1 Problem 7a, sheet 1

The problem is stated informally, and a major part of the solution is to decide on a precise interpration; is "finite" meant in the formal language of ZF, or in the metalanguage? Below, we discuss possible solutions.

Background

Let M be a model of at least the extensionality axiom. Note that M is a structure for the language with one predicate symbol ε; it consists of the universe of M (a set also denoted by the letter M) and a subset of M^2 (interpreting ε.)

Definition. *Let $A \subset M$ be any subset of M. We say that the set A is* represented *in M by an element c of M for all $m \in M$, we have*

$$M \models m \,\varepsilon\, c \text{ iff } m \in A$$

We say that A is represented in M if it is represented by some $c \in M$.

Definition. A *transitive model* (of ZFC, or of a smaller theory) is a model M whose universe is a transitive set, and such that the interpretation in M of the membership symbol is precisely the membership relation: for $m, e \in M$ we have: $M \models m \,\varepsilon\, e$ iff $m \in e$.

Exercise 1.1. *When M is a transitive model, a subset A of M is represented in M iff $A \in M$.*

Remark. The completeness and soundness theorems of logic tell us, for any sentence ϕ, that $ZF \vdash \phi$ iff for all models M of ZF we have $M \models \phi$. This is *not* true if one only considers transitive models. Gödel's theorem implies that there are statements true in every transitive model of ZF, but not provable in ZF.

1

Our first interpretation of the problem will use transitive models. The result will thus be somewhat weaker - but much simpler! - than a formulation with arbitrary models.

Let ENPU denote the theory (sub-theory of ZF) consisting of Extensionality, Null Set, Pair Set and Union.

Problem 7a - transitive model formulation. Let M be a transitive model of ENPU. Let A be a finite subset of M. Show that $\mathcal{P}(A) \in M$.

Proof. **Claim 1.** If A is a finite subset of M, then $A \in M$.

Proof. Write $A = \{a_1, \ldots, a_n\}$. Since M is transitive, each $a_i \in M$. We will show by induction that $\{a_1, \ldots, a_k\} \in M$. If $k = 0$, this means that $\emptyset \in M$; that follows from the Null Set axiom. Assume inductively that $\{a_1, \ldots, a_k\} \in M$. By the pair set axiom, the two-element set $\{a_{k+1}, \{a_1, \ldots, a_k\}\}$ lies in M. By the union axiom, $\{a_1, \ldots, a_k\} \in M$, proving the claim. □

Now given a finite $A \subset M$, by induction each subset of A is an element of M; hence $\mathcal{P}(A) \subset M$. Since $\mathcal{P}(A)$ is finite, using the claim again, $\mathcal{P}(A) \in M$.
□

1.5

We next give a fuller version, using all models; the structure of the proof is the same, but one must talk about sets being represented by elements of the model, rather than being elements of the model. Let us write $t \subseteq z$ as shorthand for $(\forall s)(s \,\varepsilon\, t \to s \,\varepsilon\, z)$, and $y = \mathcal{P}(z)$ as shorthand for: $(\forall t)(t \,\varepsilon\, y \iff t \subset z)$.

Problem 7a: formulation with arbitrary models. Let M be a model of Extensionality, Null Set, Pair Set and Union. Let $a_1, \ldots, a_n \in M$ and let a represent $\{a_1, \ldots, a_n\}$ in M. Show that $M \models (\exists y)(y = \mathcal{P}(a))$.

Proof. **Claim 1.** Any finite subset A of M is represented by some element of M.

Proof. If $A = \emptyset$, this is an assumption: A is represented by any witness to the Null Set axiom, $M \models (\exists x)(\forall t)(t \notin x)$.

We continue by induction on the cardinality of A. Say $A = \{a_1, \ldots, a_n\}$ with $n \geq 1$. Let $A' = \{a_1, \ldots, a_{n-1}\}$. Then by induction A' is represented by some e', while $\{a_n\}$ is represented by some e''. By the Pair Set and Union axioms, there exists $m \in M$ such that

$$M \models (\forall t)[(t \,\varepsilon\, m) \iff t \,\varepsilon\, e' \vee t \,\varepsilon\, e'']$$

So A is represented by m. □

Now let $A = \{a_1, \ldots, a_n\}$ be a finite subset of a model M . By Claim 1, each subset $s \subset A$ is represented by some $e_S \in M$. Since $\{e_S : S \subset A\}$ is also finite, again using Claim 1, it is represented by some $e \in M$.

Now it is possible to verify that $M \models e = P(a)$ (think through this!)

□

Remark. For each $n \in \mathbb{N}$, let $\alpha_n(x)$ be the formula asserting that x has at most n elements:

$$\alpha_n(x) = (\exists t_1) \ldots (\exists t_n)(\forall t)(t \,\varepsilon\, x \to (t = t_1 \vee t = t_2 \vee \cdots \vee t = t_n))$$

Using the completeness theorem, it follows that for any given n,

$$ENPU \vdash (\forall x)(\alpha_n(x) \to (\exists y)(y = \mathcal{P}(x))$$

Note that this is an infinite collection of sentences, not a single one.

For any fixed n, as an alternative to the above argument using the completeness theorem, it is possible to directly prove the displayed sentence; the length of such a proof will be on the order of 2^n. Think through it for $n = 2$ - it is already somewhat cumbersome, including the verification at the end.

2

The solution above, whether with transitive models or all models, treated 'finite' as stated in the metalanguage. It is also possible to interpret the word 'finite' in the problem internally. Let $\beta(x)$ be the formula expressing 'x is finite', as in the course notes. Skipping ahead a little, we will allow using also the replacement axiom in this case.

Problem 7a: formulation with internal definition of 'finite'.

Exercise 1.2 (Optional)**.** *Let $ZF-$ denote ZF without the power set axiom. Then* $ZF- \vdash (\forall x)(\beta(x) \to (\exists y)(y = \mathcal{P}(x))$.

(Hint: the replacement axiom makes it easy to complete the proof begun in class.)