B8.3: Mathematical Modelling of Financial Derivatives —Exercises—

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Exercise Sheet 2

Part A

1. Draw the expiry payoff diagrams for each of the following portfolios:

   (a) Short one share, long two calls with exercise price K.

   (b) Long one call and one put, both with exercise price K.

   (c) Long one call and two puts, all with exercise price K.

   (d) Long one put and two calls, all with exercise price K.

   (e) Long one call with exercise price K_1 and one put with exercise price K_2. Compare the three cases: $K_1 > K_2$, $K_1 = K_2$ and $K_1 < K_2$.

   (f) As (e), but also short one call and one put with exercise price K for $K_1 < K < K_2$.

Part B

1. There are n assets satisfying the following stochastic differential equations

$$dS_i = \mu_i \, S_i \, dt + \sigma_i \, S_i \, dZ_i \qquad \text{for } i = 1, \cdots, n \, .$$

Here, Z_i is a standard Brownian motion and the quadratic variation

$$[Z_i, \, Z_j] = \rho_{ij} \, t \, ,$$

where $-1 \leq \rho_{ij} = \rho_{ji} \leq 1$ is the correlation between Z_i and Z_j.

Derive Itô's Lemma for a function $f(S_1, \ldots, S_n)$ of the n assets $S_1, \ldots, S_n$. (Hint: use $dZ_i \, dZ_j = \rho_{ij} \, dt$.)

2. For $t > 0$, let

$$p(y; x, t) = \frac{1}{\sqrt{2\,\pi\,t}} e^{-(x-y)^2/2t}.$$

This can be interpreted as the probability density function for a normal random variable Y which has mean x and variance t. Show, by direct calculation, that $p(y; x, t)$ also satisfies the heat equation

$$\frac{\partial p}{\partial t} = \tfrac{1}{2}\,\frac{\partial^2 p}{\partial x^2}, \quad \text{for } t > 0,\ x \in \mathbb{R}.$$

Hence deduce that

$$u(x,t) = \mathbb{E}\big[\,f(y)\,\big] = \frac{1}{\sqrt{2\,\pi\,t}} \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/2t}\, dy$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \tfrac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\,, \quad \text{for } t > 0,\ x \in \mathbb{R}\,,$$

provided the integral converges absolutely. [Hint: you can assume that the absolute convergence means you can swap the order of partial differentiation and integration.]

Assuming that the integral converges absolutely and f is continuous at all points in $\mathbb{R}$, show that

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{\sqrt{2\,\pi\,t}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/2t}\, dy = f(x)$$

for each $x \in \mathbb{R}$. [Hint: change variables to $s = (y-x)/\sqrt{t}$ and assume that the absolute convergence allows you to interchange the order of limit and integration.]

3. Find the most general solution of the Black–Scholes PDE

$$V_t + \frac{1}{2}\sigma^2\, S^2\, V_{SS} + r\, S\, V_S - r\, V = 0 \tag{1}$$

that has the special form
(a) $V = V(S)$,
(b) $V = A(t)\, B(S)$, where V, A, B are ‘nicely-behaved’ functions.

Part C

Let the price of a stock follow the dynamics $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_t$ and let

$$\mathcal{H}(x) = \begin{cases} 0 & \text{for } x < 0\,, \\ 1 & \text{for } x \geq 0\,, \end{cases} \tag{2}$$

i.e., the Heaviside function

1. (a) What is the value of a European option struck at K and expiring at T with payoff $\mathcal{H}(S-K)$?

   (b) What is the value of a European option struck at K and expiring at T with payoff $\frac{1}{d}\left(\mathcal{H}(S-K)-\mathcal{H}(S-K-d)\right)$?