Axiomatic Set Theory: Problem sheet 1

Apologies for the lateness of this; I've been ill and everything is behind.

A.

1. Write the following as formulas of LST:
(a) $x = \langle y, z \rangle$;

In the solutions I will use symbols such as $\exists$ and $\wedge$ which are not in the language, but for which it is clear how to replace them with symbols that are in the language. I will also sometimes omit brackets, when it clear how to reinsert them.

$$\exists w \exists v \Big(w \in x \wedge v \in x \wedge \forall u \big(u \in x \leftrightarrow (u = v \vee u = w) \big) \\ \wedge \big(\forall t (t \in v \leftrightarrow t = y) \wedge \forall t (t \in w \leftrightarrow (t = y \vee t = z)) \big) \Big).$$

(b) $x = y \times z$;

In this solution, I will use the notation $\langle \cdot, \cdot \rangle$, which we already know how to eliminate, by the previous part.

$$\forall w \big(w \in x \leftrightarrow \exists a \in y \, \exists b \in z \, w = \langle a, b \rangle \big).$$

(c) $x = y \cup \{y\}$;

$$\forall z \big(z \in x \leftrightarrow (z \in y \vee z = y) \big).$$

(d) "x is a successor set";

$$\exists y \in x \, \forall z \, (\neg z \in y) \wedge \forall y \in x \, \exists z \in x \, \big(z = y \cup \{y\} \big).$$

(e) $x = \omega$.

In this solution, I will use the notation $\mathbf{S}(x)$ to refer to a formula expressing that x is a successor set.

$$\mathbf{S}(x) \wedge \forall y \, (\mathbf{S}(y) \to x \subseteq y).$$

I leave it as an exercise how to express "$x \subseteq y$".

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2. Deduce the Axiom of Pairs from the other axioms of ZF*.

Let x and y be sets.

By the Axiom of Infinity, there is a successor set. It follows that the two-element set $2 = \{\varnothing, \{\varnothing\}\}$ exists also, as it must, by definition, by an element of any successor set (being the double successor of the empty set).

Now define a class term F so that $F(\varnothing) = x$, $F(\{\varnothing\}) = y$, and F takes some value or other for any other set.

By Replacement, $\{F(\varnothing), F(\{\varnothing\})\} = \{x, y\}$ is a set.

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3. Assuming ZF, show that if a is a non-empty transitive set then $\varnothing \in a$.

Apply Foundation to a. Let m be an element of a disjoint from a.

Suppose that $z \in m$. Then $z \in a$ because a is transitive, giving a contradiction.

Therefore m is empty, as required.

(Note that Foundation is essential here.)

B.

4. Which of the Axiom of Extensionality, the Empty Set Axiom, the Powerset Axiom, and the Axiom of Infinity hold in the structure $\langle \mathbb{Q}, < \rangle$? Also, find an instance of the Separation Schema that is true in $\langle \mathbb{Q}, < \rangle$ and one that is false.

Write $\mathfrak{Q} = \langle \mathbb{Q}, < \rangle$.

Suppose that in $\mathbb{Q}$, $a < b$. Then it is not the case that $a < a$. Thus $\mathfrak{Q} \models$ Extensionality.

Since, in $\mathbb{Q}$, for all a, $a - 1 < a$, the Empty Set Axiom is false in $\mathfrak{Q}$.

Let $a \in \mathbb{Q}$. We ask what it would mean, for an element b of $\mathbb{Q}$, to say that $\mathfrak{Q} \models b = \wp(a)$.

If $a < c$, then for all $d < a$, $d < c$, so $\mathfrak{Q} \models a \subseteq c$. It is easy to see that this also works if $a = c$, and it does not work if $c < a$.

So for all d, we would need that $\mathfrak{Q} \models d \in b \leftrightarrow (d < a \vee d = a)$. But in $\mathbb{Q}$, no element has an immediate successor, so the Powerset Axiom is false in $\mathfrak{Q}$.

Since the Empty Set Axiom is false in $\mathfrak{Q}$, the Axiom of Infinity must be false also.

Now we look at the Separation Schema, as applied to an element a of $\mathbb{Q}$.

The instance of the Schema corresponding to the expression $x = x$ is true, since $\mathfrak{Q} \models \forall y\big(y \in a \leftrightarrow (y \in a \wedge y = y)\big)$, so a itself is the required "subset" of a.

Since the Empty Set Axiom is false, the instance of the Schema corresponding to the formula $\neg x = x$ must fail.

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5. Assuming ZF*, show that there exists a *transitive* set M such that

(a) $\varnothing \in M$, and

(b) if $x \in M$ and $y \in M$, then $\{x, y\} \in M$, and

(c) every element of M contains at most two elements.

Show further that if σ is an axiom of ZF*+AC other than the Axioms of Infinity, Unions and Powerset, then $\langle M, \in \rangle \models \sigma$. (It follows that if ZF* is consistent then so is this reduced set of axioms, together with the Axiom of Choice.)

Let $F(x)$ be a class term such that $y \in F(x)$ if and only if y is a subset of x having at most two elements. ($F(x)$ exists as a set by the Powerset Axiom and the Separation Schema.) Let $G(x)$ be a class term such that $G(x) = x \cup F(x)$.

Use recursion on ω to define a function g with domain ω such that $g(0) = \varnothing$, and for all $n \in \omega$, $g(n+1) = G(g(n))$.

Let $M = \bigcup \operatorname{ran} g$.

Then M is as required.

M is transitive, so it satisfies the Axiom of Extensionality. It satisfies the Empty Set Axiom and the Axiom of Pairs by construction.

Since every subset of a set with at most two elements has at most two elements, M satisfies the Separation Schema.

Since every set has at most two elements, the Replacement Schema in M is equivalent to the Axiom of Pairs, which is satisfied in M.

Suppose that a, $b \in M$. If the Axiom of Foundation is false about $\{a, b\}$ in M, then we must have $a \in a$, $b \in b$, or $a \in b$ and $b \in a$. Now we can recursively define a function $f : M \to \omega$ such that $f(\varnothing) = 0$, and $f(\{a, b\}) = \max(f(a), f(b)) + 1$. (We can use recursion on n to define $f \restriction n$, for example.) Then we find that if $a \in b$, then $f(a) < f(b)$. So failures of Foundation are impossible in M.

As for the Axiom of Choice, Suppose $A \in M$, and A is a non-empty set of non-empty sets. Suppose $A = \{a, b\}$. Let c be an element of a and let d be an element of b. Then $\{c, d\}$ is an element of M, and is a witness to this instance of the Axiom of Choice.

C.

6. (a) Assuming ZF (ie. ZF*+Foundation) prove that the following two definitions of "ordinal" are equivalent:

(i) An ordinal is a transitive set well-ordered by $\underline{\in}$.

(ii) An ordinal is a transitive set totally ordered by $\in \top =$.

It is obvious that any transtivie set well-ordered by $\underline{\in}$ is totally ordered by that relation.

Now suppose that α is a transitive set totally ordered by $\underline{\in}$.

Let S be a non-empty subset of α.

We apply Foundation to S. Suppose that β is an element of S such that $\beta \cap S = \varnothing$.

Suppose that γ is an element of S. Then since α is totally ordered by $\underline{\in}$, $\beta \underline{\in} \gamma$, or $\gamma \underline{\in} \beta$. Since $\beta \cap S = \varnothing$, it cannot be the case that $\gamma \in \beta$. So $\beta \underline{\in} \gamma$.

So β is the least element of S.

So every non-empty subset of α has a $\underline{\in}$-least element, so α is well-ordered.

(b) Prove the principle of induction for **On** using only ZF*.

Let $\phi(\alpha)$ be a statement in the language of set theory such that in V^*, $\phi(0)$ is true, that if $\phi(\alpha)$ is true, then so is $\phi(\alpha + 1)$, and that if for all α less than a limit λ, $\phi(\alpha)$ is true, then so is $\phi(\lambda)$.

Suppose that there exists an ordinal α such that $\phi(\alpha)$ is false.

Let $S = \{\beta \in \alpha + 1 : \neg\phi(\beta)\}$.

Then S is a non-empty subset of the ordinal $\alpha + 1$, which must have a least element, since $\alpha + 1$ is well-ordered. Let β be the least element of S.

Then whether β is 0, a successor, or a limit, the hypothesis about ϕ tells us that in fact $\phi(\beta)$ must be true, giving a contradiction.

The the Principle of Induction is true for **On**.

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7. (ZF) Let H_ω denote the class of *hereditarily finite sets*, ie. $H_\omega = \{x : TC(x)$ is finite$\}$. Prove that $H_\omega = V_\omega$ (and hence that H_ω is a set). Prove that $\langle V_\omega, \in\rangle \models$ the axiom of foundation, and $\langle V_\omega, \in\rangle \models \neg$ the axiom of infinity.

[It is easy, but tedious, to check that $\langle V_\omega, \in\rangle \models$ the other axioms of ZF. This shows that the other axioms of ZF do not imply the axiom of infinity.]

We first show that $V_\omega \subseteq H_\omega$, by showing, by induction on n, that for all n, that V_n is finite, and that $V_n \subseteq H_\omega$. This is trivial if $n = 0$, and if $V_n \subseteq H_\omega$ and V_n is finite, then $V_{n+1} = \wp V_n$ is finite; and if $x \in V_{n+1} \setminus V_n$, then $x \subseteq V_n$ and is therefore finite, and

$$\mathrm{TC}(x) = x \cup \bigcup_{y \in x} \mathrm{TC}(y),$$

so that $\mathrm{TC}(x)$, being a finite union of finite sets, is finite.

Now suppose that it is not the case that $H_\omega \subseteq V_\omega$. Let $W = H_\omega \setminus V_\omega$.

We apply the Axiom of Foundation to W, to find $m \in W$ such that $m \cap W = \varnothing$.

Then since $m \in H_\omega$, m is finite; say $m = \{a_0, \ldots, a_{k-1}\}$ for some natural number k. Then for all i, $a_i \in V_\omega$. Recalling that $V_\omega = \bigcup_{n \in \omega} V_n$, let n_i be such that $a_i \in V_n$. Let $n = \max_{i<k} n_i$.

Now it is the case that if $n < n'$, then $v_n \subseteq V_{n'}$.

So m is a subset of V_n, and thus an element of V_{n+1}, giving a contradiction.

So $H_\omega = V_\omega$, as required.
