. The sheafification is necessary like for a constant presheaf:

consider $X = Y \amalg Y \xrightarrow[\mathrm{id} \amalg \mathrm{id}]{} Y$, $U \subseteq Y$ open,

then $f^{-1}\mathcal{G}^{pre}(U \amalg U) = \mathcal{G}(U)$ but

$f^{-1}\mathcal{G}(U \amalg U) = \mathcal{G}(U) \times \mathcal{G}(U)$ by sheaf axioms.

Exercise: $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)} \quad \forall x \in X$.

Ex: 1) $i : S \hookrightarrow X$ open subset

$\mathcal{F} \in Sh(S)$ $\quad i_*\mathcal{F} : V \mapsto \mathcal{F}(V \cap S)$

$\mathcal{G} \in Sh(X)$ $\quad i^{-1}\mathcal{G} : \underset{S \subseteq X \text{ open}}{U} \mapsto \mathcal{G}(U)$ $\quad$ restriction $\mathcal{G}|_S$ of $\mathcal{G}$

2) $i_x : x \hookrightarrow X$ point

$\mathcal{F} \in Sh(X)$ $\quad i_x^{-1}\mathcal{F} = \mathcal{F}_x$

3) $\pi : X \to pt$

$\mathcal{F} \in Sh(X)$ $\quad \pi_*\mathcal{F} = \mathcal{F}(X) =: \Gamma(X, \mathcal{F})$ $\quad$ global sections functor

Prop. (f^{-1} is left adjoint to f_*)

There is a natural isom

$$Mor(f^{-1}\mathcal{G}, \mathcal{F}) \simeq Mor(\mathcal{G}, f_*\mathcal{F})$$

Sketch proof

$(\rightarrow)$ given $\underset{U \supseteq f(V)}{\mathrm{colim}}\, \mathcal{G}(U) \to \mathcal{F}(V)$ for $V \subseteq X$ open,

take for any $W \subseteq Y$ open

$$\mathcal{G}(W) \to \underset{U \supseteq f(V)}{\mathrm{colim}}\, \mathcal{G}(U) \to \mathcal{F}(V) = \mathcal{F}(f^{-1}W) = f_*\mathcal{F}(W)$$

pick $V = f^{-1}W$

$(\leftarrow)$ given $\mathcal{G}(W) \to \mathcal{F}(f^{-1}W)$, consider

assume $W \supseteq f(V)$ and take lim over such W

$$\begin{array}{ccc} \mathcal{G}(W) & \to & \mathcal{F}(f^{-1}W) \\ \downarrow & & \downarrow \\ \underset{U \supseteq f(V)}{\mathrm{colim}}\, \mathcal{G}(U) & \dashrightarrow & \underset{U \supseteq f(V)}{\mathrm{colim}}\, \mathcal{F}(f^{-1}U) \\ & & \downarrow \text{restriction: } f^{-1}U \supseteq V \\ & & \mathcal{F}(V) \end{array}$$