Geometric Group Theory

Cornelia Druțu

University of Oxford

Part C course HT 2024

Cayley graphs

A finitely generated group can be endowed with a geometry compatible with its algebraic structure, i.e. invariant by multiplication via a Cayley graph.

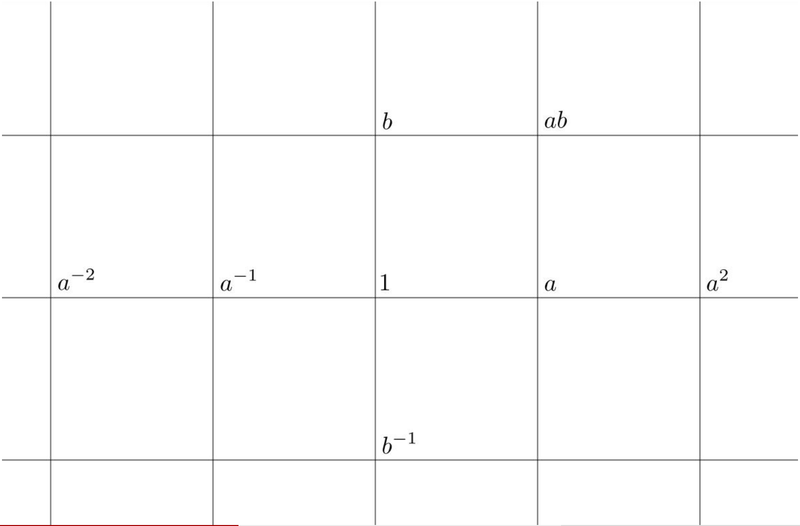
Given a countable group G and a subset S such that $S^{-1} = S$, the Cayley graph of G with respect to S, denoted $\Gamma(S, G)$, is a directed/oriented graph with

- set of vertices G;
- set of oriented edges $\{(g, gs) : g \in G, s \in S\}$;

We denote an edge $[g, gs]$. The underlying non-oriented graph is denoted $\hat{\Gamma}(S, G)$.

Examples of Cayley graphs

1. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), (0, \pm 1)\}$

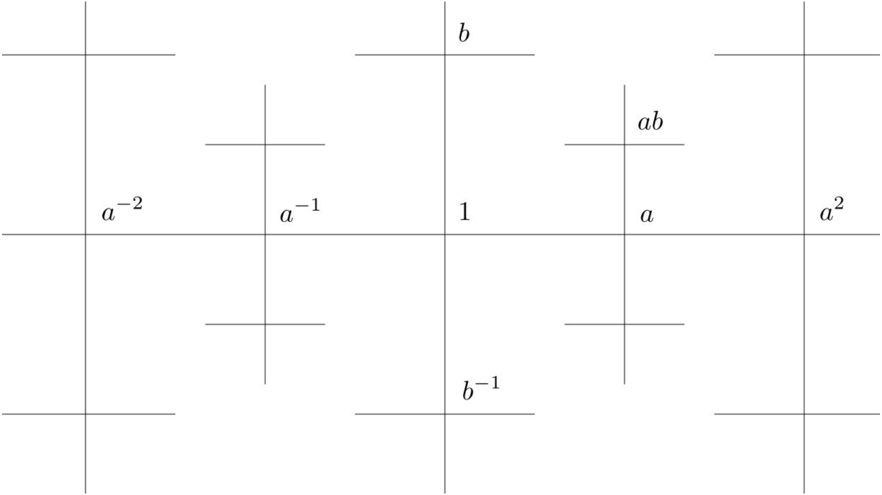


Examples of Cayley graphs

2. $\mathbb{Z}^2$ with $S = \{(\pm 1, 0), \pm(1, 1)\}$

Examples of Cayley graphs

3. $F_2 = F(\{a, b\})$ with $S = \{a^{\pm 1}, b^{\pm 1}\}$



Examples of Cayley graphs: the integer Heisenberg group

The Integer Heisenberg group:

$$H_{2n+1}(\mathbb{Z}) := \langle x_1, \dots, x_n, y_1, \dots, y_n, z\,;$$

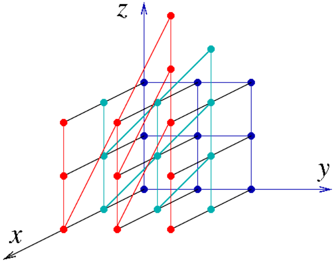
$$[x_i, z] = 1, [y_j, z] = 1, [x_i, x_j] = 1, [y_i, y_j] = 1, [x_i, y_j] = z^{\delta_{ij}}, 1 \leqslant i, j \leqslant n\rangle.$$

$$H_{2n+1}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x_1 & x_2 & \dots & \dots & x_n & z \\ 0 & 1 & 0 & \dots & \dots & 0 & y_n \\ 0 & 0 & 1 & \dots & \dots & 0 & y_{n-1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & \dots & 1 & 0 & y_2 \\ 0 & 0 & \dots & \dots & 0 & 1 & y_1 \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \;;\; x_i, y_j, z \in \mathbb{Z} \right\}$$

Examples of Cayley graphs: the Integer Heisenberg group

5. $H_3(\mathbb{Z}) := \langle x, y, z \mid [x, z] = 1, [y, z] = 1, [x, y] = z \rangle.$

2/9/23, 4:25 PM HeisenbergCayleyGraph.png (533×423)

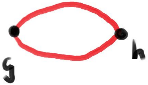


https://upload.wikimedia.org/wikipedia/commons/c/c7/HeisenbergCayleyGraph.png 1/1

Particular features of Cayley graphs

1. No monogons (edges of the form $[g, g]$) if $1 \notin S$.

2. No digons if, when $s = s^{-1}$, we do not list both s and s^{-1} in S (i.e. no repetitions in S).



   In other words, this is a simplicial graph.

3. $\Gamma(S, G)$ is connected (i.e. any two vertices can be connected by an edge path) if and only if $G = \langle S \rangle$.
4.
   a. $\Gamma(S, G)$ is regular: the valency/degree of every vertex is $|S|$.
   b. $\Gamma(S, G)$ is moreover locally finite if and only if $|S| < \infty$.

Particular features of Cayley graphs

5. If $\Gamma(S, G)$ is infinite then it always contains a bi-infinite geodesic.

6. $\Gamma(S, G)$ always contains a cycle (in fact plenty) with one exception: $\Gamma(S, G)$ does not contain a cycle (i.e. it is a simplicial tree) $\iff$ $S = X \sqcup X^{-1}$ and $G = F(X)$.

Cayley Graphs

From now on, assume that S is a finite generating set (with no repetitions), $1 \notin S$, $S = S^{-1}$. We endow $\Gamma(S, G)$ with a metric d_S:

- each edge has length 1;
- $d_S(x, g)$ is the length of a shortest path from x to g.

Proposition

The action of G on its Cayley graph is an action by isometries. The action is free on the vertices. It is free on the whole Cayley graph if and only if no $s \in S$ is of order 2.

Proof.

We have a map

$$G \to \mathrm{Isom}(\Gamma(S, G)) \quad g \mapsto L_g$$

where $L_g \in \mathrm{Isom}(\Gamma(S, G))$ extends $L_g : G \to G$, $L_g(x) = gx$ to edges. □

Cayley Graphs

Definition

The restriction of d_S to $G \times G$ is called the word metric associated to S.

Exercises

- $|g|_S := d_S(1, g)$ *is the minimum length of a word w in S such that $g =_G w$.*
- $d_S(g, h)$ *is the minimum length of a word w in S such that $gw =_G h$.*

Proposition

If $G = \langle S \rangle = \langle \bar{S} \rangle$ then d_S and $d_{\bar{S}}$ are bi-Lipschitz equivalent. That is, there exists $L > 0$ such that

$$\frac{1}{L} d_S(g, h) \leq d_{\bar{S}}(g, h) \leq L d_S(g, h)$$

for every $g, h \in G$.

Cayley Graphs. Actions on simplicial trees

A simplicial tree is a connected graph with no monogons, digons or cycles.

Theorem

$\hat{\Gamma}(S, G)$ is a simplicial tree on which G acts freely $\iff$ $S = X \sqcup X^{-1}$, $G = F(X)$.

Proof.

Oriented paths in $\Gamma(S, G)$ without spikes correspond to pairs (g, w), w a reduced word in S.
$(\Leftarrow)$: A cycle would correspond to a reduced word $w = 1$ in $F(X)$.
$(\Rightarrow)$: G acts freely $\implies$ $\forall s \in S$, $|\{s, s^{-1}\}| = 2$. For every such pair, pick one and together let these form X. X generates G and so there exists an onto homomorphism $\varphi : F(X) \to G$. Suppose $w \in F(X)$, $w \in \ker \varphi$. Since w is reduced as a word in X, it is also reduced as a word in S. So if $w \neq w_\emptyset$ then w gives a cycle in $\hat{\Gamma}(S, G)$. So $\ker \varphi = \{w_\emptyset\}$. □

Actions on simplicial trees

General theorem: G is free if and only if G acts freely by isometries on a simplicial tree T. The $(\Rightarrow)$ direction is given by the previous theorem.

For the $(\Leftarrow)$ direction, we use the following lemma.

Lemma

There exists $X \subseteq T$, X a tree, such that X contains exactly one vertex from each orbit.