# Geometric Group Theory

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Part C course HT 2024

# Tietze transformations

How to recognise when two finite presentations give the same group?

Two types of transformations (called Tietze transformations) are relevant.

(T1) Given $\langle S|R\rangle$ and $r \in \langle\langle R\rangle\rangle$, change the presentation to $\langle S|R \cup \{r\}\rangle$ (or do the inverse operation).

(T2) Given $\langle S|R\rangle$, a new symbol $a \notin S$ and $w \in F(S)$, change the presentation to $\langle S \cup \{a\}|R \cup \{a^{-1}w\}\rangle$ (or do the inverse operation).

**Theorem**

*Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.*

Proof: $(\Leftarrow)$ (T1) defines isomorphic groups because $\langle\langle R\rangle\rangle = \langle\langle R \cup \{r\}\rangle\rangle$.

# Tietze transformations

**Theorem**

*Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.*

Proof continued: For (T2), consider the homomorphisms

$$\iota : F(S) \hookrightarrow F(S \cup \{a\}) \quad \text{(injection)}$$
$$f : F(S \cup \{a\}) \twoheadrightarrow F(S) \quad f(a) = w \quad \text{(surjection)}$$

Note that $f \circ \iota = \mathrm{id}_{F(S)}$. They induce homomorphisms

$$F(S) \xrightarrow{\bar{\iota}} F(S \cup \{a\})/\langle\langle a^{-1}w \rangle\rangle \xrightarrow{\bar{f}} F(S)$$

with $\bar{f} \circ \bar{\iota} = \mathrm{id}_{F(S)}$. $\bar{\iota}$ is onto, and hence $\bar{\iota}$ and $\bar{f}$ are isomorphisms. Since also $\bar{f}^{-1}(\langle\langle R \rangle\rangle) = \langle\langle R \cup \{a^{-1}w\} \rangle\rangle / \langle\langle a^{-1}w \rangle\rangle$ we have that $\bar{f}$ induces the desired isomorphism.

# Tietze transformations

**Theorem**

*Two finite presentations define isomorphic groups if and only if they are related by a finite sequence of Tietze transformations.*

Proof continued:

$(\Rightarrow)$ Let $G_1 = \langle S_1 | R_1 \rangle$, $G_2 = \langle S_2 | R_2 \rangle$. WLOG $S_1 \cap S_2 = \emptyset$.

There exist inverse isomorphisms $\phi : G_1 \to G_2$, $\psi : G_2 \to G_1$. $\forall s \in S_1$, choose $w_s \in F(S_2)$ representing $\phi(s)$ in $G_2$. $\forall t \in S_2$, choose $v_t \in F(S_1)$ representing $\psi(t)$ in $G_1$.

Take the two subsets of $F(S_1 \cup S_2)$:

$$U_1 = \{s^{-1} w_s : s \in S_1\} \qquad U_2 = \{t^{-1} v_t : t \in S_2\}$$

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

# Tietze transformations

Claim: There exist finitely many Tietze transformations from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$.

Proof of claim: Use finitely many (T2) to get from $\langle S_1 | R_1 \rangle$ to $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$. There exists an isomorphism

$$\rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_1 | R_1 \rangle \quad \rho(s) = s, \forall s \in S_1 \quad \rho(t) = v_t, \forall t \in S_2$$

Then $\phi \circ \rho : \langle S_1 \cup S_2 | R_1 \cup U_2 \rangle \to \langle S_2 | R_2 \rangle$ is an isomorphism such that $t \overset{\rho}{\mapsto} v_t \overset{\phi}{\mapsto} t$. Also, $\forall r \in R_2$

$$\phi \circ \rho(r) = r \equiv 1 \text{ in } \langle S_2 | R_2 \rangle \Rightarrow r \in \langle\langle R_1 \cup U_2 \rangle\rangle \Rightarrow R_2 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$$

Thus $\langle S_1 \cup S_2 | R_1 \cup U_2 \rangle$ is related to $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_2 \rangle$ by a sequence of (T1) transformations. Also, $\forall s \in S_1$

$$\phi \circ \rho(s) = w_s(t_1 \dots t_k) \quad \phi \circ \rho(w_s) = \phi \circ \rho(w_s(t_1 \dots t_k)) = w_s(t_1 \dots t_k)$$

Hence, $s^{-1} w_s \in \langle\langle R_1 \cup U_2 \rangle\rangle$, which implies that $U_1 \subseteq \langle\langle R_1 \cup U_2 \rangle\rangle$. So we can apply several (T1) to get $\langle S_1 \cup S_2 | R_1 \cup R_2 \cup U_1 \cup U_2 \rangle$. □

# Properties of finite presentability

**Proposition**

*Let $G$ be a group.*

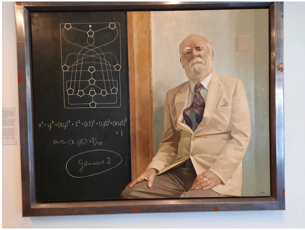
1. *$G$ finitely presented does not imply that a subgroup is finitely presented or that a quotient is finitely presented.*
2. *If $H$ is a finite index subgroup of $G$ then $G$ is finitely presented if and only if $H$ is.*
3. *If $N \trianglelefteq G$ is finitely presented and $G/N$ is finitely presented then $G$ is finitely presented.*

A proof can be found in the notes.

# Graham Higman

**Remark**

*G finitely presented does not imply that a subgroup is finitely presented.*




**Theorem**

*Every finitely generated recursively presented group can be embedded as a subgroup of some finitely presented group.*

# List of algorithmic problems of M. Dehn

Word problem: Given a finite presentation $G = \langle S|R \rangle$ design an algorithm recognising when $w \in F(S)$ satisfies $w = 1_G$ in $G$.

Conjugacy problem: Given a finite presentation $G = \langle S|R \rangle$ design an algorithm recognising when $u, v \in F(S)$ represent conjugate elements in $G$.

Remark

*The conjugacy problem implies the word problem.*

Isomorphism problem: Given finite presentations $G_i = \langle S_i|R_i \rangle, i = 1, 2$, determine if $G_1 \simeq G_2$.

Triviality problem (a particular case of the isomorphism problem): Given a finite presentation $G = \langle S|R \rangle$ determine if $G \simeq \{1\}$.

Novikov, Boone, Rabin ['56]: All of the above are unsolvable.

Fridman ['60]: There exists a group with solvable word problem, but unsolvable conjugacy problem.

# Word and conjugacy problems

**Proposition**

*If the word problem or conjugacy problem is solvable for $G = \langle S | R \rangle$ then it is solvable for any finite $\langle X | Q \rangle = G$.*

**Proof.**

WP: Given $w \in F(X)$ we run simultaneously 2 procedures:

1. List all elements in $\langle\langle Q \rangle\rangle$ (i.e. multiply conjugates $q_i^{w_i}, w_i \in F(X), q_i \in Q$ and transform into reduced word); check if $w$ is among them. If yes, stop and conclude $w = 1$.
2. 
   a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$ (i.e. enumerate all $|X|$-tuples of words in $F(S)$, then check if each $q \in Q$, rewritten by changing $x \mapsto w_x$, becomes $\equiv 1$ in $F(S)/\langle\langle R \rangle\rangle$). This can be done since the WP for $\langle S | R \rangle$ is solvable.
   b. For each $\phi$, check if $\phi(w) \neq 1$ in $F(S)/\langle\langle R \rangle\rangle$. If yes, stop and conclude $w \neq 1$.

# Word and conjugacy problems

Proof continued: CP: Given $w, v \in F(X)$, run the following 2 procedures in parallel:

1. 
   a. List all $gvg^{-1}w^{-1}$ in $F(X)$.
   b. Check if $gvg^{-1}w^{-1}$ is among the list of elements in $\langle\langle Q \rangle\rangle$. If yes, stop and conclude: "$v, w$ conjugate".
2. 
   a. List all homomorphisms $\phi : F(X)/\langle\langle Q \rangle\rangle \to F(S)/\langle\langle R \rangle\rangle$.
   b. Check if $\phi(v), \phi(w)$ are not conjugate. If yes, stop and conclude: "$v, w$ not conjugate".

□

# Residually finite groups

Idea: Approximate by finite quotients. So we will need enough of those.

**Lemma**

*TFAE*

1. $$\bigcap_{H \leq_{f.i.} G} H = \{1\}$$
2. *For all non-trivial $g \in G$, there exists $\phi : G \to F$ finite such that $\phi(g) \neq 1$.*
3. *For all $\{g_1, ..., g_n\}$ distinct, there exists $\phi : G \to F$ such that $\phi(g_1), ..., \phi(g_n)$ are distinct. In other words, every finite chunk of the infinite Cayley table of $G$ can be reproduced identically in the Cayley table of a finite quotient.*

# Residually finite groups

Proof.

The proof is based on the fact that

$$\bigcap_{H \leq_{f.i.} G} H = \bigcap_{N \trianglelefteq_{f.i.} G} N$$

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are OK.

And for (1) $\Rightarrow$ (3): $\forall i \neq j$, take $N_{ij} \not\ni g_i g_j^{-1}$ and define

$$N = \bigcap_{i \neq j} N_{ij}$$

and then consider $\phi : G \to G/N$. □

# Residually finite groups

**Examples**

1. *$GL(n, \mathbb{Z})$ is residually finite.* $\forall g \neq \mathrm{id}$:
   - a. *If $\exists i \neq j$ such that $|g_{ij}| \neq 0$, take $p > |g_{ij}|$ and reduce mod $p$.*
   - b. *If $\forall i \neq j$, $g_{ij} = 0$, then $\exists\ g_{ii} = -1$. Reduce mod 3: $g_{ii} = 2$.*
2. *Any finitely generated $G \leq SL(n, \mathbb{Q})$ (or $GL(n, \mathbb{Q})$) is RF.*