String Theory 1

Lecture #5

Chapter 1 Classical relativistic string

↳ study relativistic classical string propagating in a fixed spacetime $\mathcal{M}$

1.1 Classical relativistic point particle

1.2 Classical relativistic string: action principle

1.3 Classical solutions

1.3.1 EOM & boundary conditions

1.3.2 Conserved charges associated to the symmetries of the action

1.3.3 Solutions of EOM + bound. cond.

1.3.4 Satisfying the constraints

1.3.5 The Witt-algebra & conformal symmetries

1.3.5 The Witt-algebra & conformal symmetries

We have constructed explicitly the space of solutions of the eqs of motion ie, the phase space.

This is an infinite dimensional affine space with coordinates

$$\{ x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu \}$$

for the closed string; for the open string neglect $\tilde{\alpha}, \tilde{L}$

subject to quadratic constraints

$$\{ L_n = 0, \tilde{L}_n = 0, \forall n \in \mathbb{Z} \}$$

where L_n ($\& \tilde{L}_n$) constitute an infinite set of conserved charges corresponding to the Fourier modes of T_{ab}

$$L_m = \frac{T}{2}\frac{\ell}{\pi}\int_0^{\ell} d\sigma\, e^{2im\sigma^-}\, T_{--}(\sigma^-) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n, \qquad \tilde{L}_n = \cdots$$

This subsection: L_n & $\tilde{L}_n$ satisfy an algebra,

the algebra of the generators of conformal transformations

So far: working in Lagrangian formalism with

$$\mathcal{L} = \frac{T}{2}[\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X]$$

In a Hamiltonian formulation with canonical fields

$$X^\mu(\tau, \sigma)$$

and conjugate momenta

$$\Pi^\mu(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X_\mu(\tau,\sigma))} = T\, \partial_\tau X^\mu(\tau,\sigma)$$

We define a Hamiltonian

$$H = \int_0^\ell d\sigma \left(\partial_\tau X(\tau,\sigma)\cdot \Pi(\tau,\sigma) - \mathcal{L}\right) = T \int_0^\ell d\sigma \left(\partial_+ X \cdot \partial_+ X + \partial_- X\, \partial_- X\right)$$

$$= \begin{cases} \frac{2\pi}{\ell}(L_0 + \tilde{L}_0) & \text{closed strings} \\ \frac{\pi}{\ell} L_0 & \text{open strings} \end{cases}$$

In this formalism observables are functionals $F(X,\Pi)$.

Phase space is a Poisson manifold ie a manifold together with Poisson brackets

$$\{F, G\}_{PB} = -\{G, F\}_{PB}$$ symplectic pairing

For fields $F(\tau, \sigma)$, $G(\tau, \sigma')$, it is defined as

$$\{F, G\}_{PB} = \int d\tilde{\sigma} \left\{ \frac{\partial F(\tau,\sigma)}{\partial X^\mu(\tau,\tilde{\sigma})} \frac{\partial G(\tau,\sigma')}{\partial \Pi_\mu(\tau,\tilde{\sigma})} - \frac{\partial G(\tau,\sigma)}{\partial X^\mu(\tau,\tilde{\sigma})} \frac{\partial F(\tau,\sigma')}{\partial \Pi_\mu(\tau,\tilde{\sigma})} \right\}$$

This leads to the canonical equal time P.B.s

$$\{\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = \eta^{\mu\nu}\,\delta(\sigma-\sigma'),$$

$$\{X^\mu(\sigma), X^\nu(\sigma')\}_{PB} = 0, \quad \{\Pi^\mu(\sigma), \Pi^\nu(\sigma')\}_{PB} = 0.$$

From these we can compute the Poisson brackets of the oscillator modes by extracting the Fourier components.

$$\boxed{\begin{aligned} &\{\alpha_m^\mu, \alpha_n^\nu\}_{PB} = i m \delta_{m+n,0} \eta^{\mu\nu} \\ &\{p^\mu, x^\nu\}_{PB} = \eta^{\mu\nu} \\ &\{\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\}_{PB} = i m \delta_{m+n,0} \eta^{\mu\nu}, \qquad \{\alpha_m^\mu, \tilde{\alpha}_n^\nu\} = 0 \end{aligned}}$$

Similarly for the open string we have only one set of oscillators

Closed strings: extract Fourier modes and then compute P.B.s

$n \neq 0 \quad \frac{1}{\ell}\int_0^\ell e^{-\frac{2\pi i}{\ell} n\sigma} X^\mu(0,\sigma)\, d\sigma = i\sqrt{\frac{\alpha'}{2}}\, \frac{1}{n}\left(\alpha_n^\mu - \tilde{\alpha}_{-n}^\mu\right) \qquad n \neq 0 \quad \int_0^\ell e^{-\frac{2\pi i}{\ell} n\sigma}\, \Pi^\mu(0,\sigma)\, d\sigma = \frac{1}{\sqrt{2\alpha'}}\left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right)$

$\frac{1}{\ell}\int_0^\ell X^\mu(0,\sigma)\, d\sigma = x^\mu \qquad \int_0^\ell \Pi^\mu(0,\sigma)\, d\sigma = p^\mu$

We now use this to compute the P.B.s for the constraints, i.e., for the Fourier modes of T_{ab}:

We find

$$\{L_m, X^\mu\}_{PB} = -\frac{1}{2\pi} e^{\frac{2\pi i}{\ell} m \xi^-} \partial_- X^\mu$$

$$\{\tilde{L}_m, X^\mu\}_{PB} = -\frac{1}{2\pi} e^{\frac{2\pi i}{\ell} m \xi^+} \partial_+ X^\mu$$

and

PS2

$$\boxed{\begin{aligned} \{L_m, L_n\}_{PB} &= i(m-n) L_{m+n} \\ \{\tilde{L}_m, \tilde{L}_n\}_{PB} &= i(m-n) \tilde{L}_{m+n} \end{aligned}}$$

Witt algebra

L_m & $\tilde{L}_n$ form a Lie algebra (with Lie bracket $\{\,,\,\}_{PB}$ + Jacobi id)

Next: This is the algebra of infinitesimal conformal transformations on the world-sheet.

A conformal transformation of a (Riemannian or Lorentzian) manifold Σ is a diffeomorphism $\xi \longmapsto \tilde{\xi}(\xi)$ that preserves the metric up to rescaling

$$\gamma_{ab}(\xi) \longmapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = e^{2\Lambda(\tilde{\xi})} \gamma_{ab}(\tilde{\xi})$$

(a special case is an isometry for which $\Lambda = 0$)

The infinitesimal conformal transformations can be described explicitly: we can compute the generators of such transformations on Σ

let

$$\xi^a \longmapsto \tilde{\xi}^a = \xi^a + \epsilon^a(\xi)$$

be a general infinitesimal diffeomorphism. Then

$$\gamma_{ab} \longmapsto \gamma_{ab} + \underbrace{\nabla_a \epsilon_b + \nabla_b \epsilon_a}_{\mathcal{L}_\epsilon \gamma_{ab}}$$

This corresponds to a conformal transformation if ϵ^a satisfies

$$\nabla_a \epsilon_b + \nabla_b \epsilon_a = 2\Lambda \gamma_{ab} = (\nabla_c \epsilon^c)\gamma_{ab}$$

conformal killing equation

A solution of this equation is called a conformal killing vector

Then under an infinitesimal conformal transformation

$$\gamma_{ab} \longrightarrow \underbrace{\nabla_a \epsilon_b + \nabla_b \epsilon_a}_{\text{diffeomorphism}} - \underbrace{2\Lambda \gamma_{ab}}_{\text{Weyl}} = 0$$

for ϵ^a a KV

In the unit gauge & in the light-cone coordinates

(Recall $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$ $\quad \eta^{+-} = \eta^{-+} = -2$)

$(++)$: $\quad \partial_+ \epsilon_+ = 0 \implies \partial_+ \epsilon^- = 0 \implies \epsilon^- = \epsilon^-(\xi^-)$

$(--)$: $\quad \partial_- \epsilon_- = 0 \implies \partial_- \epsilon^+ = 0 \implies \epsilon^+ = \epsilon^+(\xi^+)$

$(+-)$: $\quad \partial_+ \epsilon_- + \partial_- \epsilon_+ = -\frac{1}{2}(\partial^a \epsilon_a) = -\frac{1}{2}(-2\partial_- \epsilon_+ - 2\partial_+ \epsilon_-)$

trivially true so no further restrictions on $\epsilon^\pm$

$\therefore$ $\epsilon^- = \epsilon^-(\xi^-)$ & $\epsilon^+ = \epsilon^+(\xi^+)$ generate infinitesimal conformal transformations

and η is left invariant.

This means that after fixing the gauge to the unit gauge the world sheet theory still has residual gauge symmetries (the conformal symmetries)

In principle one can use the residual gauge symmetries to do some further gauge fixing.
However there is no way to do this in and simultaneously preserve space-time Lorentz covariance (For example one can use the light-cone gauge at the expense of covariance: this is analogous to choosing the Coulomb gauge instead of the Lorentz gauge in EM)

One can think of the infinitesimal WS reparametrisations

$$\delta \xi^{\pm} = \epsilon^{\pm}(\xi^{\pm})$$

as being generated by $V^{\pm} \propto \epsilon^{\pm}(\xi^{\pm})\,\partial_{\pm}$ local basis for the tangent space

[Recall correspondence between tangent vector fields, say $\epsilon^{\pm}\partial_{\pm}$, and the 1 parameter group of diffeomorphisms $\xi^a \to \xi^a + \epsilon^a$]

We can pick a basis by using a complete set in terms of $e^{\frac{2\pi i}{\ell} n \xi^{\pm}}$ for $\epsilon^{\pm}(\xi^{\pm})$ which then gives a complete set of operators

$$V_n = -\frac{\ell}{2\pi} e^{\frac{2\pi i}{\ell} n \xi^-}\partial_- \qquad \tilde{V}_n = -\frac{\ell}{2\pi} e^{\frac{2\pi i}{\ell} n \xi^+}\partial_+$$

These operators have commutation relations

$$\boxed{[V_m, V_n] = i(n-m)\,V_{n+m} \quad \text{and similar for } \tilde{V}_m\text{'s}}$$

This Lie algebra gives precisely the Witt algebra

Remark

Only in 2 dim the conformal algebra is infinite dimensional

$$\text{Witt} \oplus \overline{\text{Witt}}$$

In $D>2$, the conformal algebra is

$$SO(2,D) \supseteq SO(1,D-1) \quad \text{\& special conf. transformations}$$

Thus, in $D=2$:

$$S(2,2) \simeq \underbrace{s\ell(2,\mathbb{R})}_{\{V_0,\, V_{\pm 1}\}} \times \underbrace{s\ell(2,\mathbb{R})}_{\{\tilde{V}_0,\, \tilde{V}_{\pm 1}\}}$$

which is the "global" part of $\text{Witt} \oplus \text{Witt}$

The appearance of the conformal symmetry suggests that the 2dim field theory on the WS of the string is in fact a 2dim conformal field theory.

We will look into this later

(See CFT course in Trinity term, or Polchinski vol 1 chapter 2)

Next: quantisation.

Chapter 2 Old covariant quantisation

There are several approaches (and the relation between them is not trivial)

1 Covariant BRST quantisation

modern path integral quantisation

$$Z = \int \frac{[\mathcal{D}X^\mu][\mathcal{D}\gamma]}{\mathrm{Vol}(\mathrm{Diff} \times \mathrm{Weyl})} e^{\frac{i}{\hbar} S_P [X^\mu, \gamma]}$$

This is the best quantum treatment of gauge theories

(uses Fadeev-Popov-deWitt gauge fixing & identifies BRST symmetries & currents

cancelation of Weyl anomaly requires $D = 26$)

$\rightsquigarrow$ Right thing to do but requires more experience (AQFT) and takes longer

2 Light-cone quantisation

Fix all gauge symmetry in the classical theory
(so Virasoro constraints are implemented classically)
But then the classical theory is not Poincaré invariant.

Hard work to see that anomaly cancels
(subtleties eg $\sum_{n=1}^{\infty} n = -\frac{1}{12} = \zeta(-1)$)

3 Old covariant quantisation ⟵

Start with the classical system in the conformal gauge and then quantise (promoting X^{μ} & Π^{μ} to operators, ...)
One imposes the constraints $T_{\pm\pm} = 0$ on the quantum Hilbert space

⇝ manifestly covariant

→ one need $D = 26$ to cancel anomaly in Virasoro algebra

2.1 Introduction

Classical theory

$$S_P = -\frac{T}{2}\int_\Sigma d\tau d\sigma \left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)$$

in the conformal unit gauge $\gamma_{ab} = \eta_{ab}$.

This is supplemented by the constraints

$$T_{++} = 0 \quad \& \quad T_{--} = 0.$$

The OCQ approach consists on promoting the canonical fields X^μ & their conjugate momenta $\Pi^\mu = T\partial_\tau X^\mu$ to operators and the Poisson brackets $\{\cdot,\cdot\}_{PB}$ to commutators of operators

$$\{\cdot,\cdot\}_{PB} \rightsquigarrow i[\cdot,\cdot]$$

We get the canonical equal time commutation relations

$$[\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}$$

(with $[X^\mu(\sigma), X^\nu(\sigma')] = 0$, $[P^\mu(\sigma), P^\nu(\sigma')] = 0$)

The operators X^μ & Π^μ are hermitian

$$X^\mu = (X^\mu)^\dagger, \qquad \Pi^\mu = (\Pi^\mu)^\dagger$$

this replaces the reality conditions of the classical fields

The commutation relations for the oscillator modes follow immediately from this:

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \qquad \hat{p}^\mu, \hat{x}^\nu \text{ are Hermitian}$$

(Heisenberg algebra)

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \qquad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \qquad (\tilde{\alpha}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n}$$

This forms an infinite set of harmonic oscillators ($\alpha \to \frac{1}{\sqrt{m}}\alpha$) together with the Heisenberg pair $\{x^\mu, p^\mu\}$

Now we construct the Hilbert space in the usual way

ie construct states as (irred) representations of the operators $\{\hat{x}^\mu, \hat{p}^\mu, \alpha^\mu_m, \tilde{\alpha}^\mu_m\}$

2.2 Hilbert space (without constraints)

Define the oscillator vacuum state $|0\rangle_{vac}$

$$\alpha_m^\mu |0\rangle_{vac} = \tilde{\alpha}_m^\mu |0\rangle_{vac} = 0 \qquad \forall\, m>0$$

ie the state which is annihilated by all $\alpha_m^\mu\ (\tilde{\alpha}_m^\mu)\ \forall\, m>0$

On top of $|0\rangle_{vac}$, we build the oscillator Fock spaces

ie states constructed by applying creation operators $\alpha_{-n}^\mu\ (\tilde{\alpha}_{-n}^\mu)$, $n\geq 1$

$$\mathcal{H}_{open}^{Fock} = \mathrm{Span}\left\{ \prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} |0\rangle_{vac} \right\}_{n_i\geq 1}$$

$$\mathcal{H}_{closed}^{Fock} = \mathrm{Span}\left\{ \prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i} \prod_{j=1}^{l} \tilde{\alpha}_{-n_j}^{\mu_j} |0\rangle_{vac} \right\}_{n_{i,j}\geq 1} = \underset{\mathcal{H}_{open}^{Fock}}{\overset{\mathcal{H}_R^{Fock}}{\cong}} \otimes \underset{\mathcal{H}_{open}^{Fock}}{\overset{\tilde{\mathcal{H}}_L^{Fock}}{\cong}}$$