Geometric Group Theory

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My favourite quotation

William Thurston: "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding."

Residually finite groups

Residually finite groups are groups that can be approximated by finite quotients.

Definition

A group is residually finite if one of the following equivalent properties is satisfied:

1. $$\bigcap_{H \leq_{f.i.} G} H = \{1\}$$

2. For all non-trivial $g \in G$, there exists $\phi : G \to F$ finite such that $\phi(g) \neq 1$.

3. For all $\{g_1, ..., g_n\}$ distinct, there exists $\phi : G \to F$ such that $\phi(g_1), ..., \phi(g_n)$ are distinct. In other words, every finite chunk of the infinite Cayley table of G can be reproduced identically in the Cayley table of a finite quotient.

Residually finite groups

Examples

1. *$GL(n, \mathbb{Z})$ is residually finite.*

2. *Any finitely generated $G \leq SL(n, \mathbb{Q})$ (or $GL(n, \mathbb{Q})$) is RF.*

3. *$(\mathbb{Q}, +)$ is not RF (and nor is $SL(n, \mathbb{Q})$):*

   if we have some $\phi : \mathbb{Q} \to F$, $\phi(0) = \mathrm{id}$, take $g = \phi(1)$, $n = |F|$ and then $g = \phi(1) = \phi(\frac{1}{n})^n = \mathrm{id}$.

Theorem (Mal'cev)

Let R be a commutative ring with unity. Any finitely generated $G \leq GL(n, R)$, G is residually finite.

Residually finite groups

Proposition

1. *If G is RF and $H \leq G$ then H is RF.*
2. *If H is a finite index subgroup of G then H is RF if and only if G is RF.*
3. *If two groups G and H are RF then $G \times H$ is RF.*
4. *If $G = H \rtimes K$ where H is finitely generated RF, K is RF, then G is RF.*

Proposition

For all finite or countable X, $F(X)$ is residually finite.

Proof: We have that $F_2 \simeq \langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \rangle \leq GL(2, \mathbb{Z})$. And for all X finite or countable, $F(X) \leq F_2$. □

Residually finite groups

Theorem

A finitely presented residually finite group has a solvable word problem.

Remark

Note that every finite group has a solvable word problem.

Proof.

Suppose $G = \langle S | R \rangle$. Take $w \in F(S)$. Run simultaneously two procedures:

1. List all the elements in $\langle\langle R \rangle\rangle$ and check if w is among them.
2. List all homomorphisms $\phi : F(S)/\langle\langle R \rangle\rangle \to S_n, n \in \mathbb{N}$, and check if $\phi(w) \neq 1$.

□

Residually finite groups

Definition

G is Hopf if every onto homomorphism $f : G \to G$ is an isomorphism.

Example

Every finite group is Hopf.

Theorem

A finitely generated residually finite group is Hopf.

Residually finite groups

Theorem

A finitely generated residually finite group is Hopf.

Proof.

Assume there exists an onto homomorphism $f : G \to G$ that is not 1-to-1.

Take $g \in \ker f \setminus \{1\}$. There exists $\phi : G \to F$ with $\phi(g) \neq 1$. Construct a sequence

$$g = g_0,\ \ g_1 \in f^{-1}(g_0),\ \ g_2 \in f^{-1}(g_1),\ \ \dots\ ,\ \ g_n \in f^{-1}(g_{n-1})$$

$\forall n$, $f^n(g_n) = g$ and $f^k(g_n) = 1$ for all $k > n$. Hence, for all $n > \ell$, $\phi \circ f^n(g_\ell) = 1$ and $\phi \circ f^n(g_n) \neq 1$. So the homomorphisms $\phi \circ f^n$ are pairwise distinct. But this contradicts $\mathrm{Hom}(G, F) \leq |F|^{|S|}$. □

Residually finite groups

Theorem

A finitely generated residually finite group is Hopf.

Corollary

If $F(X) = \langle A \rangle$ and $|A| = |X| = n < \infty$, then $F(X) \simeq F(A)$. (i.e. A freely generates F(X) i.e. A is a free basis for $F(X)$).

Proof.

A bijection $X \to A$ extends to $X \to F(A)$ which extends to an onto homomorphism $F(X) \to F(A)$. By Universal Property, we have a second onto homomorphism, hence an onto hom. $F(X) \to F(A) \to F(X)$. Since $F(X)$ is Hopf, the latter hom. is an isomorphism, hence all are. □

Residually finite groups. Simple groups

Theorem

A finitely generated residually finite group is Hopf.

The assumption finitely generated cannot be dropped from the theorem.

Example

- *Consider* X, Y *countable.*
- *There exists* $f : X \to Y$ *onto and not injective.*
- f *extends uniquely to an onto group homomorphism* $F(X) \to F(Y)$.

At the other extreme, we have simple groups.

Definition

G is simple if the only normal subgroups are $\{1\}$ and G.

Simple groups

Example

$\mathbb{Z}/p\mathbb{Z}$, A_n, A_∞, $PSL(n,\mathbb{Q})$, infinite f.g. due to Higman, Thompson, Olshanskii, Burger-Mozes.

Theorem

A finitely presented simple group has solvable word problem.

Proof.

Let $w \in F(S)$. Since G is simple, if $w \neq 1$ in G then $G = \langle\langle w \rangle\rangle$ and hence $\langle\langle \{w\} \cup R \rangle\rangle = F(S)$.

Two procedures:

1. Enumerate $\langle\langle R \rangle\rangle$. Check if w appears.
2. Enumerate $\langle\langle \{w\} \cup R \rangle\rangle$. Check if every $s \in S$ appears.

□

Graphs and Cayley graphs

A main method of investigation is to endow an infinite group with a geometry compatible with its algebraic structure, i.e. invariant by multiplication. This can easily be done for finitely generated groups via Cayley graphs.

Given a countable group G and a subset S such that $S^{-1} = S$, the Cayley graph of G with respect to S, denoted $\Gamma(S, G)$, is a directed/oriented graph with

- set of vertices G;
- set of oriented edges $\{(g, gs) : g \in G, s \in S\}$;

We denote an edge $[g, gs]$. The underlying non-oriented graph is denoted $\hat{\Gamma}(S, G)$.